Parametric integrals for NNLOCAL

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The story so far

First talk by Gábor [https://bodri.elte.hu/seminar/somogyi_20250204.pdf]:

- Standard Model great but not the final answer!
- Precision is key!
- Consider a hadron-hadron collision with the production of a colorless state X and m jets (e.g. Higgs + jet production @ LHC). Because of **QCD factorization**, the cross section of such a process can be written as

$$\hat{\sigma}(p_A, p_B) = \sum_{a,b} \int_0^1 \mathrm{d}x_a \, f_{a/A}(x_a, \mu_F^2) \int_0^1 \mathrm{d}x_b \, f_{b/B}(x_b, \mu_F^2) \, \sigma_{ab}(p_a, p_b; \mu_F^2)$$

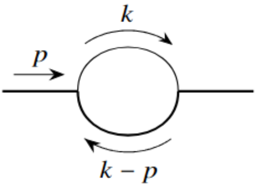
$$\sigma_{ab}(p_a, p_b; \mu_F^2) = \sum_{k=0}^{\infty} \sigma_{ab}^{\mathrm{N}^k\mathrm{LO}}(p_a, p_b; \mu_R^2, \mu_F^2)$$

At N^kLO: k additional partons emitted compared to Born-level

The story so far

Higher-order computations can generate **2** types of singularities with different origins

- UV singularities [hard momenta]

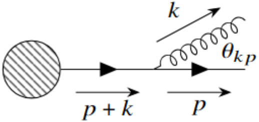


$$\int \frac{\mathrm{d}^4 k}{k^2[(k-p)^2 - m^2]} \xrightarrow{k \to \infty} \int \frac{\mathrm{d}^4 k}{(k^2)^2}$$

$\rightarrow$ Only relevant for **virtual** contributions

The story so far

- IR singularities [soft and/or collinear momenta]



$$\mathcal{M} \sim \frac{1}{(p+k)^2} = \frac{1}{2p \cdot k} = \frac{1}{E_p E_k} \frac{1}{1 - \cos\theta_{kp}}$$

$\rightarrow$ Relevant for both **virtual** and **real** contributions

The story so far

$$\sigma_{ab}^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} + \int_{m+1} \left(\mathrm{d}\sigma_{ab}^{\mathrm{RV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_1}\right) J_{m+1} + \int_{m} \left(\mathrm{d}\sigma_{ab}^{\mathrm{VV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_2}\right) J_{m}$$

Double real (RR)

- Tree-level squared MEs with $(m+2)$-parton kinematics
- No loops, so no explicit poles
- MEs diverge as one or two partons become unresolved
- Phase space integral divergent, poles up to $\mathcal{O}(\varepsilon^{-4})$

Real-virtual (RV)

- One-loop squared MEs with $(m+1)$-parton kinematics
- Explicit poles to $\mathcal{O}(\varepsilon^{-2})$
- MEs diverge as one parton becomes unresolved
- Phase space integral divergent, poles up to $\mathcal{O}(\varepsilon^{-2})$

Double virtual (VV)

- Two-loop squared MEs with m-parton kinematics
- Explicit poles to $\mathcal{O}(\varepsilon^{-4})$
- Divergences from unresolved partons screened by jet function
- Phase space integral is finite

- UV divergences treated by renormalization
- IR divergences treated by subtraction

CoLoRFulNNLO

The story so far

Subtract **approximate cross sections** that match the point-wise singularity structure of the partonic cross sections (based on **IR factorization formulae**)

$$\int_{m+2} \mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} \to \int_{m+2} \Big\{ \mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{ab}^{\mathrm{RR},A_1} J_{m+1} - \mathrm{d}\sigma_{ab}^{\mathrm{RR},A_2} J_m + \mathrm{d}\sigma_{ab}^{\mathrm{RR},A_{12}} J_m \Big\}$$

- $\mathrm{d}\sigma_{ab}^{\mathrm{RR},A_1}$ cancels the singularities coming from a single unresolved emission
- $\mathrm{d}\sigma_{ab}^{\mathrm{RR},A_2}$ cancels the singularities coming from a double unresolved emission
- $\mathrm{d}\sigma_{ab}^{\mathrm{RR},A_{12}}$ needed to avoid double subtraction in regions of phase space where single and double limits overlap

The story so far

Of course, what was subtracted needs to be added back! Furthermore, the subtraction terms need to be **integrated** over the momenta of the unresolved emissions

$$\int_{m+2} \left\{ \mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{ab}^{\mathrm{RR},A_1} J_{m+1} - \mathrm{d}\sigma_{ab}^{\mathrm{RR},A_2} J_m + \mathrm{d}\sigma_{ab}^{\mathrm{RR},A_{12}} J_m \right\}$$
$$+ \int_{m+1} \left(\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR},A_1} \right) J_{m+1} + \int_m \left(\int_2 \mathrm{d}\sigma_{ab}^{\mathrm{RR},A_2} - \int_2 \mathrm{d}\sigma_{ab}^{\mathrm{RR},A_{12}} \right) J_m$$

These integrations are performed **analytically**!

- Verify the validity of the subtraction scheme **explicitly** by checking **analytic pole cancellation** between the partonic cross sections and the approximate ones
- Better control over the final convolution integrals involving the PDFs

The story so far

Second talk by Pooja Mukherjee [https://bodri.elte.hu/seminar/mukherjee_20250211.pdf]:

- Analytic computation of the (42) master integrals for A_2 for color-singlet production ($m = 0$)
- Based on **IBP reductions** and **differential equations**
- A prophecy:

✦ Details on direct integration method : **Sam Van Thurenhout's talk in the future** .

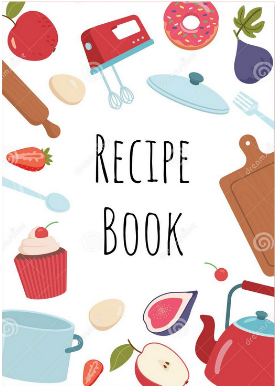
The future is now!

A large subset of the subtraction terms is integrated directly by setting up a **parametric representation**. Here we focus on A_{12} (again for $m = 0$).

Integrating the A_{12} subtraction terms

From the precise definitions of the A_{12} subtraction terms, it turns out that there are 104 basic integrals to compute!

Luckily, it turns out that the computations follow a certain fixed recipe



Plan: Give a generic overview of the necessary steps

A_{12} subtraction terms

Generically in A_{12}:

$$CT = (8\pi\alpha_s\mu^{2\varepsilon})^2\,\mathrm{Sing}_2^{(0)}\,\underbrace{|\mathcal{M}^{(0)}_{\bar{a}\bar{b},X}(\{\bar{p}\}_X;\bar{p}_a,\bar{p}_b)|^2}_{\text{2 partons removed}}$$

To start, we need to define the momentum mapping that characterizes the factorized matrix element

$$(\{p\}_{X+2};p_a,p_b)\xrightarrow{\mathrm{Map}_1}(\{\hat{p}\}_{X+1};\hat{p}_a,\hat{p}_b)\xrightarrow{\mathrm{Map}_2}(\{\bar{p}\}_X;\bar{p}_a,\bar{p}_b)$$

$\mathrm{Map}_{1,2}$: Single-unresolved momentum mappings (soft, initial-final collinear, final-final collinear). Symbolically we write

$$\bar{p}_a = \bar{x}_1\bar{y}_1 p_a \quad \text{and} \quad \bar{p}_b = \bar{x}_2\bar{y}_2 p_b$$

Integrating the subtraction terms

We now need to integrate the subtraction terms over the phase space of the unresolved emissions

$$\int CT = \int_2 \mathrm{d}\phi_{X+2}(\{p\};Q)\,\frac{1}{\omega(a)\omega(b)\Phi(p_a\cdot p_b)}CT$$

- Q: Total incoming partonic momentum $p_a + p_b$
- Φ: Partonic flux factor
- ω: Averaging over initial-state colors and spins

$$\omega(q) = 2N_c\,, \quad \omega(g) = 2(N_c^2-1)(1-\varepsilon)$$

Let us now start setting up our integration recipe.

Step I: Set up a parametric representation of the integral

All momentum mappings employed to define the subtraction terms lead to an **exact factorization** of the real emission phase space

$$\mathrm{d}\phi_{X+2}(\{p\}_{X+2}) = [\mathrm{d}\phi_2] \otimes \mathrm{d}\phi_X(\{\bar{p}\}_X; \bar{Q})$$

$\rightarrow$ Convolution of the reduced phase space of mapped momenta with an integration measure for the unresolved emissions

For example:

$$\mathrm{d}\phi_{X+2}(\{p\}_{X+2}; Q) = \int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2 \int_0^1 \mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}\phi_X(\{\bar{p}\}_X; \bar{Q}) \\ \times [\mathrm{d}\phi_2(p_1, p_2, x_1, x_2, y_1, y_2)]$$

$p_{1,2}$: Unresolved momenta

Step I: Set up a parametric representation of the integral

The unresolved phase space is of the form

$$[\mathrm{d}\phi_2(p_1, p_2, x_1, x_2, y_1, y_2)] = \frac{\mathrm{d}^d p_1}{(2\pi)^{d-1}} \delta_+(p_1^2)\delta(\bar{x}_1 - x_1)\delta(\bar{x}_2 - x_2)$$
$$\times \frac{\mathrm{d}^d p_2}{(2\pi)^{d-1}} \delta_+(p_2^2)\delta(\bar{y}_1 - y_1)\delta(\bar{y}_2 - y_2)$$

A **parametric representation** can be derived by choosing a convenient reference frame, such as the rest frame of the incoming partons.

$\Rightarrow$ Integration over unresolved energies and angles

Often better in practice to rewrite as integration over **parameters of the momentum mapping** $\{\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2\}$

Step I: Set up a parametric representation of the integral

$$[\mathrm{d}\phi_2(p_1, p_2, x_1, x_2, y_1, y_2)] \to \mathrm{d}\bar{x}_1 \mathrm{d}\bar{x}_2 \mathrm{d}\bar{y}_1 \mathrm{d}\bar{y}_2 f(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2; \varepsilon) \\ \times \delta(\bar{x}_1 - x_1)\delta(\bar{x}_2 - x_2)\delta(\bar{y}_1 - y_1)\delta(\bar{y}_2 - y_2)$$

Hence

$$\int CT = \int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2 \int_0^1 \mathrm{d}y_1\,\mathrm{d}y_2 \int_0^1 \mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2 \int_0^1 \mathrm{d}\bar{y}_1\,\mathrm{d}\bar{y}_2\,\mathrm{d}\sigma_{\bar{a}\bar{b}}(\bar{p}_a, \bar{p}_b) \\ \times g(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2; \varepsilon)\delta(\bar{x}_1 - x_1)\delta(\bar{x}_2 - x_2)\delta(\bar{y}_1 - y_1)\delta(\bar{y}_2 - y_2)$$

$$g(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2; \varepsilon) = f(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2; \varepsilon)\,\mathrm{Sing}_2^{(0)}(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$$

$$\mathrm{d}\sigma_{\bar{a}\bar{b}}(\bar{p}_a, \bar{p}_b) = \left[\frac{\alpha_s}{2\pi} S_\varepsilon \left(\frac{\mu^2}{s_{ab}}\right)^\varepsilon\right]^2 \frac{\mathrm{d}\phi_X(\{\bar{p}\}_X; \bar{Q})}{\omega(\bar{a})\omega(\bar{b})\Phi(\bar{p}_a \cdot \bar{p}_b)} |\mathcal{M}^{(0)}_{\bar{a}\bar{b},X}(\{\bar{p}\}_X; \bar{p}_a, \bar{p}_b)|^2$$

Step I: Set up a parametric representation of the integral

Due to the Dirac-delta distributions, the integration over the parameters of the momentum mapping is **trivial**: $\{\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2\} \to \{x_1, x_2, y_1, y_2\}$

$$\Rightarrow \int CT = \int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2 \int_0^1 \mathrm{d}y_1\,\mathrm{d}y_2 \int_0^1 \mathrm{d}\sigma_{\bar{a}\bar{b}}(\bar{p}_a, \bar{p}_b) g(x_1, x_2, y_1, y_2; \varepsilon)$$

Note that now

$$\bar{p}_a = x_1 y_1 p_a \quad \text{and} \quad \bar{p}_b = x_2 y_2 p_b\,.$$

To simplify the nature of the convolution, we set

$$x_1 y_1 = z_1 \quad \text{and} \quad x_2 y_2 = z_2$$

$$\Rightarrow \int CT = \int_0^1 \mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}\sigma_{\bar{a}\bar{b}}(z_1 p_a, z_2 p_b) \int_{z_1}^1 \frac{\mathrm{d}x_1}{x_1} \int_{z_2}^1 \frac{\mathrm{d}x_2}{x_2} \times g(x_1, x_2, z_1/x_1, z_2/x_2; \varepsilon)$$

Step I: Set up a parametric representation of the integral

As the cross section is **independent** of x_1 and x_2, the integration over the latter can be performed **once and for all**

$$[CT(z_1, z_2; \varepsilon)] = \int_{z_1}^{1} \frac{\mathrm{d}x_1}{x_1} \int_{z_2}^{1} \frac{\mathrm{d}x_2}{x_2} g(x_1, x_2, z_1/x_1, z_2/x_2; \varepsilon)$$

Typically, $g(x_1, x_2, z_1/x_1, z_2/x_2; \varepsilon)$ is a complicated rational function. For example, setting $(z_1, z_2) \to (\eta_a, \eta_b)$ and $(x_1, x_2) \to (\xi_a, \xi_b)$, one particular integral to evaluate is the following

$$\begin{aligned}
&\int_0^1 \mathrm{d}\xi_a \int_0^1 \mathrm{d}\xi_b \frac{(\eta_a + \eta_b - \eta_b\xi_a + \eta_a\eta_b\xi_a - \eta_a\xi_b + \eta_a\eta_b\xi_b)^{-1+2\varepsilon}}{(-\xi_a + \eta_a\xi_a - \xi_b + \eta_b\xi_b + \xi_a\xi_b - \eta_a\xi_a\xi_b - \eta_b\xi_a\xi_b + \eta_a\eta_b\xi_a\xi_b)} \\
&\times \frac{(1 + \eta_a - \xi_a + \eta_a\xi_a)^{-\varepsilon}(1 - \xi_b + \eta_b\xi_b)^{2-\varepsilon}(2 - \xi_b + \eta_b\xi_b)^{-1-\varepsilon}(1 + \eta_b - \xi_b + \eta_b\xi_b)^{-1-\varepsilon}(2 - \xi_a + \eta_a\xi_a - \xi_b + \eta_b\xi_b)^{-1+2\varepsilon}}{(\eta_a + \eta_b - \eta_a^2\eta_b - \eta_a\eta_b^2 - 2\eta_b\xi_a + 2\eta_a\eta_b\xi_a + \eta_b\xi_a^2 - 2\eta_a\eta_b\xi_a^2 + \eta_a^2\eta_b\xi_a^2 - 2\eta_a\xi_b + 2\eta_a\eta_b\xi_b + \eta_a\xi_b^2 - 2\eta_a\eta_b\xi_b^2 + \eta_a\eta_b^2\xi_b^2)} \\
&\times \Big\{ [2 - (1 - \eta_b)\xi_b - ((1 - \eta_a)\xi_a(1 - (1 - \eta_b)\xi_b))][(1 - \xi_a)(1 - (1 - \eta_b)\xi_b) + \eta_a(\eta_b + \xi_a - \xi_a\xi_b + \eta_b\xi_a\xi_b)] \\
&\times (-\eta_b^2(-1 + \xi_b)(1 + \xi_b) + (-1 + \xi_b)(\xi_a - \xi_b) + \eta_b\xi_b(-1 - \xi_a + 2\xi_b) + \eta_a(\eta_b + \xi_a - \xi_a\xi_b + \eta_b\xi_a\xi_b)) \Big\} \\
&\times (1 - \eta_a)^{1-2\varepsilon}\eta_a^{-\varepsilon}(1 - \eta_b)^{-1-2\varepsilon}\eta_b^{-\varepsilon}(1 - \xi_a)^{-\varepsilon}\xi_a^{-\varepsilon}(1 - \xi_b)^{-1-\varepsilon}\xi_b^{-1-\varepsilon}(1 - \xi_a + \eta_a\xi_a)^{-\varepsilon}(2 - \xi_a + \eta_a\xi_a)^{-\varepsilon}
\end{aligned}$$

Step II: Treat overlapping singularities

Note that the denominators have non-trivial zeroes. In particular, we need to deal with **overlapping singularities** (e.g. $\xi_a \to 0$ and $\xi_b \to 0$ at the same time)

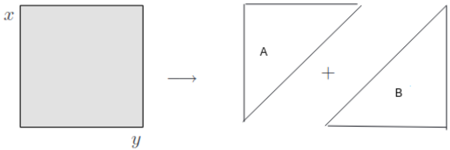
$$\frac{1}{\eta_a\eta_b\xi_a\xi_b - \eta_a\xi_a\xi_b + \eta_a\xi_a - \eta_b\xi_a\xi_b + \eta_b\xi_b + \xi_a\xi_b - \xi_a - \xi_b}$$

$$\xi_a \to 0 : \frac{1}{\xi_b(\eta_b - 1)}\,, \qquad \xi_b \to 0 : \frac{1}{\xi_a(\eta_a - 1)}$$

Disentangle using **sector decomposition** [Heinrich, 2008]

$\rightarrow$ Divide the integration region into **sectors** with the goal of **factorizing the singularities**

Step II: Treat overlapping singularities



$$\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, x^{-1-\varepsilon} y^{-\varepsilon} \frac{1}{x+y} = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, x^{-1-\varepsilon} y^{-\varepsilon} \frac{1}{x+y} \Big[\underbrace{\theta(x-y)}_{y\to tx} + \underbrace{\theta(y-x)}_{x\to ty}\Big]$$

$$= \int_0^1 \mathrm{d}x\, x^{-1-2\varepsilon} \int_0^1 \mathrm{d}t\, \frac{t^{-\varepsilon}}{1+t} + \int_0^1 \mathrm{d}y\, y^{-1-2\varepsilon} \int_0^1 \mathrm{d}t\, \frac{t^{-1-\varepsilon}}{1+t}$$

$\Rightarrow$ Representation in terms of **factorized singularities**

Typically a single step of SD does not suffice

$\Rightarrow$ Need to **iterate**!

Step II: Treat overlapping singularities

At the end of the iteration: All singularities factorized

→ Need to be regularized: Set up **subtractions**!

$$\int_0^1 \mathrm{d}x\, x^{-1-\varepsilon} f(x) \to \underbrace{\int_0^1 \mathrm{d}x\, x^{-1-\varepsilon}[f(x) - f(0)]}_{\text{regular } \textbf{per construction}} + \underbrace{\int_0^1 \mathrm{d}x\, x^{-1-\varepsilon} f(0)}_{\frac{f(0)}{\varepsilon}}.$$

After this: Integral can be evaluated **numerically** for checks

→ Monte Carlo, e.g. using `Vegas` [Lepage, 1978, Lepage, 1980]

$$\mathcal{I}(1/10, 2/10; \varepsilon) = \frac{0.361395}{\varepsilon^2} + \frac{8.11945}{\varepsilon} + 45.1866\,.$$

To continue with the analytic integration, it would be nice to have an idea of the **function space** we expect.

Intermezzo: What is the expected function space?

Generically, the integrals we encounter are **multidimensional integrations of multivariate rational functions**

$$I(x) = \int_0^1 \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\ldots\,\frac{N(x,t_1,t_2,t_3,\ldots)}{D(x,t_1,t_2,t_3,\ldots)}$$

Let's start with the t_1 integration. After **partial fractioning**, we need to compute

$$\int_0^1 \frac{\mathrm{d}t_1}{t_1^n}\,, \qquad \int_0^1 \frac{\mathrm{d}t_1}{(t_1 - f(x,t_2,\ldots))^n}$$

- Straightforward for $n > 1$: Rational function
- Non-trivial for $n = 1$: Need to introduce a new function, the **logarithm**

$$\int \frac{\mathrm{d}t_1}{t_1} = \log(t_1) + C\,, \int \frac{\mathrm{d}t_1}{t_1 - f(x,t_2,\ldots)} = \log\left(t_1 - f(x,t_2,\ldots)\right) + C$$

Intermezzo: What is the expected function space?

Next we consider the t_2 integration. After **partial fractioning** we need to compute integrals of the form

$$\int \mathrm{d}t_2 \, \frac{\log(t_2)}{(t_2 - f(x, t_3, \ldots))^n}$$

- For $n > 1$: Rational functions + logarithms, e.g.

$$\int \mathrm{d}t_2 \, \frac{\log(t_2)}{(t_2 - a)^3} = -\frac{1}{2a}\left(\frac{a\log(t_2)}{(a - t_2)^2} - \frac{1}{a - t_2} + \frac{\log(a - t_2) - \log(t_2)}{a}\right) + C$$

- Non-trivial for $n = 1$: Need to introduce a new function, the **dilogarithm**

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\mathrm{d}t}{t} \log(1 - t)$$

Intermezzo: What is the expected function space?

This reasoning continues for the subsequent integrations as well. Hence we introduce the **classical polylogarithms**

$$\mathrm{Li}_n(x) = \int_0^x \frac{\mathrm{d}t}{t}\,\mathrm{Li}_{n-1}(t)\,, \qquad \mathrm{Li}_1(x) \equiv -\log(1-x)$$

$\rightarrow$ Obey some nice functional relations

- $\mathrm{Li}_n(1/z)$ is **always** related to $\mathrm{Li}_n(z)$, e.g.

$$\mathrm{Li}_2(1/z) = -\mathrm{Li}_2(z) - \frac{1}{2}\log^2(-z) - \zeta_2\,, \qquad \zeta_n = \sum_{i=1}^{\infty}\frac{1}{i^n}$$

- The classical polylogarithm of a **square** is related to the sum of polylogarithms of the **roots**

$$\mathrm{Li}_n(z^2) = 2^{n-1}\left(\mathrm{Li}_n(z) + \mathrm{Li}_n(-z)\right)$$

Intermezzo: What is the expected function space?

The logarithms and classical polylogarithms discussed above are special cases of **generalized polylogarithms** (GPLs) [Goncharov, 1998]

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t) \,, \quad G(z) \equiv G(;z) = 1$$

$$G(0; z) = \log z \,, \qquad G(a; z) = \log\left(1 - \frac{z}{a}\right)$$

$$\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{d}t}{t} \mathrm{Li}_{n-1}(t) = -G(\underbrace{0, \ldots, 0}_{n-1}, 1; z)$$

$\vec{a}_n = (a_1, \ldots, a_n)$: Weight vector (each separate $a_i =$ **letter**)

Weight: Nr. of integrations

$\rightarrow$ Obey some nice functional relations

Intermezzo: What is the expected function space?

- Shuffle relation

$$G(\vec{a}_m; z)G(\vec{b}_n; z) = \sum_{\vec{c}_{m+n}=\vec{a}_m \sqcup\!\sqcup \vec{b}_n} G(\vec{c}_{m+n}; z)$$

  e.g.

$$G(a; z)G(b, c; z) = G(a, b, c; z) + G(b, a, c; z) + G(b, c, a; z)$$

- Fibration basis w.r.t. $(x_1, \ldots, x_n)$

$$\sum_{I=\{i_1,\ldots,i_n\}} c_I \, G(\vec{a}_{1,i_1}; x_1) \ldots G(\vec{a}_{n,i_n}; x_n)$$

  such that each weight-vector $\vec{a}_{k,i_k}$ is **independent** of x_k. E.g.

$$G(1+x; 1-y) \xrightarrow{(x,y)} -G(-1; x) + G(0; y) + G(-y; x)$$

$$G(1+x; 1-y) \xrightarrow{(y,x)} -G(-1; x) + G(0; x) + G(-x; y)$$

All this and much more implemented in the `PolyLogTools` package

[Duhr and Dulat, 2019]

Step III: Preparation of the integrand

As the integration kernels for GPLs are linear, we need to **factorize** higher-order polynomials in the denominators & arguments of GPLs

$\rightarrow$ Typically quadratic or quartic
$\Rightarrow$ Expressions can involve non-integer powers of the integration variables

Rationalize the roots!
Construct suitable transformation of the integration variable to get a **rational** expression, automated in the `RationalizeRoots` package

[Besier et al., 2020]

$$\sqrt{\eta_a\eta_b - (1-\eta_b)\xi_b(\xi_b - \eta_b(\xi_b - 2))} : \xi_b \rightarrow \frac{2\eta_b(\eta_a t + \eta_b - 1)}{\eta_b\left(\eta_a t^2 - 2\right) + \eta_b^2 + 1}$$

$$\Rightarrow \sqrt{\eta_a\eta_b}\,\frac{\eta_b(t(2-\eta_a t) - 2\eta_b t + \eta_b - 2) + 1}{\eta_a\eta_b t^2 + (\eta_b - 1)^2}$$

Step III: Preparation of the integrand

Next, we perform a **partial fraction decomposition** with respect to the integration variable. Surprisingly, this can be a bottleneck in our computations!

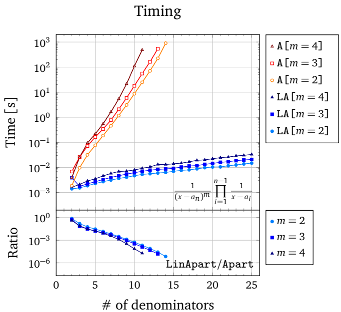
$$
\begin{aligned}
f(x_a, x_b; y) = \Big[& (-4+y)(1-y+x_b y)(2-y+x_b y)(4-y+x_b y)(1-x_a-y+x_b y)^3 \\
& \times (-1+x_a-y+x_b y)(-4-4x_b-y+x_b y)(-4x_b-y+x_b y) \\
& \times (-4x_a-4x_b-y+x_b y)(4x_a-4x_b-y+x_b y)(2+2x_b-y+x_b y)^3 \\
& \times (6+2x_b-y+x_b y)(2-4x_a+2x_b-y+x_b y)(2+4x_a+2x_b-y+x_b y) \\
& \times (-1+x_a-x_a y+x_a x_b y)(1+x_a-x_a y+x_a x_b y)(-2+2x_a-x_a y+x_a x_b y) \\
& \times (2+2x_a-x_a y+x_a x_b y)(-x_b+x_a x_b-x_a y+x_a x_b y)^3 \\
& \times (-4+2x_a+2x_a x_b-x_a y+x_a x_b y)(4+2x_a+2x_a x_b-x_a y+x_a x_b y) \\
& \times (1-2x_a+x_a^2-y-x_a y+x_b y+x_a x_b y) \\
& \times \left(2x_b-2x_a x_b+x_a y-x_b y-x_a x_b y+x_b^2 y\right)^3 \Big]^{-1}
\end{aligned}
$$

- `Apart`: $t > 10^9\,\mathrm{s}$

Intermezzo: LinApart

- Lead to the development of LinApart [Chargeishvili et al., 2025] (see also Levente's talk from last week)
- Based on a **closed-form expression** for the decomposition following from the residue theorem
- Significant speed-ups w.r.t. available tools such as Apart
  → E.g. for $f(x_a, x_b; y)$ above: $t \sim 10^{-2}$ s
- Codes (Mathematica + C) publicly available!

[https://github.com/fekeshazy/LinApart]



Step IV: Perform the integration

Use the `GIntegrate` command of `PolyLogTools`, which computes the **primitive function** of the integrand

$\rightarrow$ **Algorithmically**, based on IBP identities

$\rightarrow$ **Only** if the weight-vector is independent of the integration variable!

$\rightarrow$ Hence need to go to a **fibration basis**!

Step V: Repeat

We typically have 1 or 2 integration variables

$$[CT(z_1, z_2; \varepsilon)] = \int_{z_1}^{1} \frac{\mathrm{d}x_1}{x_1} \int_{z_2}^{1} \frac{\mathrm{d}x_2}{x_2} g(x_1, x_2, z_1/x_1, z_2/x_2; \varepsilon)$$

The result of the integration has poles to $\mathcal{O}(\varepsilon^{-2})$ and needs to be computed to $\mathcal{O}(\varepsilon^0)$. The general structure is as follows

- The $\mathcal{O}(\varepsilon^{-2})$ part is just **rational**
- The $\mathcal{O}(\varepsilon^{-1})$ part contains weight-1 GPLs (i.e. **logarithms**) involving (roots of) z_1 and z_2.
- Finally, the $\mathcal{O}(\varepsilon^0)$ part is the most complicated, with GPLs up to weight 2.
- The analytic result was tested **numerically** for different values of (z_1, z_2)

Step VI: Regulate endpoint singularities

Note that we are not quite done with the integration!

$$\int CT = \int_0^1 \mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}\sigma_{\bar{a}\bar{b}}(z_1 p_a, z_2 p_b)\,[CT(z_1, z_2; \varepsilon)]$$

Care needs to be taken with the interpretation of this object, as the integrand generically suffers from endpoint singularities!

→ Regularized by subtraction: Subtract all offending limits and add back integrated expressions

Step VI: Regulate endpoint singularities

- Single limits: $z_1 \to 1$ or $z_2 \to 1$

$$\boldsymbol{L}_2[\mathrm{CT}(z_1, z_2; \varepsilon)] = \sum_i (1 - z_2)^{c_i + d_i \varepsilon} g_i(z_1; \varepsilon)$$

$$[\boldsymbol{L}_2][\mathrm{CT}(z_1, z_2; \varepsilon)] = \sum_i \frac{g_i(z_1; \varepsilon)}{c_i + d_i \varepsilon}$$

- Double limit: $z_1 \to 1$ *and* $z_2 \to 1$

$$\boldsymbol{L}_{12}[\mathrm{CT}(z_1, z_2; \varepsilon)] = \sum_i (1 - z_1)^{a_i + b_i \varepsilon} (1 - z_2)^{c_i + d_i \varepsilon} g_i(z_1/z_2; \varepsilon)$$

$$[\boldsymbol{L}_{12}][\mathrm{CT}(z_1, z_2; \varepsilon)] = \ldots$$

- Overlaps: $z_1 \to 1$ or $z_2 \to 1$ of $\boldsymbol{L}_{12}$

$$\boldsymbol{L}_2 \boldsymbol{L}_{12}[\mathrm{CT}(z_1, z_2; \varepsilon)] = \sum_i (1 - z_1)^{a_i + b_i \varepsilon} (1 - z_2)^{c_i + d_i \varepsilon} g_i(\varepsilon)$$

$$[\boldsymbol{L}_2 \boldsymbol{L}_{12}][\mathrm{CT}(z_1, z_2; \varepsilon)] = \sum_i \frac{g_i(\varepsilon)}{c_i + d_i \varepsilon} (1 - z_1)^{a_i + b_i \varepsilon}$$

Step VI: Regulate endpoint singularities

Computation of

$$[\boldsymbol{L}_{12}][\mathrm{CT}(z_1, z_2; \varepsilon)] = \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \sum_i (1 - z_1)^{a_i + b_i\varepsilon} (1 - z_2)^{c_i + d_i\varepsilon} \times g_i(z_1/z_2; \varepsilon)$$

non-trivial!

- Overlapping singularities $\Rightarrow$ SD
- Partial fraction
- Fibration basis
- Integrate
- GPL relations for simplifications

Step VI: Regulate endpoint singularities

The result of $[\boldsymbol{L}_{12}][\mathrm{CT}(z_1, z_2; \varepsilon)]$ is just a *number*. For some specific integrated subtraction term we have

$$[\boldsymbol{L}_{12}][\mathrm{CT}(z_1, z_2; \varepsilon)] = \frac{1}{16\varepsilon^4} - \frac{\zeta_2}{4\varepsilon^2} + \frac{\mathcal{E}}{\varepsilon} + \mathcal{O}(\varepsilon^0)$$

with

$$\begin{aligned}
\mathcal{E} = &- \frac{1}{16}\log^3(2) + \frac{3}{16}G(0;2)\log^2(2) - \frac{1}{16}G(2;1)\log^2(2) - \frac{7}{24}G(0,0;2)\log(2) \\
&+ \frac{1}{2}\zeta_2\log(2) - \frac{1}{6}G\left(-\frac{1}{2};1\right)G(0;2)G(2;1) + \frac{1}{8}G(-2;1)G(0;2)G\left(-\frac{1}{2};1\right) \\
&- \frac{1}{8}G(0;2)G(-2,-2;1) - \frac{1}{48}G(0;2)G(-2,0;1) + \frac{1}{6}G(0;2)G(-2,2;1) \\
&- \frac{1}{8}G\left(-\frac{1}{2};1\right)G(-2,2;1) - \frac{1}{24}G(0;2)G(0,-2;1) + \frac{1}{48}G(-2;1)G(0,0;2) \\
&+ \frac{1}{24}G(2;1)G(0,0;2) - \frac{1}{48}G\left(-\frac{1}{2};1\right)G(0,0;2) - \frac{1}{24}G(0;2)G(0,2;1) \\
&+ \frac{1}{12}G\left(-\frac{1}{2};1\right)G(0,2;1) + \frac{1}{8}G(2;1)G\left(0,-\frac{1}{2};1\right) + \frac{1}{6}G(0;2)G(2,-2;1) \\
&- \frac{1}{24}G(0;2)G(2,0;1) + \frac{1}{12}G\left(-\frac{1}{2};1\right)G(2,0;1) - \frac{1}{8}G(0;2)G(2,2;1) \\
&+ \frac{1}{24}G\left(-\frac{1}{2};1\right)G(2,2;1) + \frac{1}{48}G(0;2)G\left(-\frac{1}{2},0;1\right) + \frac{1}{24}G(2;1)G\left(-\frac{1}{2},0;1\right) \\
&+ \frac{1}{8}G(-2,-2,2;1) + \frac{1}{48}G(-2,0,0;1) - \frac{1}{24}G(-2,0,2;1) - \frac{1}{24}G(-2,2,0;1) \\
&- \frac{1}{24}G(-2,2,2;1) + \frac{1}{24}G(0,-2,0;1) - \frac{1}{12}G(0,-2,2;1) + \frac{1}{12}G(0,0,-2;1) \\
&+ \frac{3}{16}G(0,0,0;2) - \frac{1}{12}G(0,0,2;1) - \frac{1}{12}G(0,2,-2;1) - \frac{1}{8}G(0,2,-1;1) \\
&+ \frac{1}{24}G(0,2,0;1) + \frac{1}{8}G(0,2,2;1) - \frac{1}{24}G(2,-2,0;1) - \frac{1}{24}G(2,-2,2;1) \\
&- \frac{1}{12}G(2,0,-2;1) - \frac{1}{8}G(2,0,-1;1) + \frac{1}{24}G(2,0,0;1) + \frac{1}{8}G(2,0,2;1) \\
&+ \frac{1}{12}G(2,2,-2;1) + \frac{1}{8}G(2,2,-1;1) - \frac{1}{12}G(2,2,2;1) - \frac{1}{48}G\left(-\frac{1}{2},0,0;1\right) \\
&+ \frac{1}{8}G(0,2;1)\log(3) + \frac{1}{8}G(2,-1;1)\log(3) - \frac{1}{8}G(2,2;1)\log(3) - \frac{1}{8}G(2;1)\,\mathrm{Li}_2\left(-\frac{1}{2}\right) \\
&- \frac{35}{48}G(0;2)\zeta_2 - \frac{17}{48}G(2;1)\zeta_2 - \frac{1}{16}\log(3)\zeta_2 - \frac{21\zeta_3}{16}\,.
\end{aligned}$$

Step VI: Regulate endpoint singularities

Find more compact form of $\mathcal{E}$ by applying **PSLQ** [Ferguson and Bailey, 1992]

- Evaluate $\mathcal{E}$ to high precision, say 100 digits
- Apply PSLQ using

$$\Big\{\zeta_3, \mathrm{Li}_3\left(\frac{1}{2}\right), \log^3(2), \log^2(2)\log(3), \log(2)\log^2(3),$$

$$\log^3(3), \zeta_2\log(2), \zeta_2\log(3)\Big\}$$

as a basis. This gives

$$\mathcal{E} = -\frac{21\zeta_3}{16}$$

which agrees up to (at least) 200 digits.

$$[\boldsymbol{L}_{12}][\mathrm{CT}(z_1, z_2; \varepsilon)] = \frac{1}{16\varepsilon^4} - \frac{\zeta_2}{4\varepsilon^2} - \frac{21\zeta_3}{16\varepsilon} - 5\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{19\zeta_4}{32} + \frac{5}{4}\zeta_2\log^2(2) - \frac{35}{8}\zeta_3\log(2) - \frac{5}{24}\log^4(2)$$

Step VI: Regulate endpoint singularities

The limits can be computed using **expansion by regions** [Beneke and Smirnov, 1998]

1. Determine regions with non-trivial behaviour when some asymptotic limit is approached
2. Taylor expand in each region in the appropriate variable
3. Integrate the expanded integrands **over the full integration range**

The first step is the most difficult one. We use the `Mathematica` package `asy2.m` [Pak and Smirnov, 2011, Jantzen et al., 2012] for the automatic determination of the regions.

Intermezzo: Determination of the regions

In the context of **loop integrals**, [Pak and Smirnov, 2011] starts from the α-representation of the integral, which symbolically corresponds to

$$\mathcal{I} \sim \int \left(\prod_{j=1}^{n} \mathrm{d}x_j \, x_j^{\nu_j} \right) \delta \left(1 - \sum_{i=1}^{n} x_i \right) \mathcal{U}^a \mathcal{F}^b \, .$$

- $\mathcal{U}$ and $\mathcal{F}$: Standard **Symanzik polynomials**
  $\rightarrow$ Homogeneous in x_i, order determined by loop-order (l for $\mathcal{U}$ and $l+1$ for $\mathcal{F}$)
- Scaling of (components of) loop momenta $\leftrightarrow$ scaling of x_i
  $\Rightarrow$ **Independent** of reference frame, momentum routing
  $\Rightarrow$ Inherently **covariant**!
- In practice: Consider scaling behaviour of product polynomial $\mathcal{U}\mathcal{F}$
  $\rightarrow$ Advantage: Asymptotic properties of **both** factors treated in one go
  $\rightarrow$ Disadvantage: Can lead to **large** expressions

Intermezzo: Determination of the regions

- Each term corresponds to some vector $\rho^{p_0} x_1^{p_1} \dots x_n^{p_n} \to (p_0, p_1, \dots, p_n)$
- [Pak and Smirnov, 2011] shows that the regions correspond to the points of the **convex hull** of $\{\mathcal{U}\mathcal{F}\}$
- The construction of a convex hull from M points in n dimensions is well-known and solved, e.g., by the `quickhull` algorithm [Barber et al., 1996]
  $\to$ Divide the set of points into 2 smaller subsets
  $\to$ Compute their separate convex hulls
  $\to$ Merge
- The method was generalized to more general **parametric integrals** in [Jantzen et al., 2012].

Step VI: Regulate endpoint singularities

$$\int CT = \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 \Big\{ [CT(z_1, z_2; \varepsilon)] \mathrm{d}\sigma_{\bar{a}\bar{b}}(z_1 p_a, z_2 p_b)$$

$$- \boldsymbol{L}_1[CT(z_1, z_2; \varepsilon)] \mathrm{d}\sigma_{\bar{a}\bar{b}}(p_a, z_2 p_b) - \boldsymbol{L}_2[CT(z_1, z_2; \varepsilon)] \mathrm{d}\sigma_{\bar{a}\bar{b}}(z_1 p_a, p_b)$$

$$- \Big(\boldsymbol{L}_{12}[CT(z_1, z_2; \varepsilon)] - \boldsymbol{L}_1 \boldsymbol{L}_{12}[CT(z_1, z_2; \varepsilon)] - \boldsymbol{L}_2 \boldsymbol{L}_{12}[CT(z_1, z_2; \varepsilon)] \Big) \mathrm{d}\sigma_{\bar{a}\bar{b}}(p_a, p_b)$$

$$+ [\boldsymbol{L}_1][CT(z_1, z_2; \varepsilon)] \mathrm{d}\sigma_{\bar{a}\bar{b}}(p_a, z_2 p_b) + [\boldsymbol{L}_2][CT(z_1, z_2; \varepsilon)] \mathrm{d}\sigma_{\bar{a}\bar{b}}(z_1 p_a, p_b)$$

$$+ \Big([\boldsymbol{L}_{12}][CT(z_1, z_2; \varepsilon)]) - [\boldsymbol{L}_1 \boldsymbol{L}_{12}][CT(z_1, z_2; \varepsilon)]$$

$$- [\boldsymbol{L}_2 \boldsymbol{L}_{12}][CT(z_1, z_2; \varepsilon)] \Big) \mathrm{d}\sigma_{\bar{a}\bar{b}}(p_a, p_b) \Big\}$$

The A_{12} integration recipe

The integration of all A_{12} (and also others) subtraction terms follows the steps outlined above

1. Set up a parametric representation of the integral
   $\rightarrow$ Typically leads to complicated **multivariate rational function**
2. Treat overlapping singularities
3. Prepare the integrand for integration in terms of GPLs
   - Factor higher-order polynomials and rationalize non-integer powers
   - Partial fraction
   - Fibration basis
4. Integrate
5. Repeat steps 3-4 for all integration variables
6. Regulate endpoint singularities

Summary

- **Precision** is key!
- Treat higher-order kinematic divergences using subtraction: `CoLoRFulNNLO`
- Application to color-singlet production in hadron-hadron collisions is now **within reach**
- All integrated subtraction terms have been computed **analytically**
  - ◇ Subset of the integrations based on IBP reduction and differential equations
  - ◇ Rest done **directly** using our **integration recipe**

```
************************** version  beta  **************************
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*********************** December 20th, 2024 ************************
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*                                                                  *
* https://github.com/nnlocal/nnlocal.git                           *
*                                                                  *
********************************************************************
```

- Implemented in publicly available code `NNLOCAL` [Del Duca et al., 2025]
- For now, the code is proof-of-concept: Gluon fusion Higgs production in HEFT
- Quark channels in double real radiation complete, full result will be available in the near future

`https://github.com/nnlocal`

End[1]

Thank you for your attention!

[1] Part of this work has been supported by grant K143451 of the National Research, Development and Innovation Fund in Hungary.

Appendices and references

Soft mapping

$$\begin{aligned}
\tilde{p}_a^\mu &= \lambda_r p_a^\mu \,, \\
\tilde{p}_b^\mu &= \lambda_r p_b^\mu \,, \\
\tilde{p}_n^\mu &= \Lambda(P,\tilde{P})^\mu{}_\nu \, p_n^\nu \,, \qquad n \in F \,, n \neq r \,, \\
\tilde{p}_X^\mu &= \Lambda(P,\tilde{P})^\mu{}_\nu \, p_X^\nu
\end{aligned}$$

$\Lambda(P,\tilde{P})^\mu{}_\nu$: Proper Lorentz transformation that takes the massive momentum P into a momentum of the same mass, $\tilde{P}$. One specific representation is

$$\Lambda(P,\tilde{P})^\mu{}_\nu = g^\mu{}_\nu - \frac{2(P+\tilde{P})^\mu (P+\tilde{P})_\nu}{(P+\tilde{P})^2} + \frac{2\tilde{P}^\mu P_\nu}{P^2} \,.$$

The value of λ_r is fixed by requiring that $P^2 = \tilde{P}^2$,

$$\lambda_r = 1 - \frac{s_{rQ}}{s_{ab}} \,.$$

Initial-final collinear mapping

$$\begin{aligned}
\hat{p}_a^\mu &= \xi_{a,r} p_a^\mu \,, \\
\hat{p}_b^\mu &= \xi_{b,r} p_b^\mu \,, \\
\hat{p}_n^\mu &= \Lambda(P,\hat{P})^\mu{}_\nu \, p_n^\nu \qquad \text{with } n \in F\,, n \neq r\,, \\
\hat{p}_X^\mu &= \Lambda(P,\hat{P})^\mu{}_\nu \, p_X^\nu
\end{aligned}$$

Here $\Lambda(P,\hat{P})^\mu{}_\nu$ is the same Lorentz transformation as in the soft mapping and

$$\xi_{a,r} = \sqrt{\frac{s_{ab}-s_{br}}{s_{ab}-s_{ar}} \frac{s_{ab}-s_{rQ}}{s_{ab}}}\,, \qquad \xi_{b,r} = \sqrt{\frac{s_{ab}-s_{ar}}{s_{ab}-s_{br}} \frac{s_{ab}-s_{rQ}}{s_{ab}}}\,.$$

Final-final collinear mapping

$$
\begin{aligned}
\hat{p}_a^\mu &= (1-\alpha_{ir})p_a^\mu \,, \\
\hat{p}_b^\mu &= (1-\alpha_{ir})p_b^\mu \,, \\
\hat{p}_{ir}^\mu &= p_i^\mu + p_r^\mu - \alpha_{ir} Q^\mu \,, \\
\hat{p}_n^\mu &= p_n^\mu \qquad \text{with } n \in F \,, n \neq i, r \,, \\
\hat{p}_X^\mu &= p_X^\mu
\end{aligned}
$$

The value of α_{ir} is fixed by requiring that the parent momentum, $\hat{p}_{ir}$, be massless, $\hat{p}_{ir}^2 = 0$,

$$
\alpha_{ir} = \frac{1}{2}\left[\frac{s_{(ir)Q}}{s_{ab}} - \sqrt{\frac{s_{(ir)Q}^2}{s_{ab}^2} - \frac{4s_{ir}}{s_{ab}}}\right]
$$

LinApart

Consider some proper rational function $f(x) = \frac{x^l}{Q(x)}$. We consider the decomposition over $\mathbb{C}$ such that

$$Q(x) = \prod_{i=1}^{n}(x - a_i)^{m_i}.$$

- $\{a_i\}_{i=1}^{n}$: Distinct roots of the polynomial $Q(x)$
- m_i: Multiplicity of the i-th root

Then

$$f(x) = \sum_{i=1}^{n}\left(\frac{c_{i1}}{x - a_i} + \frac{c_{i2}}{(x - a_i)^2} + \cdots + \frac{c_{im_i}}{(x - a_i)^{m_i}}\right)$$

with c_{ij} is the **residue** of $g_{ij}(x) = (x - a_i)^{j-1} f(x)$ at a_i

$$c_{ij} = (g_{ij}, a_i) = \frac{1}{(m_i - j)!}\lim_{x\to a_i}\frac{\mathrm{d}^{m_i - j}}{\mathrm{d}x^{m_i - j}}\Big((x - a_i)^{m_i} f(x)\Big)$$

LinApart

Note that

$$(x - a_i)^{m_i} f(x) = x^l \prod_{\substack{k=1\\k\neq i}}^{n} \frac{1}{(x - a_k)^{m_k}}$$

is independent of a_i $\Rightarrow$ simply set $x \to a_i$ to write c_{ij} directly in terms of the roots,

$$c_{ij} = \frac{1}{(m_i - j)!} \frac{\mathrm{d}^{m_i - j}}{\mathrm{d}a_i^{m_i - j}} a_i^l \prod_{\substack{k=1\\k\neq i}}^{n} \frac{1}{(a_i - a_k)^{m_k}} \, .$$

Hence, $f(x)$ can be expressed as

$$\begin{aligned} f(x) &= \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{c_{ij}}{(x - a_i)^j} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{1}{(x - a_i)^j} \frac{1}{(m_i - j)!} \frac{\mathrm{d}^{m_i - j}}{\mathrm{d}a_i^{m_i - j}} a_i^l \prod_{\substack{k=1\\k\neq i}}^{n} \frac{1}{(a_i - a_k)^{m_k}} \, . \end{aligned}$$

LinApart

Note: $(x - a_i)^{-j}$ is related to the $(j-1)$-st derivative of $(x - a_i)^{-1}$ with respect to a_i,

$$\frac{1}{(x - a_i)^j} = \frac{1}{(j-1)!} \frac{\mathrm{d}^{j-1}}{\mathrm{d}a_i^{j-1}} \frac{1}{x - a_i} \, .$$

Substituting, one sees that the summation over j corresponds to the general Leibniz rule for the $(m_i - 1)$-st derivative of a product of two functions,

$$f(x) = \sum_{i=1}^{n} \frac{1}{(m_i - 1)!} \frac{\mathrm{d}^{m_i - 1}}{\mathrm{d}a_i^{m_i - 1}} \left(\frac{a_i^l}{x - a_i} \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{1}{(a_i - a_k)^{m_k}} \right) \, .$$

→ Can be implemented directly in high-level language (e.g. `Wolfram Mathematica`)

LinApart

For low-level language (e.g. C): Resolve differentiation

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\prod_{j=1}^{n} h_j(x) = \sum_{j_1+\cdots+j_n=m} \binom{m}{j_1\ldots j_n} \prod_{l=1}^{n} \frac{\mathrm{d}^{j_l} h_l(x)}{\mathrm{d}x^{j_l}}$$

$$\Rightarrow f(x) = \sum_{i=1}^{n} \sum_{j_{-1}+j_0+j_1+\cdots+\hat{j}_i+\cdots+j_n=m_i-1} \binom{l}{j_{-1}} \frac{a_i^{l-j_{-1}}}{(x-a_i)^{j_0+1}} \times \prod_{\substack{k=1\\k\neq i}}^{n} \binom{m_k+j_k-1}{j_k} \frac{(-1)^{j_k}}{(a_i-a_k)^{m_k+j_k}}$$

$\hat{j}_i$: j_i is removed from the set of indices

LinApart

So far dealt with **proper** rational functions. So what about **improper** ones? Take $f(x) = \frac{x^l}{Q(x)}$ with $l \geq \deg Q$. We write

$$f(x) = x^{l-(m-1)} \frac{x^{m-1}}{Q(x)}, \qquad l \geq m$$

$m = \deg Q(x) = \sum_{i=1}^{n} m_i$: Degree of the denominator $Q(x)$

Second factor = proper rational function **by construction** $\Rightarrow$ can apply formula derived above

The resulting expression contains only rational functions of the form $g(x) = \frac{x^p}{(x-a)^q}$, and we implement the polynomial division symbolically

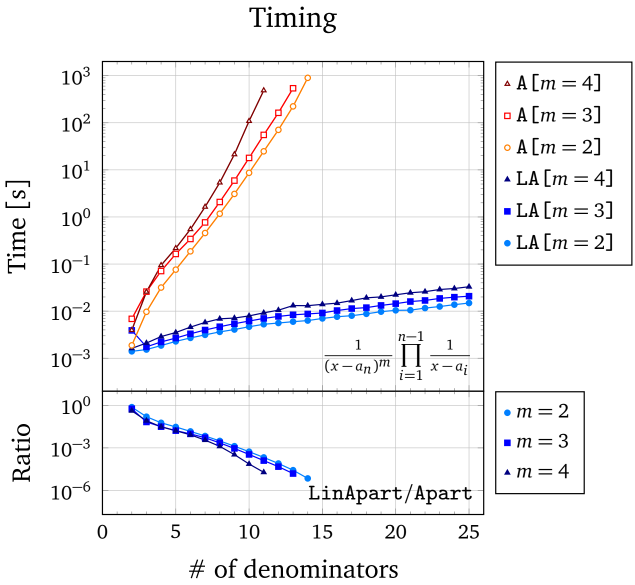
$$g(x) = \sum_{i=0}^{p-q} \binom{p-1-i}{q-1} a^{p-q-i} x^i + \sum_{i=p-q+1}^{p} \binom{p}{i} a^i (x-a)^{p-q-i}$$

LinApart

So what do we **gain** with our implementation?

How do we measure **complexity** of a rational function?

1. The number of distinct denominator factors.
2. The complexity of each individual denominator. In fact, even considering only linear denominators of the form $x - a_i$, the roots a_i may be functions of further variables and symbolic constants.
3. The multiplicity of the denominator factors.
4. The polynomial order of the numerator.


Timing
Time [s]
10^3
10^2
10^1
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10^{-1}
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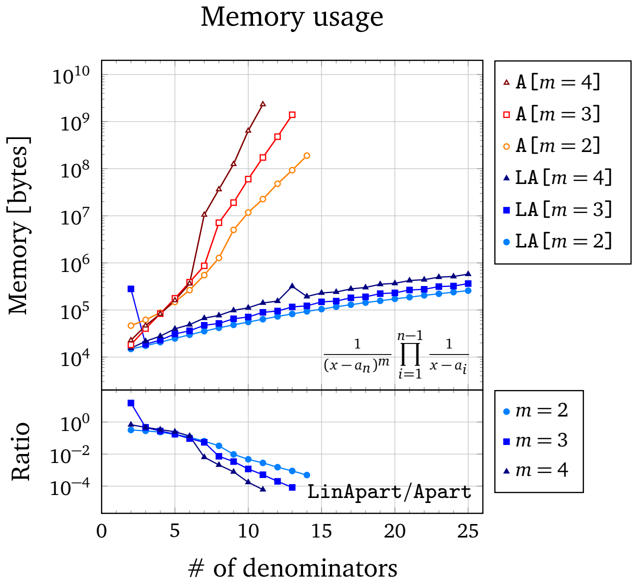
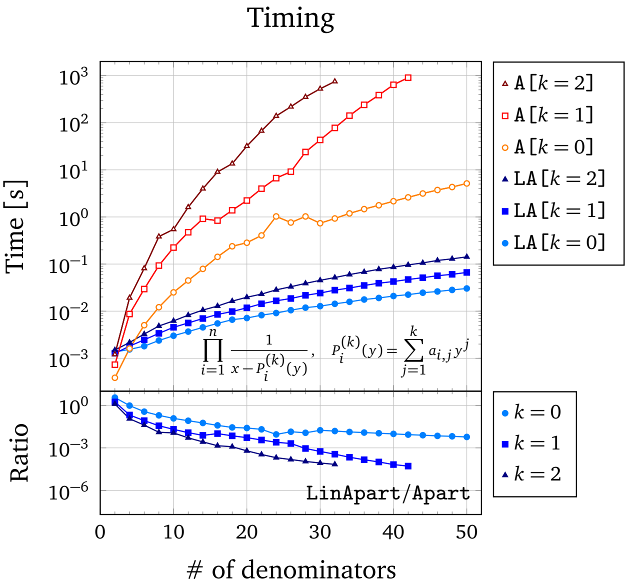
$\frac{1}{(x-a_n)^m}\prod_{i=1}^{n-1}\frac{1}{x-a_i}$
A [$m=4$]
A [$m=3$]
A [$m=2$]
LA [$m=4$]
LA [$m=3$]
LA [$m=2$]
Ratio
10^0
10^{-3}
10^{-6}
LinApart/Apart
$m=2$
$m=3$
$m=4$
0
5
10
15
20
25
# of denominators

LinApart



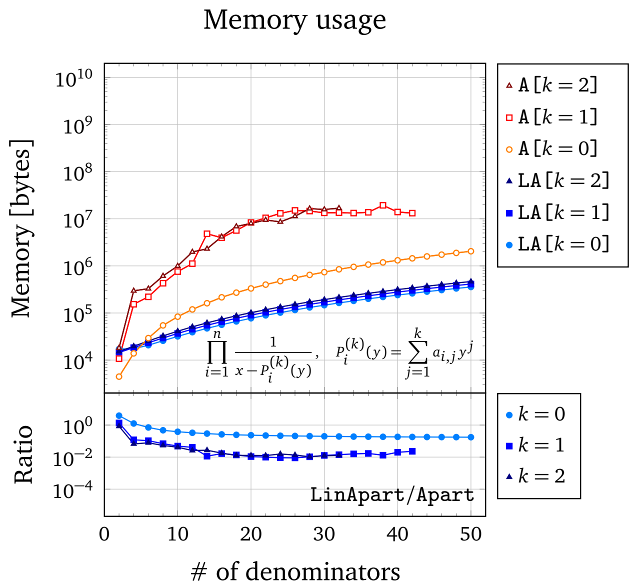
Memory usage
Memory [bytes]
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
$\frac{1}{(x-a_n)^m}\prod_{i=1}^{n-1}\frac{1}{x-a_i}$
A [m = 4]
A [m = 3]
A [m = 2]
LA [m = 4]
LA [m = 3]
LA [m = 2]
Ratio
10^{0}
10^{-2}
10^{-4}
LinApart/Apart
m = 2
m = 3
m = 4
0
5
10
15
20
25
# of denominators

LinApart



Timing
Time [s]
Ratio
# of denominators
A [k = 2]
A [k = 1]
A [k = 0]
LA [k = 2]
LA [k = 1]
LA [k = 0]
k = 0
k = 1
k = 2
LinApart/Apart
$\prod_{i=1}^{n} \frac{1}{x - P_i^{(k)}(y)}, \quad P_i^{(k)}(y) = \sum_{j=1}^{k} a_{i,j} y^j$

LinApart



Memory usage
Memory [bytes]
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
$\prod_{i=1}^{n} \frac{1}{x - P_i^{(k)}(y)}, \quad P_i^{(k)}(y) = \sum_{j=1}^{k} a_{i,j} y^j$
A [$k = 2$]
A [$k = 1$]
A [$k = 0$]
LA [$k = 2$]
LA [$k = 1$]
LA [$k = 0$]
Ratio
10^{0}
10^{-2}
10^{-4}
LinApart/Apart
$k = 0$
$k = 1$
$k = 2$
0
10
20
30
40
50
# of denominators

LinApart

$$
\begin{aligned}
f(x_a, x_b; y) = \Big[&(-4+y)(1-y+x_b y)(2-y+x_b y)(4-y+x_b y)(1-x_a-y+x_b y)^3 \\
&\times (-1+x_a-y+x_b y)(-4-4x_b-y+x_b y)(-4x_b-y+x_b y) \\
&\times (-4x_a-4x_b-y+x_b y)(4x_a-4x_b-y+x_b y)(2+2x_b-y+x_b y)^3 \\
&\times (6+2x_b-y+x_b y)(2-4x_a+2x_b-y+x_b y)(2+4x_a+2x_b-y+x_b y) \\
&\times (-1+x_a-x_a y+x_a x_b y)(1+x_a-x_a y+x_a x_b y)(-2+2x_a-x_a y+x_a x_b y) \\
&\times (2+2x_a-x_a y+x_a x_b y)(-x_b+x_a x_b-x_a y+x_a x_b y)^3 \\
&\times (-4+2x_a+2x_a x_b-x_a y+x_a x_b y)(4+2x_a+2x_a x_b-x_a y+x_a x_b y) \\
&\times (1-2x_a+x_a^2-y-x_a y+x_b y+x_a x_b y) \\
&\times \left(2x_b-2x_a x_b+x_a y-x_b y-x_a x_b y+x_b^2 y\right)^3 \Big]^{-1}
\end{aligned}
$$

- Apart: $t > 10^9$ s
- LinApart: $t \sim 10^{-2}$ s

Codes (Mathematica + C) publicly available! [https://github.com/fekeshazy/LinApart]

Where is the plus-distribution?

$$\int_0^1 \mathrm{d}\eta_a \Big\{ \mathcal{I}(\eta_a,\eta_b;\varepsilon)|\mathcal{M}^{(0)}(\eta_a p_a,\eta_b p_b)|^2 - \boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon)|\mathcal{M}^{(0)}(p_a,\eta_b p_b)|^2$$

$$+ \boldsymbol{IL_a}\,\mathcal{I}(\eta_b;\varepsilon)|\mathcal{M}^{(0)}(p_a,\eta_b p_b)|^2\delta(1-\eta_a)\Big\}$$

$$= \int_0^1 \mathrm{d}\eta_a \Big\{ [\boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon) + \{\mathcal{I}(\eta_a,\eta_b;\varepsilon) - \boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon)\}]|\mathcal{M}^{(0)}(\eta_a p_a,\eta_b p_b)|^2$$

$$- \boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon)|\mathcal{M}^{(0)}(p_a,\eta_b p_b)|^2 + \boldsymbol{IL_a}\,\mathcal{I}(\eta_b;\varepsilon)|\mathcal{M}^{(0)}(p_a,\eta_b p_b)|^2\delta(1-\eta_a)\Big\}$$

$$= \int_0^1 \mathrm{d}\eta_a \underbrace{\boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon)\Big\{|\mathcal{M}^{(0)}(\eta_a p_a,\eta_b p_b)|^2 - \mathcal{M}^{(0)}(p_a,\eta_b p_b)|^2\Big\}}_{[\boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon)]_+|\mathcal{M}^{(0)}(\eta_a p_a,\eta_b p_b)|^2}$$

$$+ \int_0^1 \mathrm{d}\eta_a\, \boldsymbol{IL_a}\,\mathcal{I}(\eta_b;\varepsilon)|\mathcal{M}^{(0)}(p_a,\eta_b p_b)|^2\delta(1-\eta_a)$$

$$+ \int_0^1 \mathrm{d}\eta_a\, [\mathcal{I}(\eta_a,\eta_b;\varepsilon) - \boldsymbol{L_a}\,\mathcal{I}(\eta_a,\eta_b;\varepsilon)]\,|\mathcal{M}^{(0)}(\eta_a p_a,\eta_b p_b)|^2$$

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