All-N structure of leading-twist alien operators in QCD

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Introduction

At HL-LHC: Statistical/systematic uncertainties $\sim 1\%$

$\Rightarrow$ Theory needs to keep up!

	Q [GeV]	$\delta\sigma^{\text{N}^3\text{LO}}$	δ(scale)	δ(PDF-TH)
$gg \to$ Higgs	m_H	3.5%	$^{+0.21\%}_{-2.37\%}$	$\pm 1.2\%$
$b\bar{b} \to$ Higgs	m_H	-2.3%	$^{+3.0\%}_{-4.8\%}$	$\pm 2.5\%$
NCDY	30	-4.8%	$^{+1.53\%}_{-2.54\%}$	$\pm 2.8\%$
	100	-2.1%	$^{+0.66\%}_{-0.79\%}$	$\pm 2.5\%$
CCDY(W^+)	30	-4.7%	$^{+2.5\%}_{-1.7\%}$	$\pm 3.2\%$
	150	-2.0%	$^{+0.5\%}_{-0.5\%}$	$\pm 2.1\%$
CCDY(W^-)	30	-5.0%	$^{+2.6\%}_{-1.6\%}$	$\pm 3.2\%$
	150	-2.1%	$^{+0.6\%}_{-0.5\%}$	$\pm 2.13\%$

Table: [Baglio et al., 2022]

$$\delta(\text{PDF-TH}) = \frac{1}{2}\frac{\left|\sigma^{\text{NNLO}}(\text{NNLO PDF}) - \sigma^{\text{NNLO}}(\text{NLO PDF})\right|}{\sigma^{\text{NNLO}}(\text{NNLO PDF})}$$

PDF scale dependence

Scale evolution of PDFs is set by the DGLAP equation [Gribov and Lipatov, 1972], [Altarelli and Parisi, 1977], [Dokshitzer, 1977]

$$\frac{\mathrm{d}f_i(x,\mu^2)}{\mathrm{d}\ln\mu^2} = \int_x^1 \frac{\mathrm{d}y}{y} P_{ij}(y) f_j\left(\frac{x}{y},\mu^2\right)$$

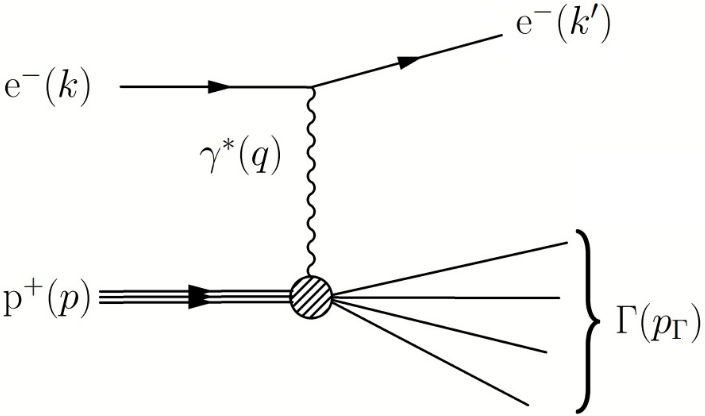
with P_{ij} the QCD splitting functions. These are perturbative quantities and can be computed as the anomalous dimensions of the leading-twist operators that define the PDFs

$$\frac{\mathrm{d}[\mathcal{O}_i]}{\mathrm{d}\ln\mu^2} = \gamma^{ij}[\mathcal{O}_j], \quad \gamma^{ij} \equiv a_s\gamma^{ij,(0)} + a_s^2\gamma^{ij,(1)} + \ldots$$

$$\gamma^{ij} = -\int_0^1 \mathrm{d}x\, x^N P_{ij}(x)$$

Where do these operators come from?

Deep-inelastic scattering (DIS)



Assumptions:

- Photon highly virtual, $Q^2 \equiv -q^2 \gg p^2$
- $s \gg m_p^2$

The DIS cross section

The physical cross section of DIS is proportional to

$$\frac{1}{q^4} L_{\mu\nu} W^{\mu\nu}$$

Here, $L_{\mu\nu}$ represents the leptonic tensor and $W_{\mu\nu}$ the hadronic one.

- $L_{\mu\nu}$ encodes the polarization information of the electrons and the off-shell photon. Applying standard techniques it is easy to find that

$$L_{\mu\nu} = \frac{1}{2}\mathrm{Tr}[\not{k}'\gamma_\mu \not{k}\gamma_\nu].$$

- $W^{\mu\nu}$ encodes the information of the $\gamma^* p^+ \to \Gamma$ process, the amplitude of which is

$$\mathcal{M}(\gamma^* p^+ \to \Gamma) \sim \langle \Gamma | J_\mu | p^+(p) \rangle$$

  with

$$J_\mu = \sum_f Q_f \overline{\psi}_f \gamma_\mu \psi_f \text{ the electromagnetic current.}$$

The DIS hadronic tensor

The hadronic tensor appearing in the DIS cross section can then be written as

$$W_{\mu\nu} = \int \mathrm{d}^4x \, e^{iq\cdot x} \left\langle p^+(p) \right| J_\mu(x) J_\nu(0) \left| p^+(p) \right\rangle .$$

Note that this is independent of the final states Γ.

Hence, the calculation of the hadronic tensor of DIS boils down to calculating the product of two current operators.

The standard formalism to deal with this type of problem is the operator product expansion (OPE).

The OPE

The OPE was first introduced by Wilson [Wilson, 1969] and later proven in perturbation theory by Zimmerman [Zimmermann, 1973].

The main idea is that the time-ordered product of two local operators $J(x)$ and $J'(y)$ can be expanded in a series of regular operators, multiplied by functions (called Wilson coefficients) encoding the singularity of the operator product as $x = y$

$$\mathcal{T} J(x) J'(y) = \sum_{n=0}^{\infty} C_n(x-y) \mathcal{O}_n\left(\frac{x-y}{2}\right).$$

To apply the OPE to the DIS hadronic tensor, we use the optical theorem to relate the rate of $\gamma^* p^+ \to \Gamma$ to the imaginary part of the forward scattering rate $\gamma^* p^+ \to \gamma^* p^+$:

$$W_{\mu\nu} = 2\,\mathrm{Im}\,T_{\mu\nu},$$

$$T_{\mu\nu} = i \int \mathrm{d}^4 x\, e^{iq\cdot x} \left\langle p^+(p)\right| \mathcal{T} J_\mu(x) J_\nu(0) \left|p^+(p)\right\rangle.$$

Application of the OPE to DIS

$T_{\mu\nu}$ can be explicitly calculated as the forward matrix element for Compton scattering, $\gamma^* q \to \gamma^* q$ (photon off-shell and no polarizations included). This gives

$$T_{\mu\nu} \sim -\bar{u}(p)\frac{\gamma_\mu(\not{p}+\not{q})\gamma_\nu}{(p+q)^2}u(p).$$

As we are interested in the regime of large Q^2, we expand the denominator for $Q^2 \gg p^2$

$$\frac{1}{(p+q)^2} = -\frac{1}{Q^2}\sum_n \left(\frac{2p\cdot q}{Q^2}\right)^n$$

such that

$$T_{\mu\nu} \sim \frac{1}{Q^2}\bar{u}(p)\gamma_\mu(\not{p}+\not{q})\gamma_\nu u(p)\sum_n \left(\frac{2p\cdot q}{Q^2}\right)^n.$$

Application of the OPE to DIS

The ingredients of the OPE, i.e. the Wilson coefficients and the operators, can be read of from the momentum expansion in a relatively straightforward manner:

- Factors of p_μ should come from factors of $i\partial_\mu$ from the operators, acting on the external states
- The dependence on the short-distance scale should be incorporated into the Wilson coefficients

This implies that the Wilson coefficients for DIS will be of the following form

$$C^{\mu_1 \cdots \mu_n} \sim \frac{2^n}{Q^{2n+1}} q^{\mu_1 \cdots \mu_n}.$$

Application of the OPE to DIS

For the extraction of the operators, it is customary to use a basis of gauge-invariant operators, meaning that ordinary derivatives are replaced by covariant ones

$$\partial_\mu \to D_\mu = \partial_\mu - i g_s A_\mu .$$

Furthermore, the OPE is dominated by leading-twist operators, where *twist* = *dimension* - *spin*. These operators are symmetric in the Lorentz indices and traceless. We can distinguish 2 sets of leading-twist operators based on their representation in the QCD flavour group.

- Flavour non-singlet quark operator

$$\mathcal{O}^{(N)}_{q\,\mathrm{NS};\mu_1\ldots\mu_N}(x) = \mathcal{S}\left[\,\overline{\psi}\lambda^{\alpha}\gamma_{\mu_1} D_{\mu_2}\ldots D_{\mu_N}\psi\,\right]$$

- Flavour singlet quark operator + gluon operator

$$\mathcal{O}^{(N)}_{g;\mu_1\ldots\mu_N}(x) = \frac{1}{2}\mathcal{S}\left[\,F^{a_1}{}_{\mu\mu_1}\, D^{a_1 a_2}_{\mu_2}\ldots D^{a_{N-2}a_{N-1}}_{\mu_{N-1}} F^{a_{N-1};\mu}{}_{\mu_N}\,\right]$$

$$\mathcal{O}^{(N)}_{q\,\mathrm{S};\mu_1\ldots\mu_N}(x) = \mathcal{S}\left[\,\overline{\psi}\gamma_{\mu_1} D_{\mu_2}\ldots D_{\mu_N}\psi\,\right]$$

Parton distribution functions

Finally, one has to consider the forward matrix elements of these operators

$$\left\langle p^+(p)\right| \mathcal{O}_{\mu_1\ldots\mu_N} \left|p^+(p)\right\rangle \sim \mathcal{M}_N(Q)\, p_{\mu_1} \ldots p_{\mu_N}.$$

The functions $\mathcal{M}_N$ can be used to define the PDFs

$$f(x) \sim \sum_n \frac{\mathrm{Im}\,\mathcal{M}_n}{x^n}.$$

To derive the scale dependence of the PDFs, we now need to compute the anomalous dimensions of the operators

$$\frac{\mathrm{d}[\mathcal{O}_i]}{\mathrm{d}\ln\mu^2} = \gamma^{ij}[\mathcal{O}_j], \quad \gamma^{ij} \equiv a_s\gamma^{ij,(0)} + a_s^2\gamma^{ij,(1)} + \ldots$$

$$\gamma^{ij} = -\int_0^1 \mathrm{d}x\, x^N P_{ij}(x)$$

Renormalization of gauge invariant operators

To extract the anomalous dimensions of interest, we now need to renormalize the operators. According to the OPE, one needs to take into account mixing of operators in the same representation. This implies that

- the non-singlet quark operators renormalize multiplicatively

$$\mathcal{O}_{q\,\mathrm{NS}}^{(N)} = Z_N[\mathcal{O}_{q\,\mathrm{NS}}^{(N)}]$$

- the singlet quark and gluon operators mix under renormalization

$$\begin{pmatrix} \mathcal{O}_{q\,\mathrm{S}}^{(N)} \\ \mathcal{O}_{g}^{(N)} \end{pmatrix} = \begin{pmatrix} Z_N^{qq} & Z_N^{qg} \\ Z_N^{gq} & Z_N^{gg} \end{pmatrix} \begin{pmatrix} [\mathcal{O}_{q\,\mathrm{S}}^{(N)}] \\ [\mathcal{O}_{g}^{(N)}] \end{pmatrix}$$

Note: Use the $\overline{\mathrm{MS}}$-scheme and $D = 4 - 2\varepsilon$ dimensional regularization.

Renormalization of gauge invariant operators

$$Z_N^{qq} = 1 + \frac{a_s}{\varepsilon}\gamma_N^{qq,(0)} + \frac{a_s^2}{2\varepsilon}\left\{\frac{1}{\varepsilon}\left[\gamma_N^{qq,(0)}(\gamma_N^{qq,(0)} - \beta_0) + \gamma_N^{qg,(0)}\gamma_N^{gq,(0)}\right] + \gamma_N^{qq,(1)}\right\}\dots$$

$$Z_N^{qg} = \frac{a_s}{\varepsilon}\gamma_N^{qg,(0)} + \frac{a_s^2}{2\varepsilon}\left\{\frac{\gamma_N^{qg,(0)}}{\varepsilon}(\gamma_N^{qq,(0)} + \gamma_N^{gg,(0)} - 2\beta_0) + \gamma_N^{qg,(1)}\right\} + \dots$$

$$Z_N^{gq} = \frac{a_s}{\varepsilon}\gamma_N^{gq,(0)} + \frac{a_s^2}{2\varepsilon}\left\{\frac{\gamma_N^{gq,(0)}}{\varepsilon}(\gamma_N^{qq,(0)} + \gamma_N^{gg,(0)} - 2\beta_0) + \gamma_N^{gq,(1)}\right\} + \dots$$

$$Z_N^{gg} = 1 + \frac{a_s}{\varepsilon}\gamma_N^{gg,(0)} + \frac{a_s^2}{2\varepsilon}\left\{\frac{1}{\varepsilon}\left[\gamma_N^{gg,(0)}(\gamma_N^{gg,(0)} - \beta_0) + \gamma_N^{gq,(0)}\gamma_N^{qg,(0)}\right] + \gamma_N^{gg,(1)}\right\}\dots$$

Renormalization of gauge invariant operators

Unfortunately, the mixing pattern of the operators is even more complicated as alluded to above when computing off-shell matrix elements. In particular, one needs to take into account mixing with non-gauge-invariant (alien) operators.

Aliens through history

- The appearance of alien[1] operators in the renormalization of the physical ones has been known since the early seventies [Gross and Wilczek, 1974]. They obtained the physical anomalous dimensions without accounting for aliens by using lightcone gauge (no ghosts).
- The origin of the issue was provided by Dixon and Taylor [Dixon and Taylor, 1974]. In particular, they showed that the bare Yang-Mills Lagrangian is invariant under a different set of gauge transformations as the renormalized one.
  $\rightarrow$ Construction of the aliens relevant for the computation of the 1-loop anomalous dimensions
- 2 years later, Joglekar and Lee worked out the general theory of the renormalization of gauge invariant operators. Their main results are summarized in 3 theorems [Joglekar and Lee, 1976]

[1]Term coined in '94 by [Collins and Scalise, 1994].

Side-step: Joglekar-Lee theorems

1. The basis of alien operators A_i that mix with the gauge invariant ones can be chosen such that they are BRST exact

$$A_i \sim \delta_{\mathrm{BRST}} B_i.$$

   Here B_i is called the ancestor of A_i.

2. Physical matrix elements of the aliens vanish.
3. The mixing matrix is triangular

$$\begin{pmatrix} [\mathcal{O}_G] \\ [\mathcal{O}_A] \\ [\mathcal{O}_E] \end{pmatrix} = \begin{pmatrix} Z_{GG} & Z_{GA} & Z_{GE} \\ 0 & Z_{AA} & Z_{AE} \\ 0 & 0 & Z_{EE} \end{pmatrix} \begin{pmatrix} \mathcal{O}_G \\ \mathcal{O}_A \\ \mathcal{O}_E \end{pmatrix}.$$

Aliens through history

- The 2-loop anomalous dimensions were computed a few years later using different gauges: [Floratos et al., 1979, Gonzalez-Arroyo and Lopez, 1980, Floratos et al., 1981] used the covariant gauge while [Furmanski and Petronzio, 1980] used the axial gauge
- The computations using covariant gauge agreed with one another but disagreed with the axial gauge one
- The issue was solved a decade later by Hamberg and van Neerven in favour of the axial gauge result [Hamberg and van Neerven, 1992]
- Unfortunately, the way forward was not clear; the generalization of the basis of aliens to higher orders in perturbation theory was unknown.
- Nevertheless, the 3-loop anomalous dimensions were computed using different methods [Vogt et al., 2004]

Aliens through history

- Finally, after 30 years, significant progress was made in the alien issue by 2 independent groups
- G. Falcioni and F. Herzog were able to derive constraints to consistently derive the aliens at fixed orders which were solved for fixed $N \leq 20$ [Falcioni and Herzog, 2022]

  $\rightarrow$ All 4-loop splitting functions now known to $N = 20$ [Falcioni et al., 2023b, Falcioni et al., 2023a, Gehrmann et al., 2024a, Falcioni et al., 2024d, Falcioni et al., 2024b, Falcioni et al., 2024a]
- On the other hand, [Gehrmann et al., 2023] developed a method to derive the counterterm Feynman rules for the aliens

  $\rightarrow$ n_f^2 contributions to the pure-singlet splitting functions at 4 loops [Gehrmann et al., 2024a]

Focus on method by Giulio and Franz in what's next

Construction of the alien operators

The complete gauge-fixed QCD action is written as

$$S = \int \mathrm{d}^D x \, (\mathcal{L}_0 + \mathcal{L}_{\mathrm{GF+G}}) \, .$$

Here $\mathcal{L}_0$ represents the classical part of the QCD Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \sum_{f=1}^{n_f} \overline{\psi}^f (i \not{D} - m_f) \psi^f \, ,$$

with

$$\mathcal{L}_{\mathsf{GF+G}} = -\frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 - \overline{c}^a \, \partial^\mu D^{ab}_\mu \, c^b$$

and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$$
$$D_\mu = \partial_\mu - i g_s T^a A^a_\mu$$
$$D^{ac}_\mu = \partial_\mu \delta^{ac} + g_s f^{abc} A^b_\mu$$

f^{abc} are the standard QCD structure constants.

Construction of the alien operators

The QCD Lagrangian can be extended to also include the leading-twist spin-N gauge-invariant operators, which we define as

$$\mathcal{O}_{\mathrm{g}}^{(N)}(x) = \frac{1}{2} F_\nu(x)\, D^{N-2} F^\nu(x)\,,$$
$$\mathcal{O}_{\mathrm{q}}^{(N)}(x) = \overline{\psi}(x) \not{\Delta}\, D^{N-1} \psi(x)\,.$$

Here Δ_μ is a lightlike vector and we introduced the notation

$$F^{\mu;a} = \Delta_\nu\, F^{\mu\nu;a}, \qquad A^a = \Delta_\mu A^{\mu;a}, \qquad D = \Delta_\mu\, D^\mu, \qquad \partial = \Delta_\mu \partial^\mu\,.$$

These physical operators now mix under renormalization with aliens, which are (a) proportional to the field EOMs and (b) contain ghosts. Schematically the complete Lagrangian is then

$$\widetilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_{\mathsf{GF+G}} + w_{\mathrm{i}}\, \mathcal{O}_{\mathrm{i}} + \mathcal{O}_{\mathsf{EOM}}^{(N)} + \mathcal{O}_c^{(N)}$$

Construction of the alien operators

The most general form of the EOM operator is [Falcioni and Herzog, 2022]

$$\mathcal{O}_{\text{EOM}}^{(N)} = \left(D \cdot F^a + g_s \overline{\psi} T^a \not{\Delta} \psi\right) \mathcal{G}^a(A^a, \partial A^a, \partial^2 A^a, ...)$$

with $\mathcal{G}^a$ a generic local function of the gauge field and its derivatives. Expanding $\mathcal{G}^a$ in a series of contributions with an increasing number of gauge fields then leads to

$$\mathcal{O}_{\text{EOM}}^{(N)} = \mathcal{O}_{\text{EOM}}^{(N),I} + \mathcal{O}_{\text{EOM}}^{(N),II} + \mathcal{O}_{\text{EOM}}^{(N),III} + \mathcal{O}_{\text{EOM}}^{(N),IV} + \ldots$$

Construction of the alien operators

$$\mathcal{O}_{\text{EOM}}^{(N),I} = \eta(N)\ \left(D\cdot F^a + g_s\overline{\psi}\,\not{\Delta}\,T^a\psi\right)\ \left(\partial^{N-2}A^a\right),$$

$$\mathcal{O}_{\text{EOM}}^{(N),II} = g_s\left(D\cdot F^a + g_s\overline{\psi}\,\not{\Delta}\,T^a\psi\right)\sum_{\substack{i+j\\=N-3}} C_{ij}^{abc}(\partial^i A^b)(\partial^j A^c),$$

$$\mathcal{O}_{\text{EOM}}^{(N),III} = g_s^2\ \left(D\cdot F^a + g_s\overline{\psi}\not{\Delta}T^a\psi\right)\sum_{\substack{i+j+k\\=N-4}} C_{ijk}^{abcd}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d),$$

$$\mathcal{O}_{\text{EOM}}^{(N),IV} = g_s^3\ \left(D\cdot F^a + g_s\overline{\psi}\not{\Delta}T^a\psi\right)\sum_{\substack{i+j+k+l\\=N-5}} C_{ijkl}^{abcde}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^l A^e).$$

Construction of the alien operators

The coefficients $C^{a_1 \ldots a_n}_{i_1 \ldots i_{n-1}}$ appearing can be written in terms of a set of independent colour tensors, each of them multiplying an associated coupling constant, as follows

$$
\begin{aligned}
C^{abc}_{ij} &= f^{abc} \kappa_{ij}, \\
C^{abcd}_{ijk} &= (f\, f)^{abcd} \kappa^{(1)}_{ijk} + d_4^{abcd} \kappa^{(2)}_{ijk} + d_{\widehat{4ff}}^{abcd} \kappa^{(3)}_{ijk}, \\
C^{abcde}_{ijkl} &= (f\, f\, f)^{abcde} \kappa^{(1)}_{ijkl} + d_{4f}^{abcde} \kappa^{(2)}_{ijkl}
\end{aligned}
$$

To avoid overcounting: κ-couplings inherit properties of the colour structures they multiply, e.g. $\kappa_{ij} = -\kappa_{ji}$

The standard gauge transformations leave $\mathcal{L}_0$ and $\mathcal{O}_i$ invariant, but not $\mathcal{O}^{(N)}_{\text{EOM}}$

$\Rightarrow$ generalized gauge transformation

$$A^a_\mu \to A^a_\mu + \delta_\omega A^a_\mu + \delta^\Delta_\omega A^a_\mu$$

Construction of the alien operators

$$A_\mu^a \to A_\mu^a + \delta_\omega A_\mu^a + \delta_\omega^\Delta A_\mu^a$$

$$\delta_\omega A_\mu^a = D_\mu^{ab} \omega^b(x),$$

$$\delta_\omega^\Delta A_\mu^a = -\Delta_\mu \Bigg[\eta(N)\, \partial^{N-1} \omega^a + g_s \sum_{\substack{i+j \\ =N-3}} \widetilde{C}_{ij}^{aa_1a_2} \left(\partial^i A^{a_1}\right) \left(\partial^{j+1} \omega^{a_2}\right)$$

$$+ g_s^2 \sum_{\substack{i+j+k \\ =N-4}} \widetilde{C}_{ijk}^{aa_1a_2a_3} \left(\partial^i A^{a_1}\right) \left(\partial^j A^{a_2}\right) \left(\partial^{k+1} \omega^{a_3}\right)$$

$$+ g_s^3 \sum_{\substack{i+j+k+l \\ =N-5}} \widetilde{C}_{ijkl}^{aa_1a_2a_3a_4} \left(\partial^i A^{a_1}\right) \left(\partial^j A^{a_2}\right) \left(\partial^k A^{a_3}\right) \left(\partial^{l+1} \omega^{a_4}\right) + \mathcal{O}(g_s^4) \Bigg]$$

Construction of the alien operators

$$\widetilde{C}_{ij}^{abc} = f^{abc}\eta_{ij},$$
$$\widetilde{C}_{ijk}^{abcd} = (f\,f)^{abcd}\eta_{ijk}^{(1)} + d_4^{abcd}\eta_{ijk}^{(2)} + d_{\widehat{4ff}}^{abcd}\eta_{ijk}^{(3)},$$
$$\widetilde{C}_{ijkl}^{abcde} = (f\,f\,f)^{abcde}\eta_{ijkl}^{(1)} + d_{4f}^{abcde}\eta_{ijkl}^{(2a)} + d_{4f}^{aebcd}\eta_{ijkl}^{(2b)}.$$

The generalized gauge symmetry implies that the couplings $\eta_{n_1 \dots n_j}^{(k)}$ are related to $\kappa_{n_1 \dots n_j}^{(k)}$

$$\eta_{ij} = 2\kappa_{ij} + \eta(N)\binom{i+j+1}{i},$$
$$\eta_{ijk}^{(1)} = 2\kappa_{i(j+k+1)}\binom{j+k+1}{j} + 2[\kappa_{ijk}^{(1)} + \kappa_{kji}^{(1)}],$$
$$\eta_{ijkl}^{(1)} = 2[\kappa_{ij(l+k+1)}^{(1)} + \kappa_{(l+k+1)ji}^{(1)}]\binom{l+k+1}{k} + 2[\kappa_{ijkl}^{(1)} + \kappa_{ilkj}^{(1)} + \kappa_{likj}^{(1)} + \kappa_{lkij}^{(1)}].$$

Construction of the alien operators

The generalized gauge transformation can now be promoted to a generalized BRST (gBRST) transformation

$$A^a_\mu \to A^a_\mu + \delta_c A^a_\mu + \delta^\Delta_c A^a_\mu$$

The ghost operator is now generated by the action of gBRST on a suitable ancestor operator [Falcioni and Herzog, 2022], giving

$$\mathcal{O}_c^{(N)} = \mathcal{O}_c^{(N),I} + \mathcal{O}_c^{(N),II} + \mathcal{O}_c^{(N),III} + \mathcal{O}_c^{(N),IV} + \ \ldots$$

Construction of the alien operators

$$\mathcal{O}_c^{(N),I} = -\eta(N)(\partial\overline{c}^a)(\partial^{N-1}c^a),$$

$$\mathcal{O}_c^{(N),II} = -g_s \sum_{\substack{i+j\\=N-3}} \widetilde{C}_{ij}^{abc}(\partial\overline{c}^a)(\partial^i A^b)(\partial^{j+1}c^c),$$

$$\mathcal{O}_c^{(N),III} = -g_s^2 \sum_{\substack{i+j+k\\=N-4}} \widetilde{C}_{ijk}^{astu}(\partial\overline{c}^a)(\partial^i A^s)(\partial^j A^t)(\partial^{k+1}c^u),$$

$$\mathcal{O}_c^{(N),IV} = -g_s^3 \sum_{\substack{i+j+k+l\\=N-5}} \widetilde{C}_{ijkl}^{abcde}(\partial\overline{c}^a)(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^{l+1}c^e).$$

Construction of the alien operators

In fact, there is another, and equivalent, approach to generate the ghost operators. Namely, we could also start from anti-gBRST, for which $\omega^a(x)$ in the generalized gauge transformation should be replaced by the anti-ghost field $\overline{c}^a(x)$

$$A^a_\mu \to A^a_\mu + \delta_{\overline{c}} A^a_\mu + \delta^{\Delta}_{\overline{c}} A^a_\mu$$

$\rightarrow$ This should lead to the same operators!
$\rightarrow$ Nevertheless, the functional form of the resulting operators is different from those derived from gBRST

$\Rightarrow$ Non-trivial identities for the η-couplings!

Construction of the alien operators

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = 0,$$

$$\eta^{(1)}_{ijk} = \sum_{m=0}^{i} \sum_{n=0}^{j} \frac{(m+n+k)!}{m!\, n!\, k!} (-1)^{m+n+k} \eta^{(1)}_{(j-n)(i-m)(k+m+n)},$$

$$\eta^{(1)}_{ijkl} = -\sum_{s_1=0}^{i} \sum_{s_2=0}^{j} \sum_{s_3=0}^{k} \frac{(s_1+s_2+s_3+l)!}{s_1!\, s_2!\, s_3!\, l!} (-1)^{s_1+s_2+s_3+l} \eta^{(1)}_{(k-s_3)(j-s_2)(i-s_1)(s_1+s_2+s_3+l)}.$$

These identities are particularly interesting as they are conjugation relations. E.g. for the class II coupling a second application of the sum gives

$$\sum_{t=0}^{i} (-1)^{t+j} \binom{t+j}{j} \eta_{(i-t)(j+t)} = -\sum_{t=0}^{i} (-1)^{t+j} \binom{t+j}{j} \sum_{s=0}^{i-t} (-1)^{s+j+t} \binom{s+j+t}{j+t} \eta_{(i-t-s)(j+t+s)}$$

and hence

$$\eta_{ij} = \sum_{t=0}^{i} \binom{t+j}{j} \sum_{s=0}^{i-t} (-1)^{s} \binom{s+j+t}{j+t} \eta_{(i-t-s)(j+t+s)}.$$

Conjugation relations

- Already encountered in the computation of the anomalous dimensions of leading-twist operators in non-forward kinematics, see e.g. [Moch and Van Thurenhout, 2021, Van Thurenhout, 2024]
- Great predictive power: Valuable information about the function space
- Analytic evaluation using principles of symbolic summation!
- Creative telescoping [Zeilberger, 1991]: Evaluate the sum of interest by rewriting it as a recursion relation using Gosper's algorithm [Gosper, 1978]
- The closed-form expression of the sum then corresponds to the linear combination of the solutions of the recursion that has the same initial values as the sum.

→ For single sums: `Sigma` [Schneider, 2004, Schneider, 2007]
→ For multiple sums: `EvaluateMultiSums` [Schneider, 2013, Schneider, 2014]

Renormalization

The complete Lagrangian is now

$$\widetilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_{\mathsf{GF+G}} + w_{\rm i}\,\mathcal{O}_{\rm i} + \mathcal{O}^{(N)}_{\mathsf{EOM}} + \mathcal{O}^{(N)}_{c}$$
$$= \mathcal{L}_0(A^a_\mu, g_s) + \mathcal{L}_{\mathsf{GF+G}}(A^a_\mu, c^a, \bar{c}^a, g_s, \xi) + \sum_{\rm k} \mathcal{C}_{\rm k}\,\mathcal{O}_{\rm k},$$

where $\mathcal{C}_{\rm k}$ labels all the distinct couplings of the operators, $\mathcal{C}_{\rm k} = \{w_{\rm i}, \eta(N), \kappa^{(i)}_{n_1\ldots n_j}, \eta^{(k)}_{n_1\ldots n_j}\}$. The UV singularities associated with the QCD Lagrangian are absorbed by introducing the bare fields/parameters

$$A^{a;\mathrm{bare}}_\mu(x) = \sqrt{Z_3}A^a_\mu(x)$$
$$c^{a;\mathrm{bare}}(x) = \sqrt{Z_c}c^a(x)$$
$$\bar{c}^{a;\mathrm{bare}}(x) = \sqrt{Z_c}\bar{c}^a(x)$$
$$g_s^{\mathrm{bare}} = \mu^\epsilon Z_g g_s$$
$$\xi^{\mathrm{bare}} = \sqrt{Z_3}\xi$$

Renormalization

This is not enough to make the OMEs finite. Instead they need an additional renormalization

$$\mathcal{O}_{\mathrm{i}}^{\mathsf{ren}}(x) = Z_{\mathrm{ij}}\,\mathcal{O}_{\mathrm{j}}^{\mathsf{bare}}(x),$$

The renormalized Lagrangian becomes

$$\begin{aligned}\widetilde{\mathcal{L}} &= \mathcal{L}_0(A_\mu^{a;\mathsf{bare}}, g_s^{\mathsf{bare}}) + \mathcal{L}_{\mathsf{GF+G}}(A_\mu^{a;\mathsf{bare}}, c^{a;\mathsf{bare}}, \bar{c}^{a;\mathsf{bare}}, g_s^{\mathsf{bare}}, \xi^{\mathsf{bare}}) \\ &\quad + \sum_{\mathrm{k}} \mathcal{C}_{\mathrm{k}}^{\mathsf{bare}}\,\mathcal{O}_{\mathrm{k}}^{\mathsf{bare}}, \\ \mathcal{C}_{\mathrm{i}}^{\mathsf{bare}} &= \sum_{\mathrm{k}} \mathcal{C}_{\mathrm{k}}\,Z_{\mathrm{k\,i}},\end{aligned}$$

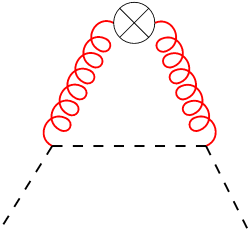
where $\mathcal{C}_{\mathrm{k}}$ is the (finite) renormalized coupling of the operator $\mathcal{O}_{\mathrm{k}}$. The UV-finite OMEs featuring a single insertion of $\mathcal{O}_{\mathrm{g/q}}^{\mathsf{ren}}$ are computed by setting the renormalized couplings $\mathcal{C}_{\mathrm{i}} = \delta_{\mathrm{i\,g/q}}$, which gives

$$\mathcal{C}_{\mathrm{i}}^{\mathsf{bare}} = Z_{\mathrm{g/q\,i}}.$$

Renormalization

⇒ The couplings of the bare operators $\eta^{\text{bare}}(N)$, ... are interpreted as the renormalization constants that mix the physical operators into the aliens

→ Extracted from the direct calculation of the singularities of the OMEs, e.g.



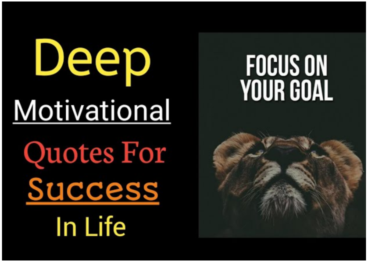
$$\eta^{\text{bare}}(N) = Z_{\mathrm{g}\,c} = -\frac{a_s}{\epsilon}\frac{C_A}{N(N-1)} + O(a_s^2)$$

We note that this quantity is known to $O(a_s^3)$

[Dixon and Taylor, 1974, Hamberg and van Neerven, 1992, Gehrmann et al., 2023]

Goal of the current work

- In [Falcioni and Herzog, 2022, Falcioni et al., 2024b], this setup was used for fixed $N \leq 20$
- The systematic study of the alien operators at arbitrary spin N was left as an open problem
- This is the subject of the present work
- We will solve the constraints on the alien couplings to leading order in g_s but for all values of N



Identities between alien couplings

- The κ couplings in the EOM operators are chosen to inherit the properties of the colour structures they multiply, e.g. $\kappa_{ij} = -\kappa_{ji}$
- Because of gBRST, the η couplings are connected to the κ ones.
- Because of anti-gBRST, there are non-trivial relations between the η-couplings ($\sim$ conjugation relations!)
- These identities allow one to restrict the function space of the couplings and hence constrain their generic N-dependence.
- During this talk: Focus on couplings coming with a string of f's

Class II couplings

$$\mathcal{O}_{\rm EOM}^{(N),II} = g_s \left(D \cdot F^a + g_s \overline{\psi} \not{\Delta} T^a \psi \right) f^{abc} \sum_{\substack{i+j \\ =N-3}} \kappa_{ij} (\partial^i A^b)(\partial^j A^c),$$

$$\mathcal{O}_{c}^{(N),II} = -g_s \, f^{abc} \sum_{\substack{i+j \\ =N-3}} \eta_{ij} (\partial \overline{c}^a)(\partial^i A^b)(\partial^{j+1} c^c)$$

$$\kappa_{ij} + \kappa_{ji} = 0, \qquad \text{[anti-symmetry of } f\text{]}$$

$$\eta_{ij} = 2\kappa_{ij} + \eta(N) \binom{i+j+1}{i}, \qquad \text{[gBRST]}$$

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = 0 \qquad \text{[anti-gBRST]}$$

Class II couplings

$$\kappa_{ij} + \kappa_{ji} = 0, \qquad \text{[anti-symmetry of } f\text{]}$$

$$\eta_{ij} = 2\kappa_{ij} + \eta(N)\binom{i+j+1}{i}, \qquad \text{[gBRST]}$$

$$\eta_{ij} + \sum_{s=0}^{i}(-1)^{s+j}\binom{s+j}{j}\eta_{(i-s)(j+s)} = 0 \qquad \text{[anti-gBRST]}$$

Combining anti-symmetry with gBRST we have

$$\eta_{ij} + \eta_{ji} = \eta(N)\left[\binom{i+j+1}{i} + \binom{i+j+1}{j}\right]$$

which gives an idea about the function space of η_{ij},

$$\eta_{ij} = \eta(N)\left[c_1\binom{i+j+1}{i} + c_2\binom{i+j+1}{j}\right]$$

Class II couplings

Substituting the Ansatz into the conjugation relation gives

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = c_1\, \eta(N) \left[(-1)^j + \binom{i+j+1}{i} \right]$$

for even values of N. Hence, we find a consistent solution if $c_1 = 0$ while c_2 remains unconstrained. Assuming that κ_{ij} lives in the same function space as η_{ij}, the full set of relations fixes both couplings uniquely

$$\eta_{ij} = \eta(N) \binom{N-2}{j},$$
$$\kappa_{ij} = \frac{\eta(N)}{2} \left[\binom{N-2}{j} - \binom{N-2}{i} \right]$$

Check: Compare with some fixed-N computations

→ Correct for $N = 4$

→ Incorrect for $N > 4$

Class II couplings

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = c_1\, \eta(N) \left[(-1)^j + \binom{i+j+1}{i} \right]$$

The RHS however suggests the inclusion of a new structure: $(-1)^j$. With

$$\eta_{ij} = \eta(N) \left[c_1 (-1)^j + c_2 \binom{i+j+1}{i} + c_3 \binom{i+j+1}{j} \right]$$

we find

$$\eta_{ij} + \sum_{s=0}^{i} (-1)^{s+j} \binom{s+j}{j} \eta_{(i-s)(j+s)} = (c_1 + c_2) \eta(N) \left[\binom{i+j+1}{i} + (-1)^j \right]$$

and hence $c_1 = -c_2$.

Class II couplings

Assuming that κ_{ij} lives in the same function space as η_{ij}, the full set of relations fixes both couplings up to 1 free parameter

$$\eta_{ij} = \eta(N)\left\{(1+2c)\left[\binom{i+j+1}{i} - (-1)^j\right] - 2c\binom{i+j+1}{j}\right\}$$

$$\kappa_{ij} = \eta(N)\left\{c\left[\binom{i+j+1}{i} - \binom{i+j+1}{j}\right] - \frac{1}{2}(1+2c)(-1)^j\right\}$$

The unknown c can be determined by the computation of 1 fixed-N matrix element computation. E.g. for $N = 6$ we have $\kappa_{30} = 1/24$ which sets $c = -3/8$

$$\eta_{ij} = -\frac{\eta(N)}{4}\left[(-1)^j - 3\binom{N-2}{i+1} - \binom{N-2}{i}\right]$$

$$\kappa_{ij} = -\frac{\eta(N)}{8}\left[(-1)^j + 3\binom{i+j+1}{i} - 3\binom{i+j+1}{i+1}\right]$$

The solution above exactly agrees with the known solution [Dixon and Taylor, 1974, Hamberg and van Neerven, 1992].

Class III couplings

$$\mathcal{O}_{\text{EOM}}^{(N),III} = g_s^2 \left(D \cdot F^a + g_s \overline{\psi} \not{\Delta} T^a \psi\right) (f\ f)^{abcd} \sum_{\substack{i+j+k \\ =N-4}} \kappa_{ijk}^{(1)} (\partial^i A^b)(\partial^j A^c)(\partial^k A^d),$$

$$\mathcal{O}_{c}^{(N),III} = -g_s^2 \, (f\ f)^{abcd} \sum_{\substack{i+j+k \\ =N-4}} \eta_{ijk}^{(1)} (\partial \overline{c}^a)(\partial^i A^b)(\partial^j A^c)(\partial^{k+1} c^d)$$

$$\kappa_{ijk}^{(1)} + \kappa_{ikj}^{(1)} = 0, \qquad \text{[anti-symmetry of } f\text{]}$$

$$\kappa_{ijk}^{(1)} + \kappa_{jki}^{(1)} + \kappa_{kij}^{(1)} = 0, \qquad \text{[Jacobi identity]}$$

$$\eta_{ijk}^{(1)} = 2\kappa_{i(j+k+1)} \binom{j+k+1}{j} + 2[\kappa_{ijk}^{(1)} + \kappa_{kji}^{(1)}], \qquad \text{[gBRST]}$$

$$\eta_{ijk}^{(1)} = \sum_{m=0}^{i} \sum_{n=0}^{j} \frac{(m+n+k)!}{m!\, n!\, k!} (-1)^{m+n+k} \eta_{(j-n)(i-m)(k+m+n)}^{(1)}. \qquad \text{[anti-gBRST]}$$

Class III couplings

The combination of the Jacobi identity with gBRST leads to

$$\eta^{(1)}_{ijk} + \eta^{(1)}_{kij} + \eta^{(1)}_{jki} = 2\kappa_{i(j+k+1)}\binom{j+k+1}{j} + 2\kappa_{k(i+j+1)}\binom{i+j+1}{i} + 2\kappa_{j(i+k+1)}\binom{i+k+1}{k}.$$

$\rightarrow$ relates the class III coupling $\eta^{(1)}_{ijk}$ to the class II coupling κ_{ij}, at one order lower in perturbation theory!

$\Rightarrow$ use it to determine the function space of the all-N expression of $\eta^{(1)}_{ijk}$

$\rightarrow$ leads to 18-dimensional function space

$$\Big\{(-1)^{i+j}\binom{i+j+1}{i}, \binom{N-2}{k+1}\binom{i+j+1}{i}, \binom{N-2}{k}\binom{i+j+1}{i}, (-1)^{j+k}\binom{j+k+1}{j}, \binom{N-2}{i+1}\binom{j+k+1}{j}, \binom{N-2}{i}\binom{j+k+1}{j}, (-1)^{i+k}\binom{i+k+1}{k}, \binom{N-2}{j+1}\binom{i+k+1}{k}, \binom{N-2}{j}\binom{i+k+1}{k} + \text{ independent permutations of } i, j \text{ and } k\Big\}.$$

Class III couplings

We assume $\kappa_{ijk}^{(1)}$ to live in the same function space. Hence in total we have 36 free parameters. Using the relations described above we are able to fix 34 of these. The final 2 free parameters are then fixed using $\kappa_{110}^{(1)} = 0$ and $\kappa_{121}^{(1)} = 13/336$, which follow from the explicit operator renormalization for $N = 6$ and $N = 8$ respectively. Our final result for $\kappa_{ijk}^{(1)}$ then becomes [new!]

$$\kappa_{ijk}^{(1)} = \frac{\eta(N)}{48}\Bigg\{2(-1)^{i+j}\binom{i+j+1}{i} + (-1)^{i+k}\binom{i+k+1}{k} + 3(-1)^{j+k+1}\binom{j+k+1}{j} + \binom{i+k+1}{i}\Bigg[2(-1)^{i+k+1} + 5\binom{N-1}{j+1}\Bigg] + \binom{j+k+1}{k}\Bigg[3(-1)^{j+k} - 10\binom{N-2}{i} + 4\binom{N-2}{i+1}\Bigg] + \binom{i+j+1}{j}\Bigg[(-1)^{i+j+1} + 5\binom{N-2}{k} - 9\binom{N-2}{k+1}\Bigg]\Bigg\}.$$

Class III couplings

We have checked that the above expression agrees with explicitly computed values, following from the renormalization of the operators, up to $N = 20$. Substituting this expression into the gBRST relation allows one to also reconstruct the full N-dependence of $\eta^{(1)}_{ijk}$ [new!]

$$
\begin{aligned}
\eta^{(1)}_{ijk} = -\frac{\eta(N)}{24}\Bigg\{ & 5(-1)^{i+j+1}\binom{i+j+1}{i} + (-1)^{i+k}\binom{i+k+1}{k} \\
& + 2(-1)^{j+k+1}\binom{j+k+1}{j} + \binom{i+k+1}{i}\left[(-1)^{i+k} + 4\binom{N-2}{j+1}\right] \\
& + \binom{j+k+1}{k}\left[5(-1)^{j+k+1} - 3\binom{N-2}{i} + \binom{N-2}{i+1}\right] \\
& + \binom{i+j+1}{j}\left[4(-1)^{i+j} - 15\binom{N-2}{k} - 5\binom{N-2}{k+1}\right]\Bigg\}.
\end{aligned}
$$

Class IV couplings

$$\mathcal{O}_{\text{EOM}}^{(N),IV} = g_s^3 \left(D \cdot F^a + g_s \overline{\psi} \not{\Delta} T^a \psi\right) (f\ f\ f)^{abcde} \sum_{\substack{i+j+k+l\\=N-5}} \kappa_{ijkl}^{(1)} (\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^l A^e),$$

$$\mathcal{O}_{c}^{(N),IV} = -g_s^3 \, (f\ f\ f)^{abcde} \sum_{\substack{i+j+k+l\\=N-5}} \eta_{ijkl}^{(1)} (\partial \overline{c}^a)(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^{l+1} c^e)$$

$$\kappa_{ijkl}^{(1)} + \kappa_{ijlk}^{(1)} = 0, \qquad \text{[anti-symmetry]}$$

$$\kappa_{ijkl}^{(1)} + \kappa_{iklj}^{(1)} + \kappa_{iljk}^{(1)} = 0, \qquad \text{[Jacobi]}$$

$$\kappa_{ijkl}^{(1)} + \kappa_{jilk}^{(1)} + \kappa_{lkji}^{(1)} + \kappa_{klij}^{(1)} = 0, \qquad \text{[double Jacobi]}$$

$$\eta_{ijkl}^{(1)} = 2[\kappa_{ij(l+k+1)}^{(1)} + \kappa_{(l+k+1)ji}^{(1)}] \binom{l+k+1}{k} + 2[\kappa_{ijkl}^{(1)} + \kappa_{ilkj}^{(1)} + \kappa_{likj}^{(1)} + \kappa_{lkij}^{(1)}], \qquad \text{[gBRST]}$$

$$\eta_{ijkl}^{(1)} = -\sum_{s_1=0}^{i} \sum_{s_2=0}^{j} \sum_{s_3=0}^{k} \frac{(s_1+s_2+s_3+l)!}{s_1!\, s_2!\, s_3!\, l!} (-1)^{s_1+s_2+s_3+l} \eta_{(k-s_3)(j-s_2)(i-s_1)(s_1+s_2+s_3+l)}^{(1)} \qquad \text{[anti-gBRST]}$$

Class IV couplings

Combining the double Jacobi identity with the gBRST one allows one to write $\eta^{(1)}_{ijkl}$ in terms of $\kappa^{(1)}_{ijk}$ appearing already in the class III operators at one order lower in perturbation theory!

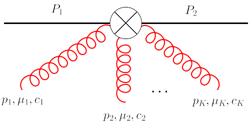
$$\eta^{(1)}_{ijkl} + \eta^{(1)}_{jilk} + \eta^{(1)}_{lkji} + \eta^{(1)}_{klij} = 2[\kappa^{(1)}_{ij(k+l+1)} + \kappa^{(1)}_{(k+l+1)ji}]\binom{k+l+1}{k} + 2[\kappa^{(1)}_{ji(k+l+1)} + \kappa^{(1)}_{(k+l+1)ij}]\binom{k+l+1}{l}$$
$$+ 2[\kappa^{(1)}_{lk(i+j+1)} + \kappa^{(1)}_{(i+j+1)kl}]\binom{i+j+1}{j} + 2[\kappa^{(1)}_{kl(i+j+1)} + \kappa^{(1)}_{(i+j+1)lk}]\binom{i+j+1}{i}.$$

Again this tells us something about the function space for $\eta^{(1)}_{ijkl}$. Taking into account all the independent permutations of the indices i, k, j and l this space is now 264-dimensional. Assuming that the functional form of $\kappa^{(1)}_{ijkl}$ is similar to the one of $\eta^{(1)}_{ijkl}$ then implies that in total we now have 528 parameters to fix. However, after implementing all of the above relations, only 8 remain in the end!

→ Explicit expressions in [Falcioni et al., 2024c]

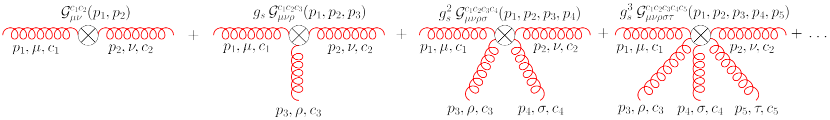
Application: Alien Feynman rules

With the couplings known, one can derive the Feynman rules of the alien operators



- For gauge invariant operators: N^4LO quark rules and N^3LO gluon rules, see e.g. [Falcioni and Herzog, 2022, Gehrmann et al., 2023, Floratos et al., 1977, Floratos et al., 1979, Mertig and van Neerven, 1996, Kumano and Miyama, 1997, Hayashigaki et al., 1997, Bierenbaum et al., 2009, Klein, 2009, Blümlein, 2001, Velizhanin, 2012, Velizhanin, 2020, Moch et al., 2017, Moch et al., 2022, Falcioni et al., 2023b, Falcioni et al., 2023a, Falcioni et al., 2024d, Moch et al., 2024, Gehrmann et al., 2024b, Kniehl and Velizhanin, 2023] and references therein. The generalization to N^KLO including total derivatives can be found in [Somogyi and Van Thurenhout, 2024]
- For aliens: Partial results up to NNLO [Hamberg and van Neerven, 1992],[Matiounine et al., 1998],[Blümlein et al., 2022], [Gehrmann et al., 2023]

Application: Alien Feynman rules



$$
\begin{aligned}
\mathcal{G}^{c_1c_2c_3c_4c_5}_{\mu\nu\rho\sigma\tau}(p_1,p_2,p_3,p_4,p_5) = {}& \frac{1+(-1)^N}{2}\, i^{N-1} f^{c_1c_2x} f^{xc_3y} f^{yc_4c_5} \Bigg\{ \\
& - g_{\mu\rho}\Delta_\nu\Delta_\sigma\Delta_\tau \sum_{i+j=N-3} \kappa_{ij} (\Delta\cdot p_4)^i (\Delta\cdot p_5)^j + \Delta_\rho\Delta_\sigma\Delta_\tau [(p_1+2p_2)_\mu \Delta_\nu \\
& - (\Delta\cdot p_2) g_{\mu\nu}] \sum_{i+j+k=N-4} \kappa^{(1)}_{ijk} (\Delta\cdot p_3)^i (\Delta\cdot p_4)^j (\Delta\cdot p_5)^k + [p_1^2 \Delta_\mu \\
& - p_{1\mu}(\Delta\cdot p_1)]\Delta_\nu\Delta_\rho\Delta_\sigma\Delta_\tau \sum_{i+j+k+l=N-5} \kappa^{(1)}_{ijkl} (\Delta\cdot p_2)^i (\Delta\cdot p_3)^j (\Delta\cdot p_4)^k (\Delta\cdot p_5)^l \Bigg\} \\
& + \frac{1+(-1)^N}{2}\, i^{N-1} d_{4f}^{c_1c_2c_3c_4c_5} \Bigg\{ \Delta_\mu\Delta_\nu\Delta_\rho [(p_4+2p_5)_\sigma \Delta_\tau \\
& - (\Delta\cdot p_5) g_{\sigma\tau}] \sum_{i+j+k=N-4} \kappa^{(2)}_{ijk} (\Delta\cdot p_1)^i (\Delta\cdot p_2)^j (\Delta\cdot p_3)^k + [p_1^2 \Delta_\mu \\
& - p_{1\mu}(\Delta\cdot p_1)]\Delta_\nu\Delta_\rho\Delta_\sigma\Delta_\tau \sum_{i+j+k+l=N-5} \kappa^{(2)}_{ijkl} (\Delta\cdot p_2)^i (\Delta\cdot p_3)^j (\Delta\cdot p_4)^k (\Delta\cdot p_5)^l \Bigg\} \\
& + \text{permutations}
\end{aligned}
$$

Application: Alien Feynman rules

- Ghost vertices:
  (a) Agreement with [Gehrmann et al., 2023] for 0- and 1-gluon vertices and $(f\,f)$, d_4 parts of the 2-gluon vertex
  (b) $d_{\widehat{4ff}}$ part of 2-gluon vertex new!
  (c) 3-gluon vertex new!
- Alien gluon vertices:
  (a) Agreement with [Blümlein et al., 2022, Gehrmann et al., 2023] for 2- and 3-gluon vertices; agreement with [Gehrmann et al., 2023] for $(f\,f)$, d_4 parts of the 4-gluon vertex
  (b) $d_{\widehat{4ff}}$ part of 4-gluon vertex new!
  (c) 5-gluon vertex new! [Recently also obtained in [Gehrmann et al., 2024c], comparison in progress]
- Alien quark vertices:
  (a) Agreement with [Gehrmann et al., 2023] for 0-, 1- and 2-gluon vertices
  (b) 3- and 4-gluon vertices new!

Conclusions and outlook

- One way to reconstruct the functional form of the alien operators is based on the use of generalized gauge symmetry, which is then promoted to a generalized (anti)-BRST symmetry
- One then finds classes of EOM and ghost operators, the couplings of which obey interesting consistency relations
- Bootstrap: Complicated higher-order couplings in terms of simpler lower-order ones
- We used these relations to reconstruct the full N-dependence of the 1-loop alien couplings necessary to perform the operator renormalization to 4 loops
- This should be useful in the reconstruction of the full N-dependence of the 4-loop splitting functions!
- Next steps: Generalization to higher orders

End [2]

Thank you for your attention!

[2]Part of this work has been supported by grant K143451 of the National Research, Development and Innovation Fund in Hungary.

Appendices and references

Colour structures

f^{abc} are the QCD structure constants. The other colour structures are in turn defined as

$$
\begin{aligned}
(f\,f)^{abcd} &= f^{abe} f^{cde}, \\
(f\,f\,f)^{abcde} &= f^{abm} f^{mcn} f^{nde}, \\
d_4^{abcd} &= \frac{1}{4!}[\mathrm{Tr}(T_A^a T_A^b T_A^c T_A^d) + \text{symmetric permutations}], \\
d_{4ff}^{abcd} &= d_4^{abmn} f^{mce} f^{edn}, \\
d_{\widehat{4ff}}^{abcd} &= d_{4ff}^{abcd} - \frac{1}{3} C_A d_4^{abcd}, \\
d_{4f}^{abcde} &= d_4^{abcm} f^{mde}.
\end{aligned}
$$

Solving conjugation relations

- To take full advantage of the anti-gBRST conjugation relations, one needs to be able to evaluate them analytically
- Use principles of symbolic summation!
- Creative telescoping [Zeilberger, 1991]: evaluate the sum of interest by rewriting it as a recursion relation using Gosper's algorithm [Gosper, 1978]
- The closed-form expression of the sum then corresponds to the linear combination of the solutions of the recursion that has the same initial values as the sum.

→ For single sums: `Sigma` [Schneider, 2004, Schneider, 2007]

→ For multiple sums: `EvaluateMultiSums` [Schneider, 2013, Schneider, 2014]

Classical telescoping and Gosper's algorithm

The telescoping algorithm is a well-known method for evaluating finite sums. Suppose we want to evaluate the following sum

$$\sum_{k=a}^{N} f(k)$$

with $a, N \in \mathbb{N}$ and $a \leq N$. Now, if we can find a function $g(N)$ such that

$$f(k) = \Delta g(k) \equiv g(k+1) - g(k)$$

then

$$\begin{aligned}\sum_{k=a}^{N} f(k) &= \sum_{k=a}^{N} g(k+1) - \sum_{k=a}^{N} g(k) \\ &= g(N+1) - g(a).\end{aligned}$$

Here, Δ represents the finite difference operator. The telescoping function $g(N)$ can be found by application of Gosper's algorithm [Gosper, 1978].

Classical telescoping and Gosper's algorithm

Suppose

$$\frac{g(N)}{g(N-1)}$$

is a rational function in N. The algorithm consists of three main steps. Assume we want to calculate the telescoping function for some sequence $\{a_N\}$

$$a_N = \Delta b(N).$$

It is assumed that $\{a_N\}$ is a hypergeometric sequence, that is

$$\frac{a_{N+1}}{a_N} = q(N)$$

with $q(N)$ a rational function of N. The steps of Gosper's algorithm can then be summarized as follows

Classical telescoping and Gosper's algorithm

1. Determine three functions $f(x), g(x)$ and $h(x)$ such that
$$q(x) = \frac{f(x+1)}{f(x)} \frac{g(x)}{h(x+1)}$$
and
$$\gcd[g(x), h(x+n)] = 1 \quad (n \in \mathbb{N}_0).$$
2. Solve the so-called Gosper equation,
$$f(x) = g(x)y(x+1) - h(x)y(x),$$
for the polynomial $y(x)$.
3. If such a polynomial solution does not exist, it means that the sum in question does not have a hypergeometric closed form. Otherwise, the telescoping function is determined by
$$t(x) = \frac{h(x)}{f(x)} y(x) \qquad \text{with } b(N) = t(N)a(N)$$

More details can e.g. be found in [Kauers and Paule, 2011]

Creative telescoping

Classical telescoping works when dealing with sequences that depend on one variable only. When we want to determine a closed form for a summation of a sequence depending on two variables, we can use the creative telescoping algorithm by Zeilberger [Zeilberger, 1991]. The idea is similar to that of classical telescoping. Suppose we want to evaluate

$$\sum_{k=a}^{b} f(N,k) \equiv S(N).$$

The way to go about this is by attempting to find d functions $c_0(N), \ldots, c_d(N)$ and a function $g(N,k)$ such that

$$g(N,k+1) - g(N,k) = c_0(N) f(N,k) + \ldots + c_d(N) f(N+d,k).$$

Summing both sides, and applying classical telescoping to the left-hand side then gives

$$g(N,b+1) - g(N,a) = c_0(N) \sum_{k=a}^{b} f(N,k) + \ldots + c_d(N) \sum_{k=a}^{b} f(N+d,k).$$

Creative telescoping

This leads to an inhomogeneous recursion relation for the original sum of the form

$$q(N) = c_0(N)S(N) + ... + c_d(N)S(N+d).$$

Typically, one starts this procedure at $d = 0$, which is equivalent to classical telescoping. The value of d is then increased stepwise until a solution is found. The creative telescoping algorithm can be applied when the sequence under consideration is holonomic. A sequence $\{a_N\}$ is said to be holonomic if there exist polynomials $p_0(x), \ldots, p_r(x)$ such that the following recursion relation is obeyed [Kauers and Paule, 2011]

$$p_0(N)a_N + p_1(N)a_{N+1} + \cdots + p_r(N)a_{N+r} = 0 \quad (N \in \mathbb{N}, p_r(N) \neq 0).$$

For example, the harmonic numbers $\{S_1(N)\}$ form a holonomic sequence as they obey

$$(N+1)S_1(N) - (2N+3)S_1(N+1) + (N+2)S_1(N+2) = 0.$$

More details on the summation algorithms reviewed here can e.g. be found in the excellent books [Graham et al., 1989, Petkovŝek et al., 1996].

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