Baryon number cumulant ratios in strong coupling lattice QCD

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Terukazu Ichihara, Kenji Morita, and Akira Ohnishi
arXiv:1507.04527 ; PTEP accepted
### Related talks in Sign 2015

#### Schedule @ Sign 2015

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<td>Luigi Scorzato: The Lefschetz thimble and the sign problem</td>
<td>Kurt Langfeld: Dense matter QFT with the density-of-states method</td>
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<td>14.30 – 15.20</td>
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<td>Tin Sulejmanpasic: Particle interactions and scattering phase-shifts from finite density and dual variables</td>
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*Social Dinner*
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  - Physical motivation
  - Sign problem
- Formalism & Purpose - Strong coupling lattice QCD
- Results
- Summary & Outlook
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QCD phase diagram

- Critical point (CP) at finite mass
- The location of CP is important information to know the QCD phase diagram

Beam energy scan program (freeze-out line)
**QCD phase diagram**

- **Experiment**
  - **Beam energy scan program**
    Examining finite mu region by heavy ion collision

- **Theory**
  - Theoretical arguments
  - Explicit calculations
    - Model analyses
      - Model dependent
    - **Lattice QCD**
      - **Sign problem** at finite $\mu$
        not easy to study

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Beam energy scan program (freeze-out line)
Heavy ion collision

- **Beam energy scan program**
  see example: the STAR Collaboration, STAR Notes, SN0598

- **Fluctuation has important information**

- **Higher-order cumulants of conserved charges**
  
  - Information on early stages of heavy ion collision
  
  - Divergent correlation length around CP
  
  - Sign change of 3rd cumulant
Non-Gaussian fluctuation

- Higher-order cumulant ratios of net baryon number e.g. $S\sigma$, $\kappa\sigma^2$, ...

\[ S\sigma = \frac{\chi^{(3)}_\mu}{\chi^{(2)}_\mu} \quad \kappa\sigma^2 = \frac{\chi^{(4)}_\mu}{\chi^{(2)}_\mu} \]
\[ \chi^{(n)}_\mu = \frac{1}{VT^3} \left( \frac{\partial^n \log Z}{\partial (Nc\mu/T)^n} \right) \]

1. Experimental data: net proton
   Non-monotonic behavior
   (Skellam distribution:
   Poisson - Poisson)
   may be lots of mechanism

2. Theory
   Useful way to study origins of the non-monotonic behavior

QCD phase diagram

• **Experiment**
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      : Model dependent
    - **Lattice QCD**
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Theoretical arguments

• Origins of non-monotonic $\kappa \sigma^2$
  
  • Z(2) criticality
  
  Around CP, $\kappa \sigma^2$ can be negative

• (Remnant effects of) O(4) criticality
  
  $\kappa \sigma^2$ is affected by the divergent part of the derivatives of free energy density
  
  $(f = f_{\text{regular}} + f_{\text{singular}})$

• Other mechanisms?

• Necessity of explicit calculation
  
  - confirm how mach criticality is realized
    Singular behavior can be smeared by finite volume, finite mass, and regular part of free energy density (or its derivatives)

singular part of $\kappa \sigma^2$ at finite $\mu$
**QCD phase diagram**

- **Experiment**
  - Beam energy scan program
    - Examining finite mu region by heavy ion collision

- **Theory**
  - Theoretical arguments
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    - Model analyses
      - Model dependent
    - Lattice QCD
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        - not easy to study

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Lattice QCD ~ Sign problem

- Partition function
  \[ Z = \int \mathcal{D}U \det D(\mu) \exp(-S_g[U]) \]
  - Zero chemical potential (\(\mu = 0\))
    - underlined part is regarded as the probability in MC simulation
    - configurations are generated by the probability distribution
  - Finite chemical potential (\(\mu \neq 0\))
    - Complex fermion determinant
      - \(\det D(\mu) = [\det D(-\mu^*)]^*\)
        \(\rightarrow\) complex values at finite chemical potential
    - underlined part cannot be regarded as the probability
  - Many methods to avoid sign problem
    - Reweighting, Taylor expansion, Analytic continuation from imaginary \(\mu\), Canonical approach, Density of state, Fugacity expansion, Complex Langevin, Lefschetz Thimble, Dual variables, …
Why strong coupling expansion?

- Lattice QCD action with sufficiently large coupling
  Wilson (’74), Creutz (’80), Munster (’81), Kawamoto, Smit (’81), Ichinose (’84), Karsch, Mutter (’89), Bilic, Karsch, Redlich (’92), Faldt, Petersson (’86), Damggar, Fukushima (’03), Nishida (’03), de Forcrand, Fromm (’10), de Forcrand, Unger (’11), Nakano, Miura, Ohnishi (’10,’11), Tomboulis (’13,’14),…

Approximation in lattice QCD (g ≫ 1)

- Characteristic - Integration procedure
  1. (Spatial) Link variables [Grassmann variables (lattice QCD)]
  2. Grassmann variables [Link variables (lattice QCD)]

- Action is written in terms of color singlet states

- We expect that smaller phases can be realized in the path integral
  - Deviation from energy eigen states could generate complex phase
  - Color singlet states may be close to energy eigen states of QCD

See also Wolfgang Unger’s talk
Strong coupling lattice QCD

• Recent developments

  • **Fluctuations beyond MF**
    Monomer-Dimer-Polymer (MDP) simulation
    Auxiliary field Monte-Carlo (AFMC) method

  • **Sign problem**
    MDP - from baryon loop
    AFMC - from bosonization
    See also Wolfgang Unger’s talk

  • Less severe weight cancellation than lattice QCD in strong coupling limit
    → QCD phase diagram

Strong coupling and chiral limit

Average phase factor $8^4$ lattice: AFMC

Arrows show $T_c$
cf) Strong coupling lattice QCD - NLO

- Recent developments
  - Next-to-leading order (NLO) effects beyond MF
    de Forcrand, Langelage, Philipsen, Unger, PRL113('14)152002
    A. Ohnishi’s talk (Lattice 2015)

MDP

AFMC

de Forcrand, Langelage, Philipsen, Unger, PRL113('14)152002
A. Ohnishi’s talk (Lattice 2015)
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Formalism

- Lattice QCD action in strong coupling limit (SCL): no plaquette term
  - unrooted staggered fermion with \( d(=3)+1 \) dimension for \( N_c=3 \)
  - anisotropic lattice assumption: \( a_\tau = a_s/\gamma^2 \) due to quantum correction. \( T_c(\mu=0) \) does not depend on \( \gamma \) in MF. N. Bilic, F. Karsch, and K. Redlich, Phys. Rev. D 45, 3228 (1992); N. Bilic and J. Cleymans, Phys. Lett. B 355, 266 (1995)

\[
S_{\text{SCL}} = \frac{1}{2} \sum_{x,\nu=0}^{d} [\eta_{\nu,x}^+ \bar{x}_x U_{\nu,x} x_+ - \eta_{\nu,x}^- (H.C.)] + m_0 \sum_x \bar{x}_x x_x \eta_{\nu,x}^\pm = (\gamma e^{\pm \mu a_\tau}, (-1)^{x_1+\cdots+x_\nu})
\]

- Effective action after integrating out spatial link variables and \( 1/d \) expansion
  H. Kluberg-Stern, A. Morel, B. Petersson, (1983), etc.

\[
S_{\text{eff}} = \frac{1}{2} \sum_x [V_x^+ - V_x^-] - \frac{1}{4N_c} \sum_{x,j} M_x M_{x+j} + m_0 \sum_x M_x V_x^\pm = \gamma e^{\pm \mu a_\tau/\gamma^2} \bar{x}_x U_{0,x} x_{x+0}
\]


Bosonization

Auxiliary fields are integrated by Monte-Carlo technique (AFMC method)
Effective action

- **Bosonization** - extended Hubbard-Stratonovich transformation
  

  - bosonization when \((A \neq B)\)
  - an imaginary number → **sign problem**

  \[
e^{\alpha AB} = \int d\varphi d\phi e^{-\alpha \left\{ [\varphi - (A + B)/2]^2 + [\phi - i(A - B)/2]^2 \right\} + \alpha AB}
  \]

- **Auxiliary field Monte-Carlo (AFMC) method**
  
  after Grasmann and \(U_0\) integral

  - integrating over auxiliary fields\((\sigma, \pi)\) by MC
  - **Mesonic fluctuation effects taken into account**

  \[
  m_x = m_0 + \frac{1}{4N_c} \sum_j \left( \sigma + i\varepsilon \pi \right)_{x \pm j}
  \]

  \[\epsilon_x = (-1)^{x_0 + \cdots + x_d} \gamma_5\]

  \[
  S_{\text{eff}}^{AF} = \sum_{k, \tau, f(k) > 0} \frac{L^3 f(k)}{4N_c} \left[ |\sigma_{k, \tau}|^2 + |\pi_{k, \tau}|^2 \right] - \sum_x \log R(x)
  \]

  \[
  R(x) = X_{N\tau}(x)^3 - 2X_{N\tau}(x) + 2 \cosh\left( N_c\mu / T \right)
  \]

  \[X_{N\tau} : \text{known function} \quad \text{G. Faldt and B. Petersson, Nucl. Phys. B 265, 197 (1986)}\]

  \[X_{N\tau} = 2 \cosh\left( N\tau \arcsinh \left( m_x / \gamma \right) \right) : \text{Mean Field case} \]
Sign problem in AFMC in SCL

- **Origin**
  - **Bosonization procedure:** extended Hubbard-Stratonovich transformation
  - **Diagonal parts of fermion determinant**
    \[ m_x = m_0 + \frac{1}{4N_c} \sum_j (\sigma + i\varepsilon \pi)_{x\pm j} \]
    \[ \epsilon_x = (-1)^{x_0 + \cdots + x_d} \gamma_5 \]
  - not chemical potential (different origin from lattice QCD)

- **Severity**
  - Expect that **sign problem is not so severe in the case where we study phase transition phenomena** (in which long wave length phenomena dominate)
    - Phase on one site tends to be canceled by phases on the nearest neighbor site when \( \pi \sim \text{const.} \)
    - Phases are almost canceled out when \( \pi \sim \text{const.} \)
    - High momentum modes contribute sign problem
High momentum modes & QCD phase diagram

- High momentum modes
  - contribution to sign problem
  - Analysis
    - cut off high momentum modes by cut-off parameter $\Lambda$
    - neglected by $\sum_j \sin^2 k_j > \Lambda$
  - Results
    - average phase factor becomes unity when setting $\Lambda = 3$ (spatial dimension)

- contributions of high momentum modes to statistical weight cancellation

- QCD phase diagram
  - although we have the sign problem as shown, we expect that we could investigate phase transition phenomena as long as long wave dynamics dominate

Let’s investigate higher-order cumulant ratios with strong coupling approach!

How about higher order net baryon number cumulants (Normalized Skewness and Kurtosis) at strong coupling ???

• Purpose

• To study behavior of higher order net baryon number cumulants (explicit calculations) in the strong coupling limit
  • including (mesonic) field fluctuation effects
  • starting from lattice QCD action
  • in the chiral limit
    • criticality is smeared only by finite size effects (regular part contribution is implicitly included)
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Results

- Set up
  unrooted staggered fermion: O(2) symmetry
  chiral and strong coupling limit
  Auxiliary field Monte-Carlo method (mesonic field fluctuation effects)
  on $4^4$, $6^3 \times 4$, $6^4$, $8^4$ lattices for various $\mu/T$ lines

- Higher-order cumulant ratios
  - Normalized skewness
    $$S\sigma = \frac{\chi^{(3)}_{\mu}}{\chi^{(2)}_{\mu}}$$
  - Normalized kurtosis
    $$\kappa\sigma^2 = \frac{\chi^{(4)}_{\mu}}{\chi^{(2)}_{\mu}}$$
    $$\chi^{(n)}_{\mu} = \frac{1}{VT^3} \frac{\partial^n \log Z}{\partial (Nc_{\mu}/T)^n}$$
μ, T dependence

- **Small μ/T**
  - Both cumulant ratios
    - have small positive peak around phase boundary.
    - stay positive,

- **Large μ/T**
  - Kurtosis (Skewness)
    - shows oscillatory behavior.
    - has two (one) positive peaks and one negative valley.
    - has large amplitude approaching to TCP.
Small $\mu/T$

Small $\mu/T$

$\mu/T = 0.2$

Large $\mu/T$

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Size dependence at small $\mu/T$

- **Amplitude**
  - becomes slightly larger with increasing lattice size.

- **Positive peak(s) and negative valley**
  - $S\sigma$ stays positive and grows slightly
  - Negative $\kappa\sigma^2$ valley appears on larger lattices (bigger than $6^3 \times 4$)
    - Singular part may overcome regular part (derivatives of the free energy density ($f = f_{\text{regular}} + f_{\text{singular}}$))

\[ T_{c} = T_{c}(\mu=0.6^4 \text{lattice}) \]

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arXiv:1507.04527; PTEP accepted
Large $\mu/T$

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Small $\mu/T$

Large $\mu/T$

$\mu/T=0.8$

TCP
Size dependence at large $\mu/T$

- **Amplitude**
  - divergent behavior
    - becomes larger with increasing lattice size

- **Positive peaks and negative valley of kurtosis**
  - shrink on larger lattices

- **Positive divergent behavior in the chiral limit**
  which is consistent with $O(4)$ scaling function analysis.
  
  - Qualitative behavior is similar in $O(4)$ and $O(2)$


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8^4 lattice results

- Qualitative behavior is the same as results on smaller lattices

\[ S_\sigma, \mu/T=0.2 \]

\[ \text{Tc} = \text{Tc(}\mu=0.8^{4}\text{lattice)} \]
Negative kurtosis region?

- Explicit calculation in the strong coupling limit - how mach criticality is realized

- Origins of non-monotonic $\kappa \sigma^2$ behavior
  - $Z(2)$ criticality

- (Remnant effects of) $O(4)$ criticality

- Other mechanisms?
Negative normalized kurtosis region in the QCD phase diagram

- **Set up**
  - **Chiral and strong coupling limit**
  - Lattice size: $6^4$
  - Subtracting artifact at high $T$
  - No spatial baryonic hopping

- **Phase boundary**
  - Determined by chiral susceptibility peak at $\mu/T < 0.8$
  - (would be 2nd order)

- **Negative kurtosis valley - shaded area**
  - (error bars not taken into account)
  - Black dots: boundary of $\kappa_2$
  - Shaded area: just connecting by black dots
  - **Consistent with phase boundary**
  - **Expected to shrink in the thermodynamic limit**

- **Important next steps**
  - **Finite mass simulation**
  - Negative region may survive in the thermodynamic limit

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Summary

• We investigate normalized kurtosis and skewness in the chiral and strong coupling limit. (~$8^4$ lattice)
  
  • We find
    
    • the oscillatory behavior due to the criticality which may be the origin of non-monotonic behavior of $\kappa\sigma^2$
    
    • oscillatory behavior at high $\mu/T$ and negative kurtosis valley due to the finite size effect (~$8^4$ lattice).
    
    • that peak heights of skewness and kurtosis increase and negative valley of kurtosis shrinks on larger lattices (~$8^4$ lattice).
    
    • consistency with O(4) scaling analysis (~$8^4$ lattice).

• Important steps
  
  • on larger lattices ($10^3 \times 4, \ldots$) and finite size scaling analyses of higher-order cumulants.
  
  • with finite mass.
    → Negative region may be modified in the thermodynamic limit.
  
  • Next-to-leading effects: density fluctuation effects are explicitly included
I have studied one more thing, …

Although we have used complexfied theories in CLM or Lefshetz thimble approaches, we have poor understanding of complexified theories...

Nearest neighbor distribution tells us the nature of complexfied theory?

• Plenty time for discussion
  • I can show some preliminary results of Complex Langevin method if we have some time after discussion for my talk, cumulant ratios in strong coupling lattice QCD
Results

- $\mu^{\text{tilde}}=3$, $m^{\text{tilde}}=2$
- Chiral Random Matrix
  - different distribution
  - close to Ginibre (Wrong)
  - close to Wigner (Correct)

Wrong

Correct
Question
How to interpret?
Please discuss!
Thank you for your attention!
Back up

Reference material for design (approved license)
http://www.slideshare.net/yutamorishige50/how-to-present-better
Zoom in on results

$S\sigma, \mu/T=0.2$

$\kappa\sigma^2, \mu/T=0.2$

$Tc = Tc(\mu=0.8^{4}\text{lattice})$
$\mu$, $T$ dependence on $8^4$ lattice

$S_\sigma, 6^3 \times 6$

$S_\sigma/3, \mu/T = 0.8$

$\mu/T = 0.2$, $0.3$, $0.6$

$0.4 \leq T/T_c \leq 1.1$

$S_\sigma, 8^3 \times 8$

$S_\sigma/2, \mu/T = 0.6$, $\mu/T = 0.8$

$\mu/T = 0.2$, $0.3$

$0.5 \leq T/T_c \leq 1.1$

$\kappa \sigma^2, 6^3 \times 6$

$\kappa \sigma^2/3, \mu/T = 0.6$, $\mu/T = 0.8$

$0.4 \leq T/T_c \leq 1.1$

$\kappa \sigma^2, 8^3 \times 8$

$\kappa \sigma^2/30, \mu/T = 0.6$, $\kappa \sigma^2/300, \mu/T = 0.8$

$\mu/T = 0.0$, $0.2$, $0.4$

$0.5 \leq T/T_c \leq 1.1$

$T_c = T_c(\mu = 0.6^{4\text{ lattice}})$

$T_c = T_c(\mu = 0.8^{4\text{ lattice}})$
Back up-2
• Density fluctuation effect are explicitly included at next-to-leading order (NLO) effects of strong coupling expansion
  • order parameter: linear combination of chiral condensate and baryon number density

• It is not easy to investigate 1st order phase transition region with NLO effects in Auxiliary field Monte Carlo

• Shifted path to imaginary direction: pass through MF value to reduce sign problem
  • It is difficult to simulate when two local minima exist (~ similar problem in Lefschetz thimble)
• I expect that Complex Langevin (CL) may work even at 1st order phase transition with NLO effects

• Recently, complexfied theory has been attracted much attention
  • Need to understand nature of complexfied filed theory
Outlook - Intro

• Chiral Random Matrix model

• Convergence depends on parametrization Mollgaard, Splittorff (2013,2014)

\[ Z = \int d\Phi_1 d\Phi_2 (\det(\mathcal{D} + m))^{N_f} e^{-N \text{ Tr } (\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2)} \]

\[ \mathcal{D} + m = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} + m \]

Wrong result

\[
\Phi_1 = a + ib \quad \Phi_2^\dagger = \alpha + i\beta \\
X_{ij} = i \cosh(\mu) (a_{ij} + ib_{ij}) + \sinh(\mu) (\alpha_{ij} + i\beta_{ij}) \\
Y_{ij} = i \cosh(\mu) (a_{ji} - ib_{ji}) + \sinh(\mu) (\alpha_{ji} - i\beta_{ji})
\]

Correct result

\[
(\Phi_1)_{ij} = r_{1,ij} e^{i\theta_{1,ij}} \quad (\Phi_2)_{ji} = r_{2,ji} e^{i\theta_{2,ji}} \\
X_{ij} = e^{\mu+i\theta_{1,ij}} r_{1,ij} - e^{-\mu-i\theta_{2,ji}} r_{2,ji} \\
Y_{ij} = -e^{-\mu-i\theta_{1,ji}} r_{1,ji} + e^{\mu+i\theta_{2,ij}} r_{2,ij}
\]

or \( re^{i\theta} \)

Linear Transformation
Nearest Neighbor spacing distribution

- Nearest Neighbor spacing distribution at finite $\mu$  
  Markum, Pullirsch, Wettig (1999)
  - Fluctuation effects of Dirac spectrum density
  - Hermitian ($\mu \neq 0$)  : Wigner surmise
  - Non-Hermitian ($\mu = 0$)  : Ginibre ensemble

- Are convergences of CL characterized by symmetry the model has?
Results

- $\mu^{\tilde{\text{d}}} = 3$, $m^{\tilde{\text{d}}} = 30$
- almost same distribution
- close to Ginibre

Correct

**Preliminary**