Klasszikus térelmélet

pontmechanika		térelmélet
$q_i(t)$ ↖ index, véges sok	$\Longrightarrow$	$\phi_{k,\vec{x}}(t) = \phi_k(\vec{x},t) = \phi_k(x)$
$S = \int dt\, L$	$\Longrightarrow$	$S = \int d^4x\, \mathcal{L}(\phi_k, \partial_\mu \phi_k, \cancel{\ldots})$

Tfh. magasabb deriváltaktól nem függ

Mozgásegyenletek

$$S = \int \mathcal{L}(\phi_k, \partial_\mu \phi_k)\, d^4x$$

Tfh. $\phi_k(x)$, $\partial_\mu \phi_k(x)$ a végtelenben lecsengenek

$$\delta S = \int \left[\sum_k \frac{\partial \mathcal{L}}{\partial \phi_k} \delta\phi_k + \sum_k \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \delta(\partial_\mu \phi_k) \right] d^4x$$

$$\int \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \delta(\partial_\mu \phi_k)\, d^4x = \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \delta\phi_k \right) d^4x - \int \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \right) \delta\phi_k\, d^4x$$

$$\Longrightarrow \delta S = \int \sum_k \left(\frac{\partial \mathcal{L}}{\partial \phi_k} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \right) \delta\phi_k\, d^4x$$

$$\delta S = 0 \quad \forall \delta\phi_k \Longrightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi_k} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} = 0} \quad \forall k$$

$\mathcal{L}$ csak egy teljes négyesdivergencia erejéig meghat.

$$\mathcal{L}' = \mathcal{L} + \partial_\mu f^\mu(\phi, \partial_\mu \phi) \Longrightarrow S' = S \text{ ha } f^\mu \text{ lecseng}$$

Példák ① valós skalár $S = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \right]$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -V'(\phi) = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi = \Box \phi$$

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{g_3}{3!}\phi^3 + \frac{g_4}{4!}\phi^4 \Longrightarrow \Box\phi = -m^2\phi - \frac{g_3}{2}\phi^2 - \frac{g_4}{6}\phi^3$$

② komplex skalár $\Phi = \frac{1}{\sqrt{2}}(\underbrace{\phi_1 + i\phi_2}_{\in \mathbb{R}})$ $\Phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$ ②

$$S' = \int d^4x \left[(\partial_\mu \Phi)(\partial^\mu \Phi^*) - V(\Phi^* \Phi)\right]$$

$$= \int d^4x \left[\frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - V\left(\frac{\phi_1^2 + \phi_2^2}{2}\right)\right]$$

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \Rightarrow \partial_\mu \partial^\mu \phi_1 = -V'(|\phi|^2)\,\phi_1 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \phi_2} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \Rightarrow \partial_\mu \partial^\mu \phi_2 = -V'(|\phi|^2)\,\phi_2 \qquad (2)$$

Trükk: Φ és Φ^*-ra flen mezőként tekintünk

$$\frac{\partial \mathcal{L}}{\partial \Phi^*} = -V'(|\Phi|^2)\,\Phi = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} = \partial_\mu \partial^\mu \Phi \Leftarrow (1) + i(2)$$

$$\frac{\partial \mathcal{L}}{\partial \Phi} = -V'(|\Phi|^2)\,\Phi^* = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = \partial_\mu \partial^\mu \Phi^* \Leftarrow$$ előző komplex konjugáltja

Noether-tétel

globális folytonos szimmetria $\Rightarrow$ megmaradó áram j_μ

$$\partial^\mu j_\mu = 0$$

folytonos $\Rightarrow$ vegyük az infinitezimális verziót

$$\phi_k(x) \to \phi_k'(x) = \phi_k(x) + \delta\phi_k(x)$$

amire a hatás invariáns:

$$\delta \mathcal{L} = \partial_\mu \mathcal{J}^\mu$$

másrészt

③

$$\delta\mathcal{L} = \sum_k \frac{\partial\mathcal{L}}{\partial\phi_k}\delta\phi_k + \sum_k \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\partial_\mu\delta\phi_k$$

$$= \underline{\sum_k \frac{\partial\mathcal{L}}{\partial\phi_k}}\delta\phi_k + \sum_k \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\delta\phi_k\right) - \underline{\sum_k\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\right)}\delta\phi_k$$

mozgásegyenlet

$$= \partial_\mu\left(\sum_k \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\delta\phi_k\right)$$

$$\Rightarrow \boxed{\partial_\mu j^\mu = 0} \quad \text{ahol} \quad \boxed{j^\mu = \sum_k \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\delta\phi_k - \mathcal{J}^\mu}$$

Noether-áram

Megjegyzés: van egy kis paraméter amit ki lehet még emelni

példa: komplex skalár

$$\mathcal{L} = (\partial_\mu\Phi)(\partial^\mu\Phi^*) - V(|\Phi|^2)$$

U(1) szimmetria (belső) $\quad \Phi \to \Phi e^{i\delta\chi}$

infinitezimális verzió:
$$\left.\begin{aligned}\Phi' &= \Phi + i\Phi\delta\chi \\ \Phi^{*\prime} &= \Phi^* - i\Phi^*\delta\chi\end{aligned}\right\} \quad \text{vagy} \quad \left.\begin{aligned}\phi_1' &= \phi_1 - \phi_2\delta\chi \\ \phi_2' &= \phi_2 + \phi_1\delta\chi\end{aligned}\right\}$$

$$\mathcal{J}^\mu = 0$$

$$\left.\begin{aligned}j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\delta\phi_1 + \frac{\partial\mathcal{L}}{\partial(\partial\phi_2)}\delta\phi_2 = \phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1 \\ j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\delta\Phi^* = i\left(\Phi\partial^\mu\Phi^* - \Phi^*\partial^\mu\Phi\right)\end{aligned}\right\} \text{ugyanaz}$$

Energia-impulzus tenzor

$$x^\nu \to x^\nu - \delta a^\nu \Rightarrow \phi(x) \to \phi(x+a) \cong \phi(x) + \delta a^\nu (\partial_\nu \phi)$$

$$\delta\phi = \delta a^\nu \partial_\nu \phi = \delta a_\nu \partial^\nu \phi$$

$$\mathcal{J}^\mu = \mathcal{L}\,\delta a^\mu$$

$\mathcal{L}$ is skalár $\Rightarrow \delta\mathcal{L} = \delta a^\nu \partial_\nu \mathcal{L} \overset{!}{=} \partial_\mu \mathcal{J}^\mu$

$$\Rightarrow j^\mu = \sum_k \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \delta a_\nu \partial^\nu \phi_k - \mathcal{L}\,\delta a^\mu$$

$$j^\mu = \delta a_\nu \underbrace{\left[\sum_k \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \partial^\nu \phi_k - \eta^{\mu\nu}\mathcal{L}\right]}_{\boxed{T_{\mu\nu}}}$$

$\boxed{T_{\mu\nu}}$ energia-impulzus tenzor

(emlékeztető: tömegpontra $p_k = \dfrac{\partial L}{\partial \dot{q}_k}$ $\quad H = \sum_k \dfrac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L$)

$$T^{00} = \sum_k \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_k)}(\partial_0 \phi_k) - \mathcal{L} = \mathcal{H} = \text{energiasűrűség}$$

Megmaradás

$$\partial^\mu T_{\mu\nu} = 0 \Rightarrow 0 = \int_{x_0=\text{fix}} (\partial_\mu T^{\mu\nu})\, d^3x = \partial_0 \int_{x_0=\text{fix.}} T^{0\nu} d^3x + \int_{x_0=\text{fix}} \underbrace{\partial_i T^{i\nu}}_{\text{3-as div.}} d^3x$$

$$= \partial_0 \int_{x_0=\text{fix.}} T^{0\nu} d^3x + \cancelto{0}{\oint T^{i\nu} d^3x}$$

(végtelenben lecsengő mezők)

$$\Rightarrow \partial_0 \int_{x_0=\text{konst.}} T^{0\nu} d^3x = 0$$

$$p^\nu = \int T^{0\nu} d^3x = \text{konst.} \leftarrow \text{négyesimpulzus}$$

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T^{ij} jelentése? $\partial_\mu T^{\mu\nu}=0 \implies \partial_0 \int_V T^{0j}\, d^3x = -\oint T^{ij}\, df_i \neq 0$ ha nem zárt rendszer

T^{ij}: impulzusáram sűrűség

T^{0i}: energiaáram sűrűség

Mennyire egyértelmű $T^{\mu\nu}$?

$$T'^{\mu\nu} = T^{\mu\nu} + \partial_\rho \Lambda^{\rho\mu\nu} \quad \text{ahol} \quad \Lambda^{\rho\mu\nu} = -\Lambda^{\mu\rho\nu}$$

szintén megmarad: $\partial_\mu T'^{\mu\nu}=0$ hiszen $\underbrace{\partial_\mu\partial_\rho}_{\text{szimm.}}\underbrace{\Lambda^{\rho\mu\nu}}_{\text{antiszimm.}}=0$

$\Lambda^{\rho\mu\nu}$-vel általában elérhető, hogy $T^{\mu\nu}$ szimmetrikus legyen

<u>Imp. momentum</u>

tömegpont rendszerre: $J^{\mu\nu} = \sum_i \left(x^\mu_{(i)} p^\nu_{(i)} - x^\nu_{(i)} p^\mu_{(i)}\right)$

most: $\sum_i \to \int d^3x$ $\quad p^\mu_{(i)} \to T^{0\mu}$ imp. sűrűség

$$\implies (\#) \boxed{J^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu})\, d^3x} = \int M^{0\mu\nu}\, d^3x$$

$$M^{\alpha\mu\nu} = x^\mu T^{\alpha\nu} - x^\nu T^{\alpha\mu}$$

eddigiek szerint: $\partial_\alpha M^{\alpha\mu\nu} = 0 \iff J^{\mu\nu}$ megmaradó mennyiség

$M^{0\mu\nu}$: imp. mom. sűrűség

$M^{i\mu\nu}$: imp. mom. áramsűrűség

$$\partial_\alpha M^{\alpha\mu\nu} = \partial_\alpha (x^\mu T^{\alpha\nu} - x^\nu T^{\alpha\mu}) = \delta^\mu_\alpha T^{\alpha\nu} + x^\mu \overbrace{\partial_\alpha T^{\alpha\nu}}^{=0} - \delta^\nu_\alpha T^{\alpha\mu} - x^\nu \overbrace{\partial_\alpha T^{\alpha\mu}}^{=0}$$
$$= T^{\mu\nu} - T^{\nu\mu}$$

Tehát szimmetrikus $T^{\mu\nu}$, melyre $\partial_\mu T^{\mu\nu}=0$ olyan anyagi rendszert ír le, ahol teljesül az energia, imp. és impulzusmomentum megmaradás, és az impulzusm. a várt (#) alakban írható

HF $\delta x^\nu = \delta\Omega^{\mu\nu} x_\nu$ szimm.

$\Rightarrow$ $j^\mu_{Noether} = \delta\Omega_{\alpha\beta} M^{\alpha\beta\mu}$ $\Rightarrow$ $\partial_\alpha M^{\alpha\mu\nu}=0$

Példa: valós skalármező $T^{\mu\nu}$ tenzora $\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-V(\phi)$

$$T^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi-\eta^{\mu\nu}\mathcal{L}=(\partial^\mu\phi)(\partial^\nu\phi)-\eta^{\mu\nu}\left[\frac{1}{2}(\partial_\rho\phi)(\partial^\rho\phi)-V(\phi)\right]$$

HF: komplex skalár

EM tér Lagrange-sűrűsége

Használjuk az A_μ tereket (ezzel egyből a forrásmenteset megoldjuk)

Invariánsok: $F_{\mu\nu}F^{\mu\nu}$ $\quad$ $\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$

$\uparrow$ teljes divergencia (HF)

$\Rightarrow$ nem ad járulékot a mozgásegyenletekbe

$\Rightarrow$ Legyen $\mathcal{L}=aF_{\mu\nu}F^{\mu\nu}$

kölcsönhatás: 1 töltésre $S_{k}=S_{kh}=-Q\int A_\mu dx^\mu$

általánosítása $S_{kh}=-\int A_\mu j^\mu d^4x$

$$\mathcal{L} = a F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu$$

mértékinvariancia? $A_\mu \to A_\mu + \partial_\mu f \Rightarrow \mathcal{L} \to \mathcal{L} - j^\mu \partial_\mu f$

$$= \mathcal{L} - \partial_\mu (j^\mu f) + f \underbrace{\partial_\mu j^\mu}_{=0}$$

teljes div. tér el

mennyi az a? mozgásegyenletek összevetése Maxwell egyenleteivel

$$\mathcal{L} = a (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - A_\mu j^\mu$$

$$\Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}}{\partial A_\nu} \Rightarrow \partial_\mu \left(a\, 4 (\partial^\mu A^\nu - \partial^\nu A^\mu)\right) = -j^\nu$$

$$\partial_\mu F^{\mu\nu} = -\frac{1}{4a} j^\nu = \frac{1}{\varepsilon_0} j^\nu$$

$$\Rightarrow a = -\varepsilon_0 / 4$$

$$\boxed{\mathcal{L} = -\frac{\varepsilon_0}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu}$$

EM tér energia-impulzus tenzora

1) töltések nélkül ($j^\mu = 0$)

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \partial^\nu A_\sigma - \eta^{\mu\nu} \mathcal{L} = -\varepsilon_0 F^{\mu\sigma} \partial^\nu A_\sigma - \eta^{\mu\nu} \mathcal{L} = -\varepsilon_0 F^{\mu\sigma} \partial^\nu A_\sigma + \frac{\varepsilon_0}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

nem mértékinvariáns $A_\mu \to A_\mu + \partial_\mu f$

$$\Rightarrow T^{\mu\nu} \to T^{\mu\nu} + (-\varepsilon_0 F^{\mu\sigma} \partial^\nu \partial_\sigma f) \underset{\text{mozg. egy.}}{\Longrightarrow} T^{\mu\nu} + \partial_\sigma (-\varepsilon_0 F^{\mu\sigma} \partial^\nu f)$$

teljes 4-es div.

integrálás után eltűnik, ha a terek lecsengenek a ∞-ben nem szimmetrikus

$$T^{\mu\nu} - T^{\nu\mu} = -\varepsilon_0 F^{\mu\sigma}\partial^\nu A_\sigma + \varepsilon_0 F^{\nu\sigma}\partial^\mu A_\sigma$$

$$\Lambda^{\mu\rho\nu} = +\varepsilon_0 F^{\mu\rho} A^\nu$$

$$T^{\mu\nu} \to T^{\mu\nu} + \partial_\rho \Lambda^{\mu\rho\nu}$$ ← szimmetrizálja

$$-\varepsilon_0 F^{\mu\sigma}\partial^\nu A_\sigma + \partial_\sigma(\varepsilon_0 F^{\mu\sigma} A^\nu) \underset{\uparrow}{=} \varepsilon_0(-F^{\mu\sigma}\partial^\nu A_\sigma + F^{\mu\sigma}\partial_\sigma A^\nu)$$

$$= \varepsilon_0 F^{\mu\sigma} F_\sigma{}^\nu \qquad \partial_\sigma F^{\mu\sigma} = 0$$

$$\boxed{\Theta^{\mu\nu} = \varepsilon_0 F^{\mu\sigma} F_\sigma{}^\nu + \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}}$$

EM mező szimm. energia-impulzus tenzora

(ez már mértékinv. is)

$$\Theta^{00} = \frac{\varepsilon_0}{2}(\vec{E}^2 + \vec{B}^2) = u$$

$$\Theta^{0i} = \varepsilon_0 (\vec{E}\times\vec{B})_i = g_i \qquad \text{Poynting-vektor}$$

$$\Theta^{ij} = -\varepsilon_0\left(E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}(\vec{E}^2 + \vec{B}^2)\right) = -T^{(M)}_{ij}$$

Maxwell-féle feszültség tenzor

$$\Theta^{\mu\nu} = \left(\begin{array}{c|c} u & \vec{g} \\ \hline \vec{g} & -T^{(M)}_{ij} \end{array}\right)$$

Megmaradási törvény: $\partial_\mu \Theta^{\mu\nu} = 0$

$\nu = 0$: $\frac{\partial u}{\partial t} + \operatorname{div}\vec{g} = 0$ energia megm.

$\nu = i$: $\frac{\partial g_i}{\partial t} - \sum_j \frac{\partial T^{M}_{ij}}{\partial x_j} = 0$ impulzus megm.

Megmaradási törvények töltések jelenlétében

$$\partial_\mu F^{\mu\nu} = \frac{1}{\varepsilon_0} j^\nu \neq 0 \quad \Rightarrow \quad \boxed{\partial_\mu \Theta^{\mu\nu} = -F^{\nu\sigma} j_\sigma}$$

Lorentz-erő sűrűség

Biz.: $\partial_\mu \Theta^{\mu\nu} = \varepsilon_0 \, \partial_\mu (F^{\mu\sigma} F_\sigma{}^\nu) + \frac{\varepsilon_0}{4} \partial^\nu (F_{\alpha\beta} F^{\alpha\beta})$

$$= \varepsilon_0 \left[F_\sigma{}^\nu \partial_\mu F^{\mu\sigma} + F^{\mu\sigma} \partial_\mu F_\sigma{}^\nu + \frac{1}{2} F_{\alpha\beta} \partial^\nu F^{\alpha\beta} \right]$$

$(-1)\ \sigma \leftrightarrow \nu$

$$\partial_\mu \Theta^{\mu\nu} + \varepsilon_0 F^\nu{}_\sigma \underbrace{\partial_\mu F^{\mu\sigma}}_{\frac{1}{\varepsilon_0} j^\sigma} = \frac{\varepsilon_0}{2} \left[F_{\mu\sigma} \partial^\mu F^{\sigma\nu} + F_{\mu\sigma} \partial^\mu F^{\sigma\nu} + F_{\mu\sigma} \partial^\nu F^{\mu\sigma} \right]$$

$$= \frac{\varepsilon_0}{2} \Big[\partial^\mu F^{\sigma\nu} + \underbrace{\partial^\mu F^{\sigma\nu} + \partial^\nu F^{\mu\sigma}}_{= \partial^\sigma F^{\mu\nu}} \Big] F_{\mu\sigma}$$

forrásmentes Maxwell

$$= \frac{\varepsilon_0}{2} \underbrace{F_{\mu\sigma}}_{\substack{\text{antiszimm.} \\ \mu,\sigma\text{-ban}}} \underbrace{\left[\partial^\mu F^{\sigma\nu} + \partial^\sigma F^{\mu\nu} \right]}_{\text{szimm.}} = 0$$

$\square$

$$f_L^\nu = F^{\nu\sigma} j_\sigma = (\vec{j}\,\vec{E},\ \rho\vec{E} + \vec{j} \times \vec{B})$$

Emlékeztető:
$$\frac{d\vec{p}_{\text{részecskék}}}{dt} = \vec{F}_L = \int \vec{f}_L \, d^3x$$

A mezőre:
$$\int \partial_\mu \Theta^{\mu\nu} d^3x = \frac{d}{dt} \int \Theta^{0\nu} d^3x + \underbrace{\int \partial_i \Theta^{i\nu} d^3x}_{\oint \Theta^{i\nu} dF \stackrel{\text{lecseng}}{=} 0}$$

$x_0 = \text{fix}$

$$= \frac{d}{dt} p^\nu_{\text{mező}}$$

$$0=\int(\partial_\mu \Theta^{\mu\nu}+f_L^\nu)d^3x=\frac{d}{dt}\left(p^\nu_{mező}+p^\nu_{részecskék}\right)=0$$

A mező és a részecskék teljes 4-es impulzusa megmarad

Imp. momentum

$$M^{\alpha\mu\nu}=x^\mu\Theta^{\alpha\nu}-x^\nu\Theta^{\alpha\mu}$$

$$\partial_\alpha M^{\alpha\mu\nu}=x^\mu\partial_\alpha\Theta^{\alpha\nu}-x^\nu\partial_\alpha\Theta^{\alpha\mu}$$
$$+\cancel{\delta^\mu_\alpha\Theta^{\alpha\nu}}-\cancel{\delta^\nu_\alpha\Theta^{\alpha\mu}}$$
$$=-\left(x^\mu f_L^\nu-x^\nu f_L^\mu\right)$$

$$0=\int\left(\partial_\alpha M^{\alpha\mu\nu}+x^\mu f_L^\nu-x^\nu f_L^\mu\right)d^3x$$
$$=\frac{d}{dt}\left(J^{\mu\nu}_{mező}+J^{\mu\nu}_{részecskék}\right)$$