Universität Regensburg

Numerical study of complex instantons in the Gross-Witten U(N) matrix model

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Sign 2015, Debrecen, Hungary

Gross-Witten matrix model is defined as follows:

$$\mathcal{Z} = \int\limits_{U(N)} dW \exp\left[\frac{N}{\lambda}\mathrm{Tr}(W + W^{-1})\right] \qquad \lambda = g^2 N$$

Model describes one-plaquette world in 2 dimension.

There is a 3rd order phase transition in the limit of infinitely large N:

Phys.Rev.D 21 446

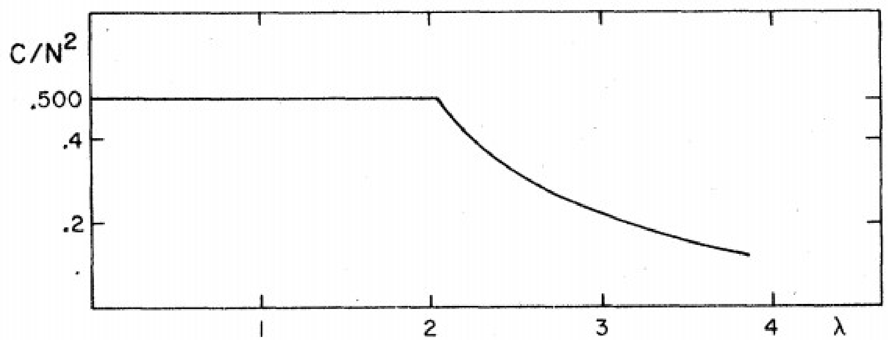


FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

There are weak and strong couping regimes separated by transition point $\lambda = 2$ **.**

The 1/N expansion was extensively studied.

It was argued that phase transition is caused by condesation of instantons.
(Neuberger, Nucl.Phys. B 179 253-282)

Model has rich physical content and it is exactly solvable:

$$\mathcal{Z} = \det(I_{j-i}(2N/\lambda))|_{i,j=1\ldots N}$$

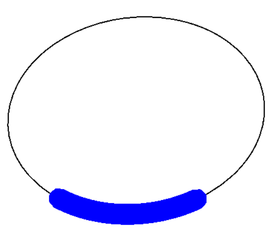
Instantons in the Gross-Witten model

One can reduce partition function of the Gross-Witten model to the integral over phases of matrix eigenvalues ρe^{iz} :

$$\mathcal{Z} = \prod_i \int_{-\pi}^{\pi} dz_i \prod_{i<j} \sin^2\left(\frac{z_i - z_j}{2}\right) \exp\left(\frac{2N}{\lambda}\cos(z_i)\right)$$

$$\mathcal{Z} = \int dz \exp(S(z)) \qquad S(z) = \frac{2N}{\lambda}\sum_i \cos(z_i) + \sum_{i<j} \ln\sin^2\left(\frac{z_i - z_j}{2}\right)$$

In the weak coupling phase there are 2 saddle points: one stable and one unstable. Instanton can be associated with tunneling between them: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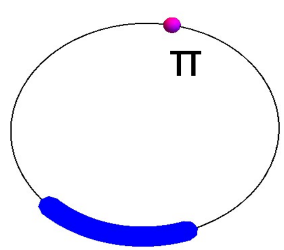
stable



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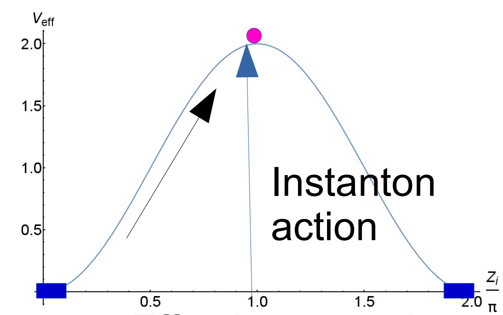
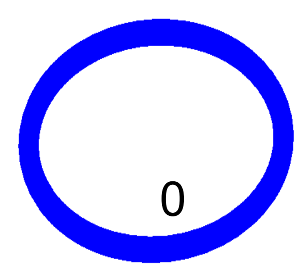
unstable



Effective potential

$\lambda < 2$

In contrast, in the strong coupling regime eigenvalues cover entire circle, so there is only one saddle point and no instantons:



$\lambda > 2$

Gross, Witten, Phys.Rev.D 21 446

Let us consider a double scaling limit in this model which is defined as:

$$g_s \to 0, \lambda \to 2, \kappa = g_s^{-2/3}(2-\lambda) \text{ fixed,}$$

then asymptotic behavior of free energy will be given by a solution of the string equation:

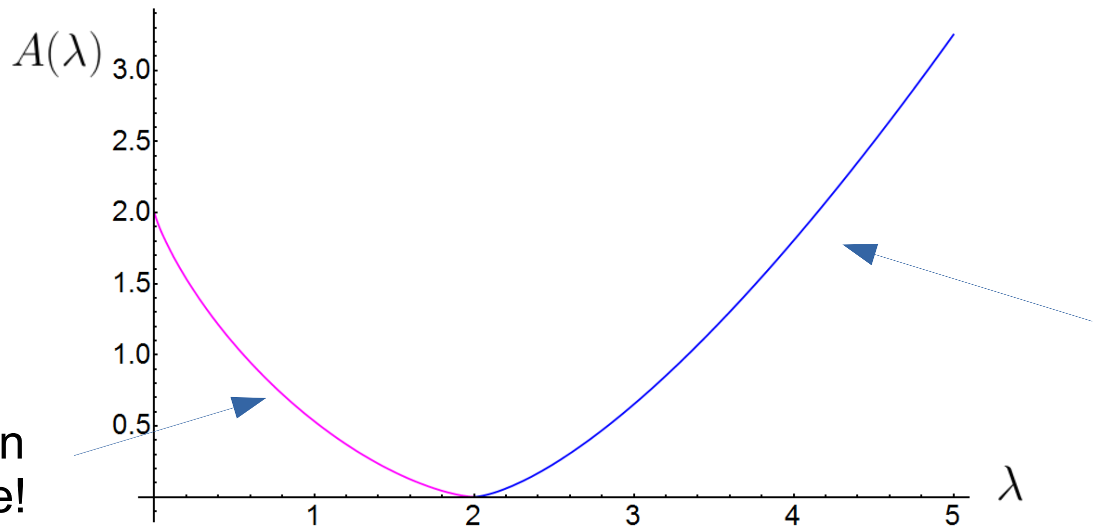
$$F''_{ds}(\kappa) = u^2(\kappa)$$

$$u''(\kappa) - 2u^3(\kappa) + 2\kappa u(\kappa) = 0 \qquad \text{(Painleve II)}$$

Solutions have a formal trans-series form:

$$u(\kappa) = \sqrt{\kappa} \sum_{l=0}^{\infty} C^l \kappa^{-3l/4} e^{-lA\kappa^{3/2}} \epsilon^{(l)}(\kappa)$$

It is possible to fix the shape of the function A(k) from both sides of the phase transition:



Precisely instanton action from prev. slide!

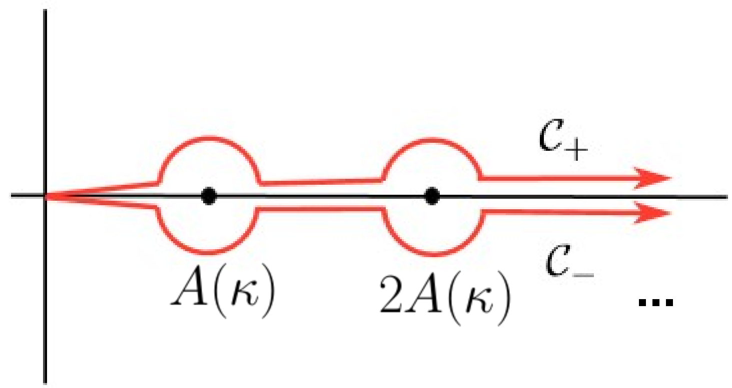
?

Marino, JHEP, 0812:114, 2008

In fact, the 1/N expansion of the Gross-Witten model is factorially divergent:

$$F_g \sim (2g)!$$

and its Borel transformation produces many poles along the real axis:



Poles on the Borel plane

Ambiguities of imaginary part caused by these poles are related to the instanton action.

This phenomenon is known as resurgence phenomenon and it was actively studied previously in many quantum mechanical problems, CP^N model and other models (G. Dunne, M. Unsal, ...)

This relation can be explicitly shown in the weak coupling regime, but in the strong couling regime it remains unclear what instantons are?

We would like to answer this question.

Lefcshetz thimbles

It is natural to study contribution of instantons using Morse theory. By virtue of this theory, partition functions can be expressed as a sum over all saddle points of complexified action S(z):

$$\mathcal{Z} = \sum_\sigma n_\sigma \mathcal{Z}_\sigma \quad \mathcal{Z}_\sigma = \int_{\mathcal{J}_\sigma} dz \exp(NS(z)) \sim \frac{\exp(NS(z_\sigma))}{\sqrt{-\det \partial^2 S(z)/\partial z^2}} \sum_{k=0}^{\infty} c_k N^{-k}$$

where $\mathcal{J}_\sigma$ is the steepest descent contour (Lefcshetz thimble) in the complex plane originating from a given saddle point:

$$\mathcal{J}_\sigma : \frac{dz(t)}{dt} = -\frac{\delta \bar{S}(z)}{\delta \bar{z}}$$

and n_σ is a number of intersections of upward flow $\mathcal{K}_\sigma$ with the real axis:

$$\mathcal{K}_\sigma : \frac{dz(t)}{dt} = +\frac{\delta \bar{S}(z)}{\delta \bar{z}} \qquad n_\sigma =< \mathcal{K}_\sigma, \mathbb{R} > \qquad n_\sigma \in \mathbb{Z}$$

Downward/Upward flow is defined in such a way that real part ReS(z) is decreasing/increasing along the flow and imaginary is constant.

Witten, arXiv:1001.2933

Therefore, our program is:

1. **Find and study all saddle points in the complex plane.**

2. **Inspect eigenvalues of Hessian matrix** $H_{ij} = \partial^2 S(z)/\partial z_i \partial z_j$ **at these points.**

3. **Try to count intersection numbers** n_σ**.**

Critical point equation: $\partial S(z)/\partial z_i = 0$

It might be quiet tricky to solve this equation analytically in the complex domain without some input guess.

Let's solve it numerically!

Simple Newton iterations:

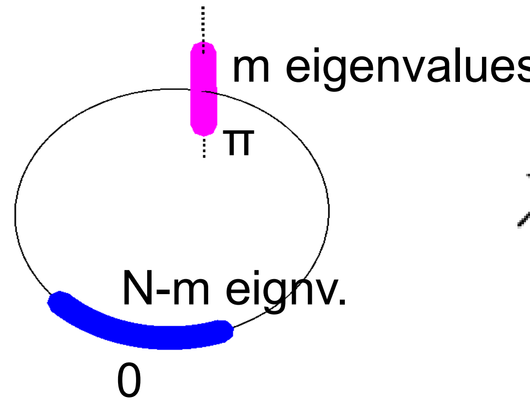
$$z^{n+1} = z^n - (\partial S(z)/\partial z_i \partial z_j)^{-1}(z^n)S(z^n)$$

with random choice of initial vector z0 on the complex plane.

(In order to improve convergence of iterations, we actually use second order Halley iterations, which are an improvement of Newton iterations)

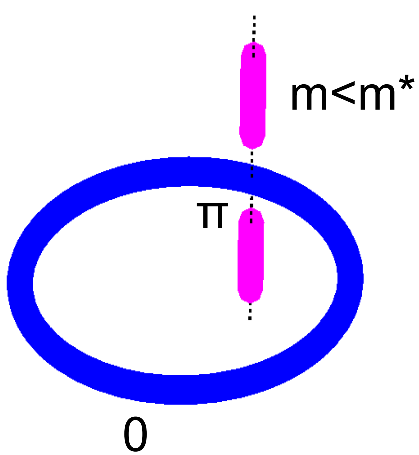
What have we found? **Making degrees of freedom complex, we promote unit circle to a cylinder and allow eigenvalues to move along it. Transversal direction of the cylinder represents imaginary part of eigenvalues.**

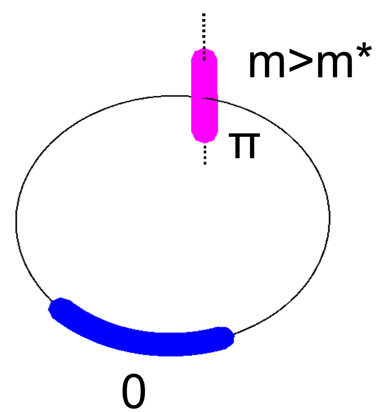
Situation changes dramatically: in both phases there are N saddle points and therefore many instantons.



$\lambda < 2$

In the strong coupling regime we have 2 distinct types of saddle points:





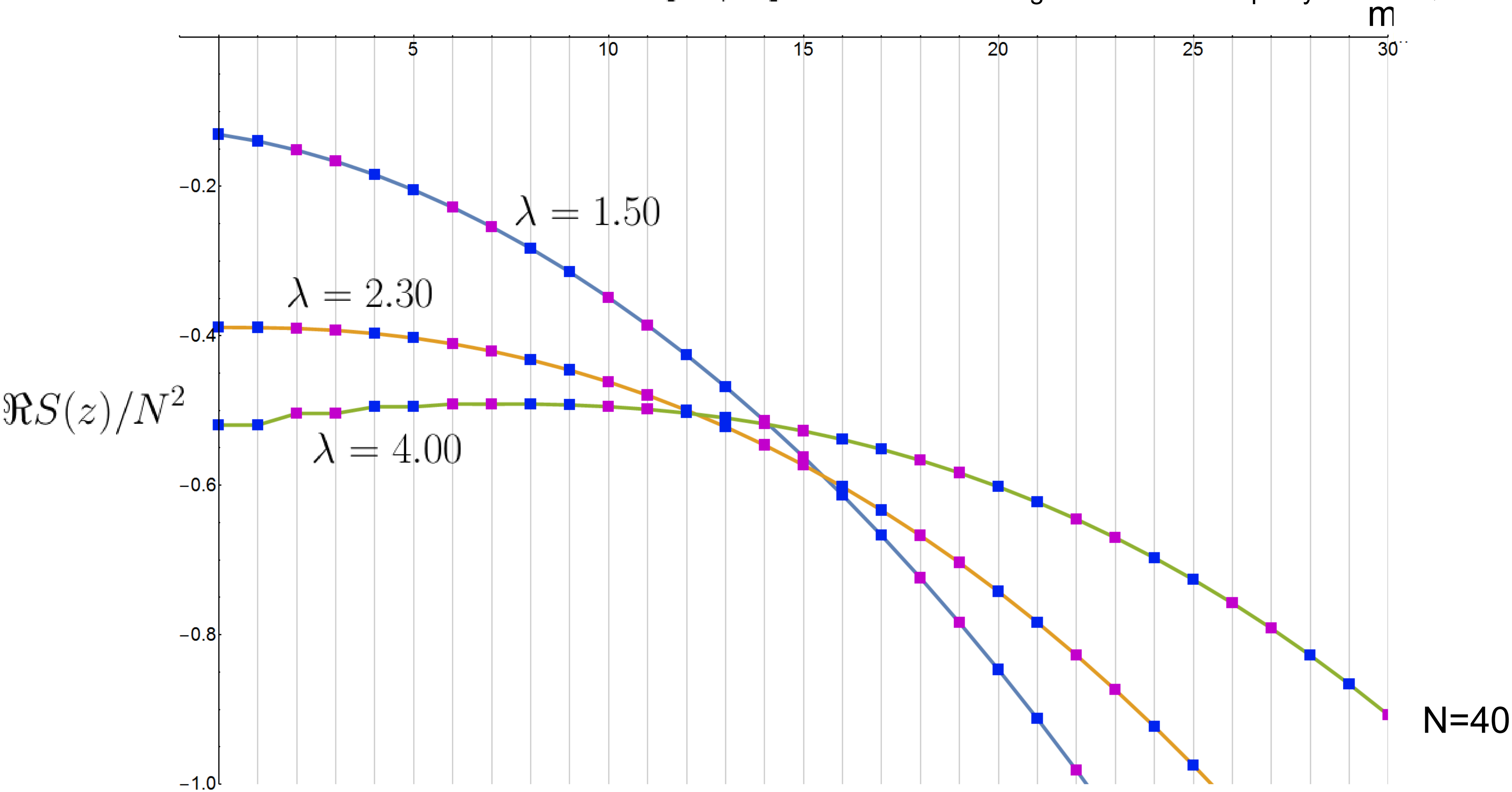
$\lambda > 2$

There is some "critical" number of complex eigenvalues m* which distinguishes between two different types of saddle points.

Weight of saddle points is always real, but the sign varies:

$$\mathrm{Im}S(z) = \pi n, \quad n = [m/2]$$

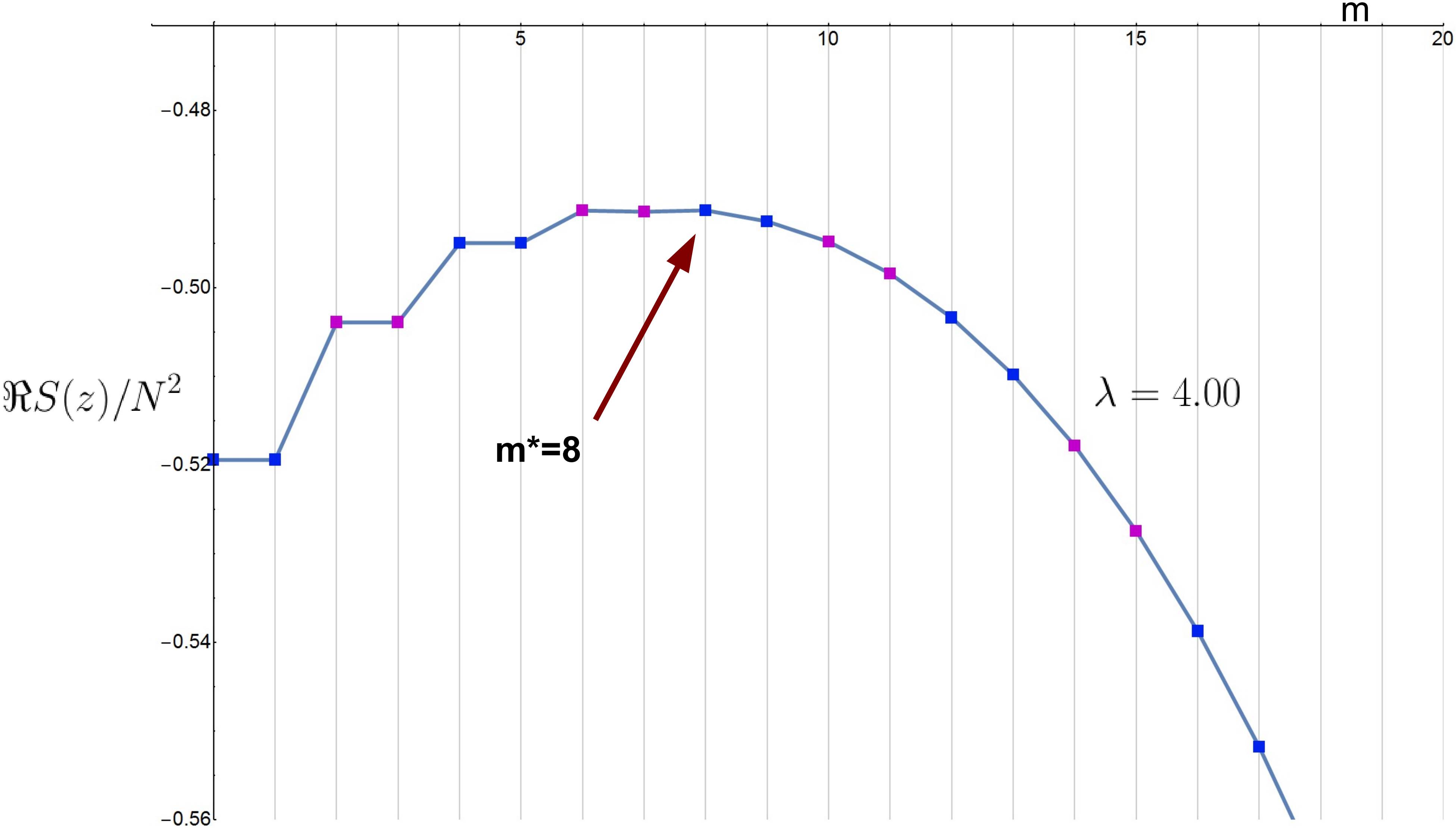
See M. Ünsal, T. Sulejmanpasic, ... arXiv:1507.04063 for something similar in N=2 Supersymmetric QM



Blue dots → weight is +exp(ReS(z)) Red dots → weight is -exp(ReS(z))

Dilute instanton gas in the weak coupling regime. We observe indications for condensation of instantons at the transition point.

In the strong coupling regime configuration with maximal weight has "instanton" number m = m* > 1



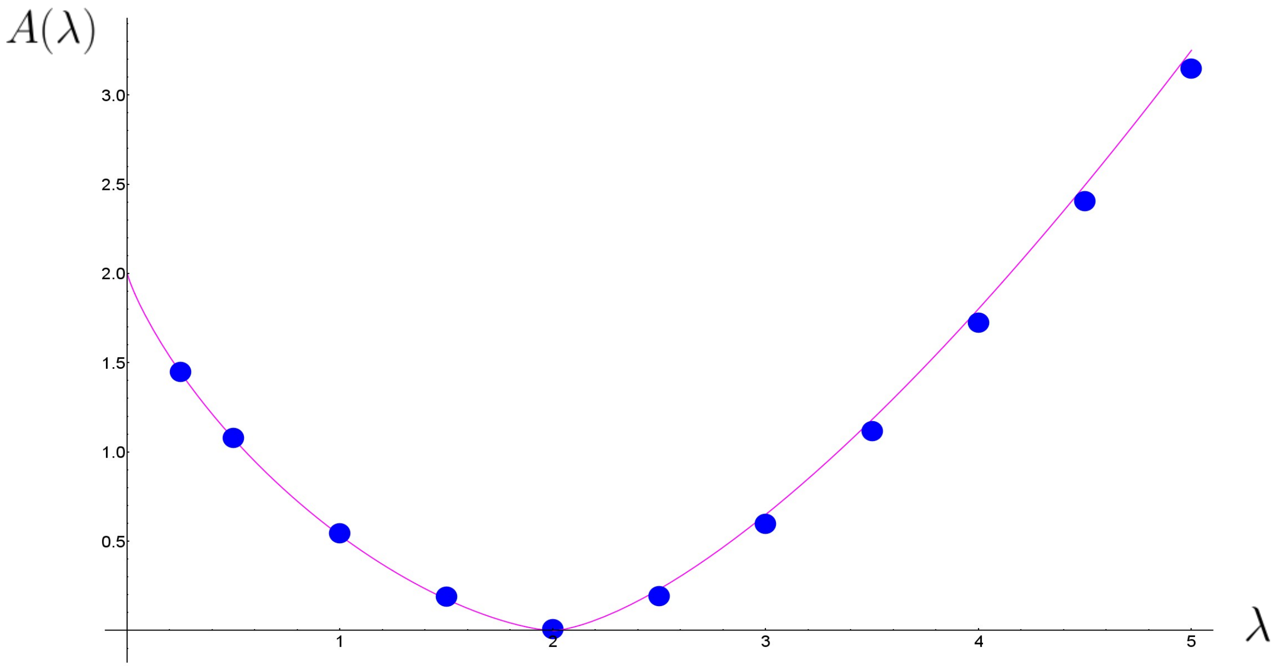
N=40

Next, we would like to address the question of instanton action in the strong coupling.

Assuming that complex saddle points z1 with m = 2 and z2 with m = 0 contribute to the path integral (intersection numbers $n_1 \neq 0, n_2 \neq 0$), we can approximate instanton action as

$$A(\lambda) = \frac{\lambda}{2N}(S(z_1) - S(z_2))$$

Then we compare our one-instanton action to analytical result obtained from purely algebraic consideration without any knowledge about what instantons are in the strong coupling regime:

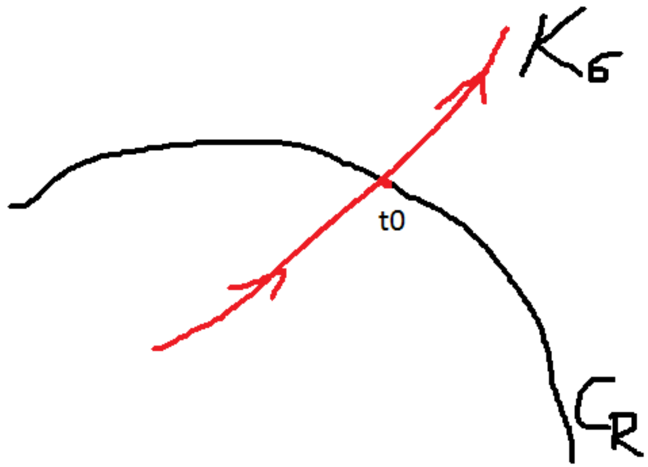


Blue dots – numerical data, Solid line – Marino, JHEP, 0812:114, 2008

Missing ingredient – complex instantons – is found?..

Let us sketch an argument why (probably) $n_\sigma = 0$ for all complex saddle points.

Consider the upward flow and its intersection with the real cycle:



Construct the flow as power series:

$$z_i(t) = z_i(t_0) + \frac{dz_i}{dt}(t_0)(t - t_0) + O((t - t_0)^2)$$

If $z = x + iy$ and $S(z) = u(x,y) + iv(x,y)$ then the flow equations are:

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial u(x,y)}{\partial x_i} \\ \dfrac{dy_i}{dt} = \dfrac{\partial u(x,y)}{\partial y_i} \end{cases} \quad \text{and} \quad y(t_0) = 0$$

In the GW model also $\dfrac{\partial u(x,y)}{\partial y_i}(t_0) = 0$ $\quad \dfrac{\partial^k u(x,y)}{\partial y_{j_1} \partial x_{j_2} \cdots \partial x_{j_k}}(t_0) = 0$

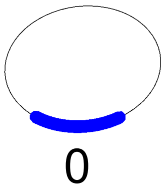
From the above it follows that $y(t) \equiv 0$ => $n_\sigma = 0$

We have studied eigenvalues of Hessian matrix at saddle points:

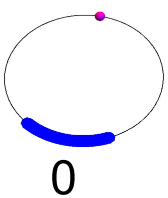
$$H_{ij} = \partial^2 S(z)/\partial z_i \partial z_j$$

In the weak coupling regime $\lambda < 2$ we have found what we expected:

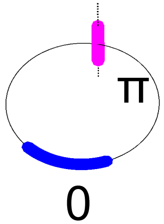
1. Real-valued configuration with m=0 delivers global maximum on the unit circle, no zero modes:



2. Configuration with one tunneled eigenvalue has 1 unstable direction associated with instanton:

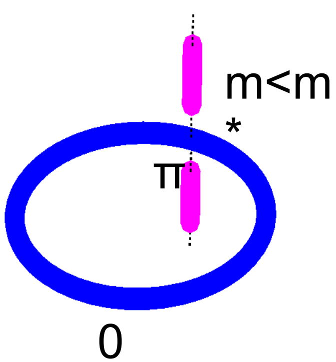


3. All complex saddle points are non-degenerate:



But in the strong coupling regime $\lambda > 2$ we came across to something very interesting:

All saddles of this type (including ground sate)

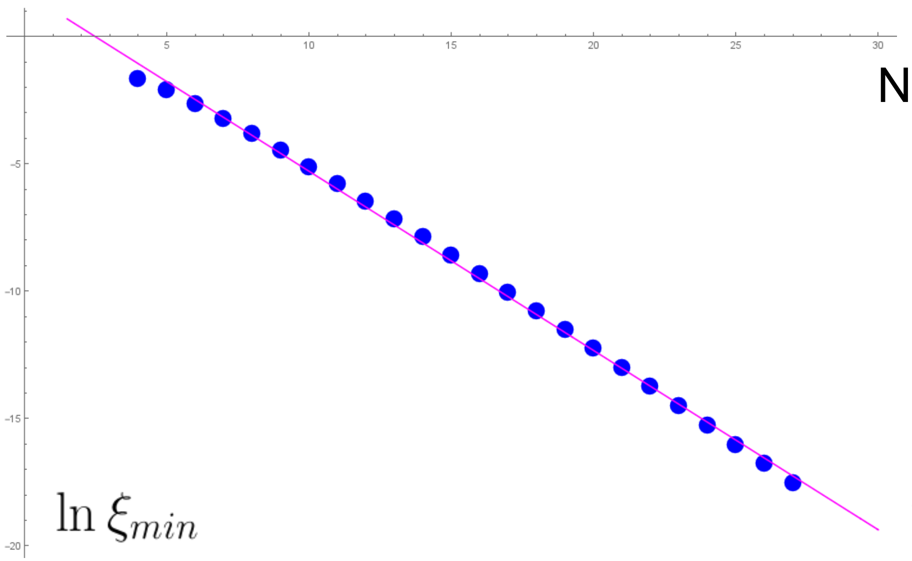


have precisely one zero eigenvalue of Hessian matrix in the limit of large N

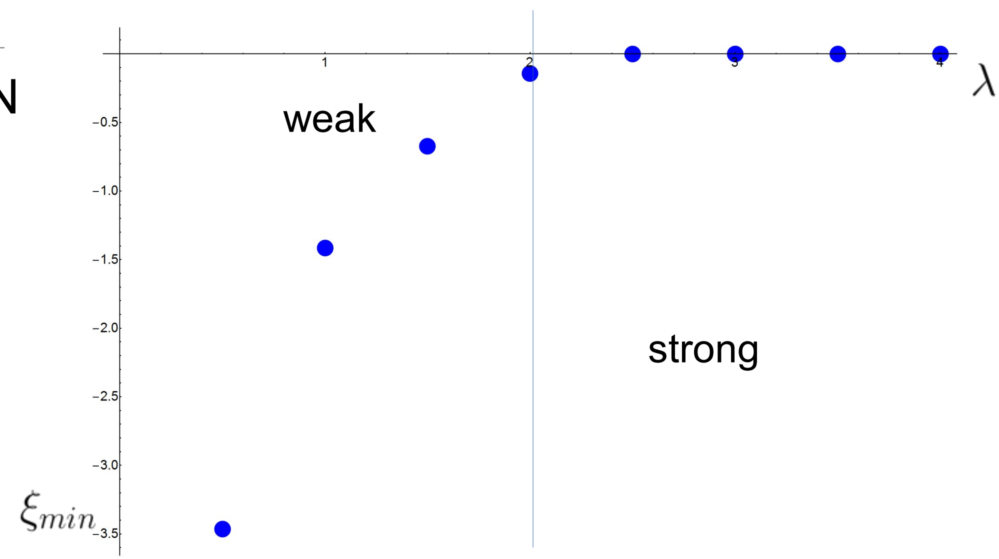
We'd like to address the issue of zero mode in more details.

To do so, we have studied lowest eigenvalue ξ_{min} of Hessian matrix of suspicious configurations and found that it decays exponentially with N in the strongly coupled phase:

$$\xi_{min} \sim e^{-cN}$$



Lowest eigenvalue in a Log-scale versus N at $\lambda = 6$

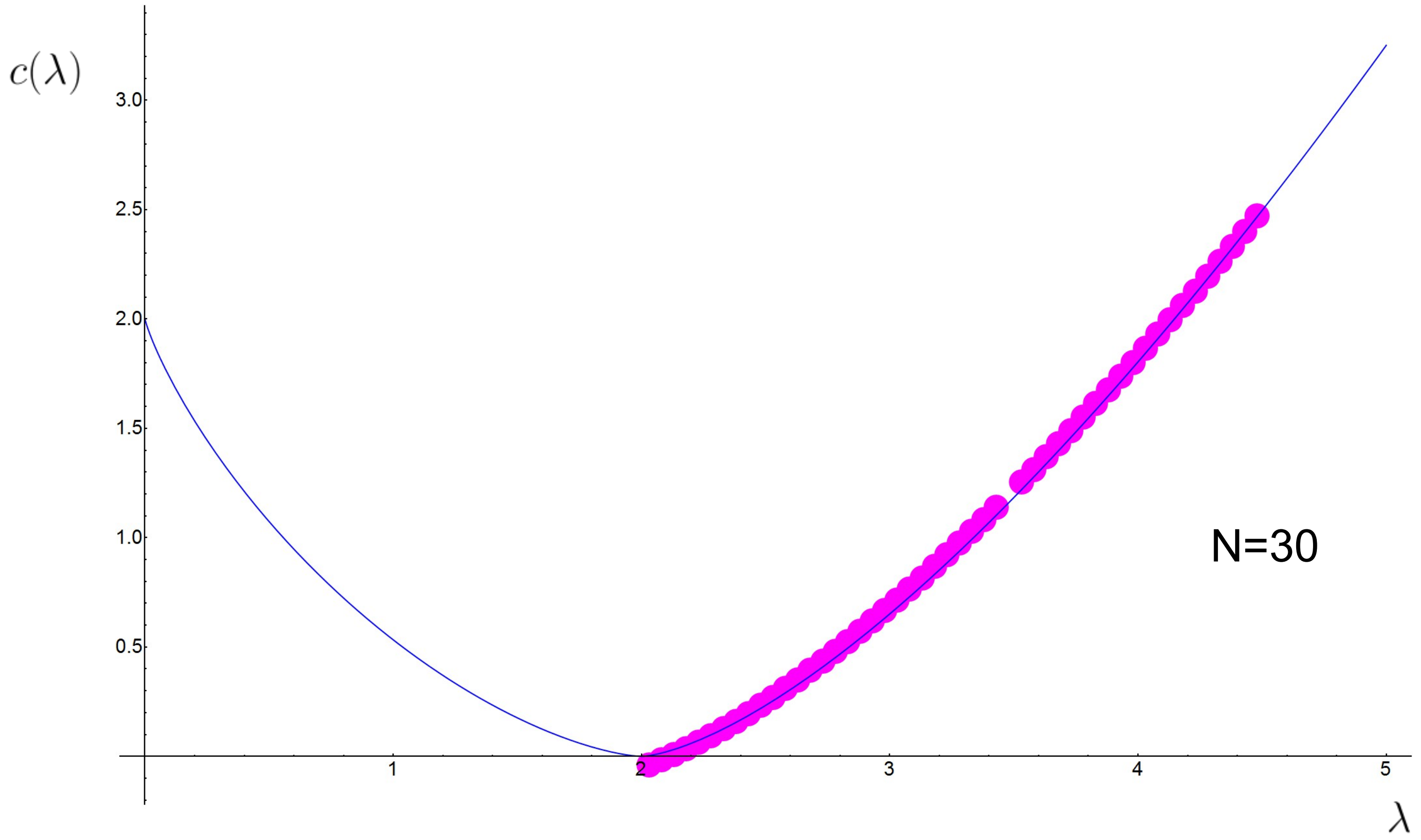


Lowest eigenvalue versus λ at N=400

To deal properly with this eigenvalue in the limit of large N, one can pick it from determinant and include to the action of corresponding saddle:

$$\int_{\mathcal{J}_0} dz \exp(S(z)) \sim \frac{\exp(S(z_0))}{\sqrt{-\det H_{ij}}} = \frac{\exp(S(z_0) + cN/2)}{\sqrt{-\det H_{ij}/\xi_{min}}}$$

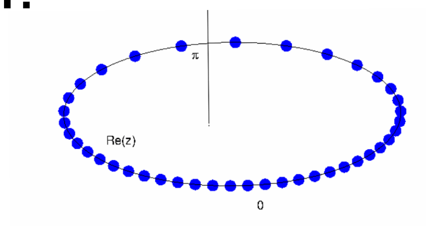
And again, let us compare the contribution of this eigenvalue as function of coupling constant and algebraic expression for the instanton action:

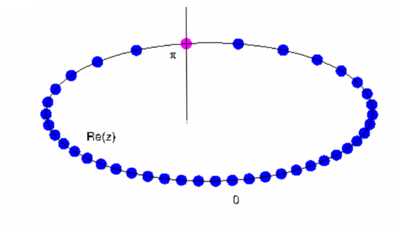


Magenta dots – numerical data, Solid line – Marino, JHEP, 0812:114, 2008

Numerical data is slightly shifted to the left in order to compare with analytic result

The origin of this zero mode is not trivial: at any finite N in the strong coupling phase there are two distinct saddle points on the unit circle: one z1 with m=0 and z2 with m=1:





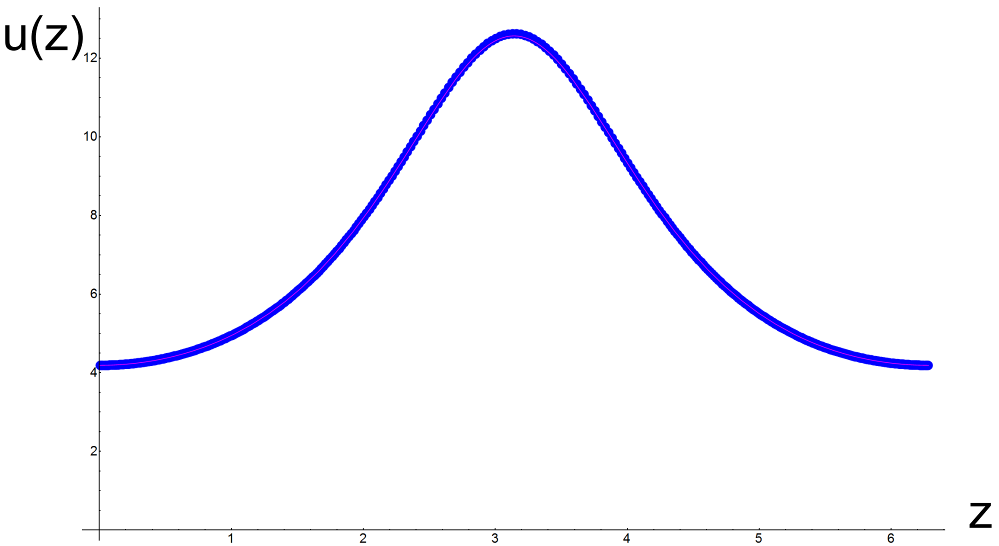
and the difference between their actions vanishes as we increase N.

Since in the large N limit these saddles are indistinguishable, they probably simply merge and as a result the zero mode appears.

It is possible to find the zero mode explicitly in this limit:

$u(z) \sim z_i - z_{i+1} \sim 1/\rho(z)$ **where** $\rho(z)$ **is density function in the large N limit.**

Comparison of the exact expression for the zero mode (magenta) and numerically calculated eigenvector (blue) of the Hessian matrix at N=400:



Conclusions

1) We have found many saddle points in the complex plane and studied their structure.

2) We managed to reproduce known instanton action in both phases. This observation allows us to identify complex saddle points which most likely govern divergences in the 1/N expansion (this was not known).

3) However, there is an argument that complex saddle points do not contribute to the partition function (according to naive Morse theory).

4) On the other hand, in the strong coupling regime we observe emergence of zero mode and its contribution is surprisingly similar to the instanton action.