Primordial Magnetic Fields

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Introduction

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  - Amplification of weak "seed magnetic fields" by the rotating galactic plasma
  - But what is the **origin** of this seed field?

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  - ▷ But what is the **origin** of this seed field?
- Evidence for weak galactic fields outside of galaxies (in voids)
  - ▷ Observed in TeV blazar data [Neronov et al., 2010]
  - ▷ Diffuse image caused by magnetic fields between the blazar and observer
  - ▷ Result: $B_{\text{void}} \sim \mathcal{O}(10^{-16})$ G at $\lambda_{\text{void}} \sim 1$ Mpc correlation length
  - ▷ Compare: $\lambda_{\text{void}} \sim 10^{21}$ m $\longrightarrow$ Weak field but enormous volume

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- Weak fields with large scale homogeneity $\to$ Origin?
  1. Collective magnetic effects of many galaxies surrounding voids
  2. From topological defects in the electroweak theory [Vachaspati, 1991]

Magnetic fields from galaxies

Magnetic fields from galaxies

- Galactic magnetic fields → external field strength $B|_{r=7\,\mathrm{kpc}} \approx 10^{-6}\,\mathrm{G}$ [Unger et al., 2024]
- Voids → regions in space with much fewer than expected number of galaxies
  - Size: $r_{\mathrm{void}} \sim \mathcal{O}(10\,\mathrm{Mpc})$
  - Galaxy density outside of voids: $d_{\mathrm{inter-galactic}} \approx 1\,\mathrm{Mpc}$

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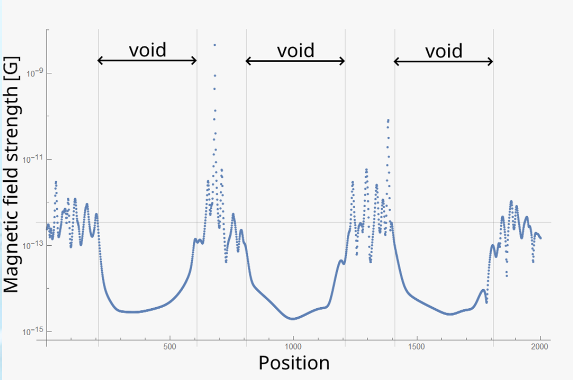
  - ▷ Size: $r_{\mathrm{void}} \sim \mathcal{O}(10\,\mathrm{Mpc})$
  - ▷ Galaxy density outside of voids: $d_{\mathrm{inter-galactic}} \approx 1\,\mathrm{Mpc}$
- Naive calculation [Garg et al., 2025]:
  1. Galactic **B** fields modeled as dipoles
  2. Voids are completely empty
  3. Only a fraction f of all galaxies contribute to the void fields (fudge factor)
  4. Galaxies are dispersed on the surface of a spherical void

$$\langle B_{\mathrm{void}} \rangle \simeq \underbrace{B|_{r_{\mathrm{g}}} \left(\frac{r_{\mathrm{g}}}{r_{\mathrm{void}}}\right)^3}_{\text{contribution of 1 galaxy}} \times \underbrace{\sqrt{\frac{4\pi r_{\mathrm{void}}^2 f}{d_{\mathrm{inter-galactic}}^2}}}_{\sqrt{\text{nr. of galaxies}}} \approx 10^{-15}\,\mathrm{G}\,, \quad \text{if} \quad f = 10\%$$

Naive numerical example

- Can be more realistic:
  1. Uniformly random distribution of galaxies outside the void
  2. Uniformly random distribution of dipole directions
  3. Gaussian distribution of dipole strengths



Where the naive estimate falls short...

- Voids are not *completely* empty!
  - They are filled with a (dilute) plasma
  - Plasma screens the magnetic fields effectively

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- Reality is more complicated due to galactic outflows but it does not change the qualitative result [Brandenburg et al. 2026]
- Plasma screening with negligible flow: [Jedamzik and Sigl, 2011]

$$\partial_z \mathbf{B}_{\mathbf{k}}(z) = \frac{2\mathbf{B}_{\mathbf{k}}(z)}{1+z} + \eta(z)\frac{1+z}{H(z)}|\mathbf{k}|^2 \left[\mathbf{B}_{\mathbf{k}}(z) - 4\pi \mathbf{M}_{\mathbf{k}}^{\perp}(z)\right]$$

  - z: redshift (takes the place of time t)
  - $\eta(z)$: plasma resistivity
  - $H(z)$: Hubble expansion rate
  - $\mathbf{M}_{\mathbf{k}}^{\perp}(z) = \mathbf{M}_{\mathbf{k}} - (\hat{\mathbf{k}} \cdot \mathbf{M}_{\mathbf{k}})\hat{\mathbf{k}}$: orthogonal part of the magnetization vector

Plasma screening

- Solution to the screening equation under mild assumptions: [Seller and Sigl, 2026]
  1. Temperature and redshift are simply related: $T(z) = T_0(1+z)^{n_T}$ ($n_T^{\mathrm{R}} = 1$ or $n_T^{\mathrm{NR}} = 2$)
  2. Resistivity depends on plasma temperature as: $\eta(T) \propto T^{-n_\eta}$ (e.g., $n_\eta^{\mathrm{Spitzer}} = 3/2$)
  3. Single component dominated Universe: $H(z) = H_0(1+z)^{n_H}$ (e.g., $n_H^{\mathrm{matter}} = 3/2$)
  4. Magnetization depends on redshift as: $\mathbf{M}_{\mathbf{k}}^{\perp}(z) \simeq (1+z)^2 \mathbf{M}_{\mathbf{k}}^{\perp}(0)$

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  4. Magnetization depends on redshift as: $\mathbf{M}_{\mathbf{k}}^{\perp}(z) \simeq (1+z)^2 \mathbf{M}_{\mathbf{k}}^{\perp}(0)$
- Analytic solution today ($z = 0$):

$$\mathbf{B}_{\mathbf{k}}(0) = 4\pi \left[1 - \exp\left(-\frac{k^2}{k_{r,0}^2}\right)\right] \mathbf{M}_{\mathbf{k}}^{\perp}(0)\,, \quad \text{where } k_r(z) = \underbrace{\left[\int_z^\infty \mathrm{d}\tilde{z}\, \frac{\eta(\tilde{z})(1+z)}{H(z)}\right]^{-1/2}}_{(\text{Effective screening length})^{-1}}$$

Plasma screening

$$\text{Close to the source:} \quad \lim_{k \gg k_{r,0}} \mathbf{B}_{\mathbf{k}}(0) \simeq 4\pi \mathbf{M}_{\mathbf{k}}^{\perp}(0)$$

$$\text{Far away from the source:} \quad \lim_{k \ll k_{r,0}} \mathbf{B}_{\mathbf{k}}(0) \simeq 4\pi \frac{k^2}{k_{r,0}^2} \mathbf{M}_{\mathbf{k}}^{\perp}(0)$$

- Given that $k_{r,0}^{-1} \ll r_{\text{void}}$ by many orders of magnitude
  $\rightarrow$ Galactic contribution to the large-scale void fields is negligible

Returning to real space

- To reconnect with the naive estimate we can FT $\mathbf{B}_{\mathbf{k}}$ with assuming a dipole source:

$$\mathbf{B}_{\text{dipole}}(\mathbf{u}) = -k_{r,0}^3 \frac{\mathbf{m}|\mathbf{u}|^2\mathcal{F}_{\text{s}}(|\mathbf{u}|;2) - 3\mathbf{u}(\mathbf{m}\cdot\mathbf{u})\mathcal{F}_{\text{s}}(|\mathbf{u}|;6)}{|\mathbf{u}|^5},$$

where $\mathbf{u} = k_{r,0}\mathbf{r}$ and the screening function is:

$$\mathcal{F}_{\text{s}}(u;n) = 1 - \text{erf}\left(\frac{u}{2}\right) + \frac{u}{\sqrt{\pi}}\exp\left(-\frac{u^2}{4}\right)\left(1+\frac{u^2}{n}\right)$$

- Estimate for the void fields (screening length $l_{r,0} = 2\pi/k_{r,0}$):

$$\mathbf{B}_{\text{void}} \approx (10^{-5}\,\text{G}) \times \text{e}^{-r_{\text{void}}^2/4l_{r,0}^2} \quad \rightarrow \quad \text{would need } l_{r,0} \sim 2\,\text{Mpc (unrealistic)}$$

Primordial Magnetic Fields

Topology of the electroweak model

- Topological defects are stable, localized field configurations that arise from nontrivial topology of the vacuum manifold $\mathcal{M}$
  - ▷ Classification of defects based on homotopy [equivalence classes based on continuous deformations]
  - ▷ $\pi_n(\mathcal{M}) = \{\text{homotopy classes of (based) maps } S^n \to \mathcal{M}\} \leftarrow$ Homotopy group
  - ▷ Non-trivial topology = one or more of $\pi_n(\mathcal{M}) \neq \mathbb{I}$ (in physics: $n = 0, 1, 2$)
    - ▷ $\pi_0(\mathcal{M}) = \mathbb{Z}_2 \quad \to \quad$ domain walls
    - ▷ $\pi_1(\mathcal{M}) = \mathbb{Z} \quad \to \quad$ topological strings (cosmic strings)
    - ▷ $\pi_2(\mathcal{M}) = \mathbb{Z} \quad \to \quad$ monopoles
  - ▷ In general $\pi_n(S^m) = \mathbb{I}$ for $m > n$
- Electroweak model has a vacuum manifold that is $\mathcal{M}_{\text{EW}} \cong S^3 \;\to\; \pi_{n\leq 2}(S^3) = \mathbb{I}$ $\longrightarrow$ No defects!

$$\mathcal{M}_{\text{EW}} : \; |\hat{\Phi}|^2 = 1 \quad \leftrightarrow \quad \sum_{i=1}^{4} \hat{\phi}_i^2 = 1 \quad \to \quad \mathcal{M}_{\text{EW}} \cong S^3$$

Electroweak dumbbells

- Dumbbell = a U(1) string with SU(2) monopoles attached to both ends

  [$\pi_1(S^1) = \mathbb{Z}$ and given SU(2)→U(1) symmetry breaking pattern $\pi_2[SU(2)/U(1)] = \pi_1[U(1)] = \pi_1(S^1) = \mathbb{Z}$]
- The EW model technically admits such solutions, although they will not be stable

  [Nambu, 1977]
- Modern view [Hindmarsh, Vachaspati, 't Hooft...]:
  - ▷ "Semilocal" limit of EW model: set either $g \to 0$ or $g' \to 0$
  - ▷ Gauge orbits will be S^1 or S^2 → right topology for strings or monopoles
  - ▷ EW model has $g > g' > 0$ but mathematically a fibered structure exists:

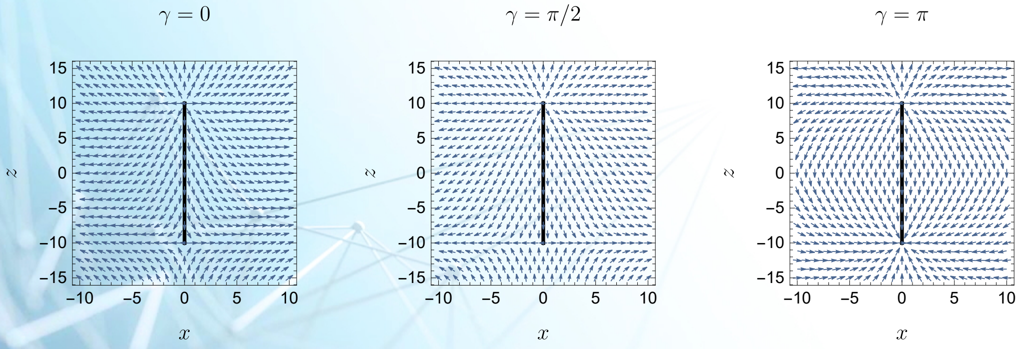
    → Hopf fibration: $S^3 \to S^2 \times S^1$ [map from S^3 to S^2 with S^1 fibers]
- Non-trivial structure of S^3 fibration → unstable dumbbell configurations

  [Patel and Vachaspati, 2023–2024]

Electroweak dumbbells visualized

- Phase plot for a monopole-antimonopole pair connected by a (twisted) string

  [twist denoted as γ, black line indicates where the phase is undefined ($\phi = 0$) $\leftrightarrow$ location of defects]



Primordial Magnetic Fields [Nambu, 1977]

$$\mathcal{L}_{\text{SM}} = |D_\mu \Phi|^2 + V(\Phi) + \dots$$

$$\Phi_{\min} : \quad D_\mu \Phi_{\min} \equiv \left(\partial_\mu - \frac{\mathrm{i}g}{2}\tau^a A^a_\mu - \frac{\mathrm{i}g'}{2} A^0_\mu\right)\Phi_{\min} = 0 \quad \text{and} \quad V(\Phi_{\min}) = 0\,.$$

$$g' A^0_\mu = -\mathrm{i}\sin^2\theta_{\mathrm{w}}\left(\Phi^\dagger \cdot \partial_\mu \Phi - \partial_\mu \Phi^\dagger \cdot \Phi\right)$$

$$g A^a_\mu = -\mathrm{i}\cos^2\theta_{\mathrm{w}} n^a \left(\Phi^\dagger \cdot \partial_\mu \Phi - \partial_\mu \Phi^\dagger \cdot \Phi\right) - \epsilon^{abc} n^b \partial_\mu n^c \qquad \left(n^a = \Phi^\dagger \cdot \tau^a \cdot \Phi\right)$$

$$\mathcal{F}_{\mu\nu} = \cos\theta_{\mathrm{w}} F^0_{\mu\nu} - \sin\theta_{\mathrm{w}} F^a_{\mu\nu} n^a = -\frac{2\mathrm{i}}{g}\sin\theta_{\mathrm{w}}\left(\partial_\mu \Phi^\dagger \partial_\nu \Phi - \partial_\nu \Phi^\dagger \partial_\mu \Phi\right).$$

$$\mathbf{B} = -\mathrm{i}\frac{2\sin\theta_{\mathrm{w}}}{g}\left(\boldsymbol{\nabla}\Phi^\dagger \times \boldsymbol{\nabla}\Phi\right) \quad \text{and} \quad \mathbf{A} = -\mathrm{i}\frac{2\sin\theta_{\mathrm{w}}}{g}\Phi^\dagger \boldsymbol{\nabla}\Phi$$

Primordial Magnetic Fields II

- Sufficiently far away from defects $\rightarrow \hat{\Phi} \in \mathcal{M}_{\mathrm{EW}} \leftrightarrow |\hat{\Phi}| = 1$
  - ▷ Parametrize $\hat{\Phi}$ on S^3 with Hopf angles α, β, γ:

$$\hat{\Phi} = \begin{pmatrix} \cos\alpha \exp(\mathrm{i}\beta) \\ \sin\alpha \exp(\mathrm{i}\gamma) \end{pmatrix} , \quad \text{where} \quad \beta, \gamma \in [0, 2\pi] \quad \text{and} \quad 2u \equiv \cos(2\alpha) \in [-1, 1]$$

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  - ▷ Recall: Hopf fibration

$$\pi_{\mathrm{Hopf}}[\Phi] \equiv \chi = (|\Phi_1|^2 - |\Phi_2|^2 \,,\, 2\Phi_1\Phi_2^*)^T \in \mathbb{R} \times \mathbb{C} \;\longrightarrow\; |\chi|^2 = 1 \to \chi \in S^2$$

$$\hat{\Phi} = \exp(\mathrm{i}\gamma) \begin{pmatrix} \cos(\alpha)\exp(\mathrm{i}\varphi) \\ \sin(\alpha) \end{pmatrix} \longrightarrow \pi_{\mathrm{Hopf}}[\hat{\Phi}] = \begin{pmatrix} \cos(2\alpha) \\ \sin(2\alpha)\exp(i\varphi) \end{pmatrix} \equiv \frac{1}{\sqrt{2}r^{1/4}} \begin{pmatrix} \sqrt{r+z} \\ \frac{(x+iy)}{\sqrt{r+z}} \end{pmatrix}$$

Singular at $z = -r$ (negative z axis) $\to$ string defect

Magnetic fields from EWPT

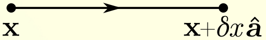
Lattice Setup I.

- Idea based on [Vachaspati et al., 2025]
- Cubic lattice of size N^3 with lattice spacing δx and $L = N\delta x = V^{1/3}$
- Choose random Hopf-angles α, β, γ at each lattice site:

  Random realization of the scalar field: $\hat{\Phi}(\mathbf{x}_{ijk})$, where $\mathbf{x}_{ijk} = i\hat{\mathbf{x}} + j\hat{\mathbf{y}} + k\hat{\mathbf{z}}$

- Let $2\sin\theta_{\mathrm{w}}/g \to 1$ so that $\mathbf{A} = -\mathrm{i}\hat{\Phi}^\dagger \boldsymbol{\nabla} \hat{\Phi}$
- Define the vector potential through the **link variable**:

$$T_a(\mathbf{x}) \equiv \mathbf{A}(\mathbf{x}) \cdot (\delta x\, \hat{\mathbf{a}}) \simeq \mathfrak{Im}\left[\hat{\Phi}^\dagger(\mathbf{x}) \cdot \hat{\Phi}(\mathbf{x} + \delta x\, \hat{\mathbf{a}})\right] \equiv$$



- Important step: **Periodic boundary condition** when defining link variables!

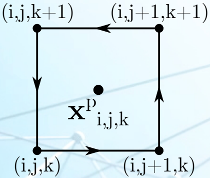
Lattice Setup II.

- Magnetic field defined via the loop integral:

$$\mathbf{B}(\mathbf{x}^p)\delta x^2 = \hat{\mathbf{p}} \oint_{\partial p} \mathrm{d}\mathbf{l} \cdot \mathbf{A}(\mathbf{x})$$

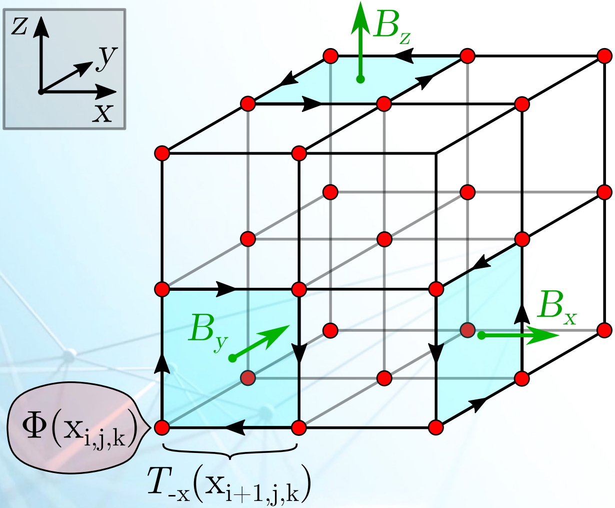
- On the lattice this reduces to a sum of 4 link variables, *e.g.*:

$$B_x(\mathbf{x}^p_{i,j,k})\delta x^2 =$$

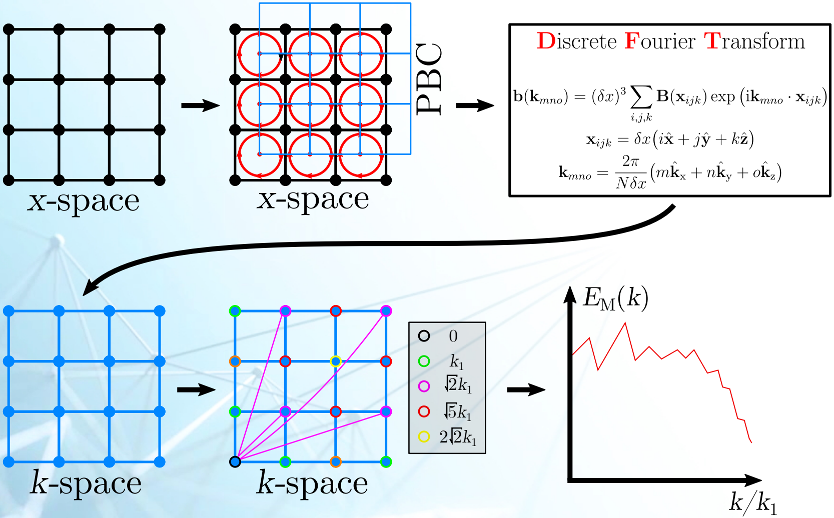


$$= T_y(\mathbf{x}_{i,j,k}) + T_z(\mathbf{x}_{i,j+1,k}) + T_{-y}(\mathbf{x}_{i,j+1,k+1}) + T_{-z}(\mathbf{x}_{i,j,k+1})$$

Visualizing the Lattice



Basic Simulation Approach



An analytic approach for the spectra

"Simulation on Paper"

- Basic simulation approach: Slow for large lattice size N and very noisy for small k (large distances)
  - Personal problem: cannot run on a laptop due to limited RAM for $N \gtrsim 256$

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- **We can do much better!** [Seller and Sigl, 2026]
  - ▷ We can exploit the randomness of $\Phi(\mathbf{x}_{ijk})$ and the law of large numbers (nr. of lattice points $\ggg 1$)
  - ▷ We can also use general results in the mathematics of Fourier transform

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- Approach:
  1. Find the spatial correlation function → $\xi(\mathbf{r})$
  2. Average over polar angles to get the isotropic correlation function → $\xi_{\text{iso}}(r)$
  3. Use relationship between power spectrum and correlation function → $E_{\text{M}}(k)$

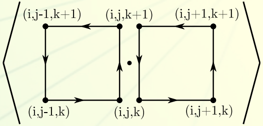
Analytic Approach for the Spectrum

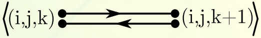
- Approach: [Seller and Sigl, 2026]
  - ▷ Evaluate the correlation function $\xi(\mathbf{r})$ for the lattice:

$$\xi(\mathbf{r}) = \frac{1}{N^3} \sum_{i=1}^{3} \sum_{\mathbf{x}} B_i(\mathbf{x}) B_i(\mathbf{x}+\mathbf{r}) \simeq \sum_{i=1}^{3} \langle B_i(\mathbf{x}) B_i(\mathbf{x}+\mathbf{r}) \rangle_{\mathbf{x}}$$

  - ▷ Using $\langle T_a(\mathbf{x}) T_b(\mathbf{y}) \rangle = 0$ unless $T_a(\mathbf{x})$ and $T_b(\mathbf{y})$ are the “same” link:

$$\langle B_x(\mathbf{x}^p_{i,j-1,k}) B_x(\mathbf{x}^p_{i,j,k}) \rangle =$$





$$= \langle T_z(\mathbf{x}_{i,j,k}) T_{-z}(\mathbf{x}_{i,j,k+1}) \rangle \equiv -\langle [T_z(\mathbf{x}_{i,j,k})]^2 \rangle$$

Analytic Approach – The Correlation Function

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  - ▷ Obviously, correlations vanish for $r > \delta x$ as there will be no shared links:

$$\xi(\mathbf{r}) = \frac{1}{\delta x^4} \sum_{i=1}^{3} \left(\delta_{\mathbf{r},\mathbf{0}} - \frac{1}{4} \sum_{j=1}^{3} \delta_{\mathbf{r},\pm\delta x \hat{\mathbf{e}}_j} (1 - \delta_{ij}) \right)$$

Discrete approach I.

- Isotropic limit on the lattice:
  - ▷ Introduce the set of lattice points that are at a given distance:

$$\mathcal{P}(r) = \left\{ \mathbf{r} \,\middle|\, |\mathbf{r}| = r;\, \mathbf{r} \in \text{lattice} \right\} \qquad \rightarrow \qquad |\mathcal{P}(0)| = 1\,, \;\; |\mathcal{P}(\delta x)| = 6$$

  - ▷ We average over the direction-dependence of $\xi(\mathbf{r})$:

$$\boxed{\xi_{\text{iso}}(r) = \frac{3}{\delta x^4 |\mathcal{P}(r)|}\left(\delta_{r,0} - \delta_{r,\delta x}\right) = \frac{3}{\delta x^4}\left(\delta_{r,0} - \frac{1}{6}\delta_{r,\delta x}\right)}$$

  - ▷ The power spectrum is the DFT of the correlation function:

$$\langle |\mathbf{b}(\mathbf{k})|^2 \rangle = V\delta x^3 \sum_{\mathbf{r}} \xi(\mathbf{r}) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) = \delta x^2 N^3 \sum_{i=1}^{3} \left[1 - \cos\left(k_i \delta x\right)\right]$$

Discrete approach I.

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  - ▷ We neglect the direction-dependence of the 2nd term in $\xi(\mathbf{r})$:

$$\xi_{\text{iso}}(r) = \frac{3}{\delta x^4 |\mathcal{P}(r)|}\left(\delta_{r,0} - \delta_{r,\delta x}\right) = \frac{3}{\delta x^4}\left(\delta_{r,0} - \frac{1}{6}\delta_{r,\delta x}\right)$$

  - ▷ The power spectrum is the DFT of the correlation function:

$$\langle |\mathbf{b}(\mathbf{k})|^2 \rangle = \delta x^2 N^3 \sum_{i=1}^{3}\left[1 - \left(1 - \frac{k_i^2 L^2}{2N^2} + \mathcal{O}(k_i^4 L^4)\right)\right] = \frac{N}{2}\delta x^2 (kL)^2 \left[1 + \mathcal{O}\left((kL)^2\right)\right]$$

Discrete Approach II.

- The energy spectrum is obtained from the power spectrum:

$$8\pi^2 \frac{E_\mathrm{M}(k)}{k^4} = \delta x \left[1 + \mathcal{O}\big((kL)^2\big)\right] \qquad \rightarrow \qquad E_\mathrm{M}(k) \propto k^4$$

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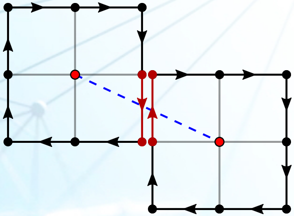
- More detailed spectrum → generalized coarse-grain algorithms
  - ▷ Effectively: averaging over $n \times n$ sized plaquettes
  - ▷ Correlations will exist up to larger distances for n^2-plaquette averaged fields, *e.g.*:



Discrete Approach II.

- The energy spectrum is obtained from the power spectrum:

$$8\pi^2 \frac{E_{\mathrm{M}}(k)}{k^4} = \delta x \left[1 + \mathcal{O}\left((kL)^2\right)\right] \quad \rightarrow \quad E_{\mathrm{M}}(k) \propto k^4$$

- More detailed spectrum → generalized coarse-grain algorithms
  - ▷ Effectively: averaging over $n \times n$ sized plaquettes
  - ▷ Correlations will exist up to larger distances for n^2-plaquette averaged fields
  - ▷ More complicated correlation function and power spectrum...

$$\xi_{\mathrm{iso}}^{(4\mathrm{PA})}(r) = \frac{2 \cdot 3}{(2\delta x)^4 |\mathcal{P}(r)|} \left[\delta_{r,0} + \delta_{r,\delta x} - \delta_{r,2\delta x} - \delta_{r,\sqrt{5}\delta x}\right]$$

  - ▷ Scaling of $E_{\mathrm{M}}(k)$ remains the same for small k

Continuous Approach

- Starting point is the same $\xi(\mathbf{r})$ lattice correlation function:

$$\xi_{\text{iso}}(r) = \frac{3}{\delta x^4 |\mathcal{P}(r)|} \left(\delta_{r,0} - \delta_{r,\delta x}\right)$$

  - ▷ Lattice: $|\mathcal{P}(r)|$ counts nr. of neighbors
  - ▷ Lattice: $\delta_{r,a}$ Kronecker-delta

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  - ▷ Lattice: $\delta_{r,a}$ Kronecker-delta → Continuum: $\delta_{r,a} \rightarrow \delta x\, \delta(r - a)$

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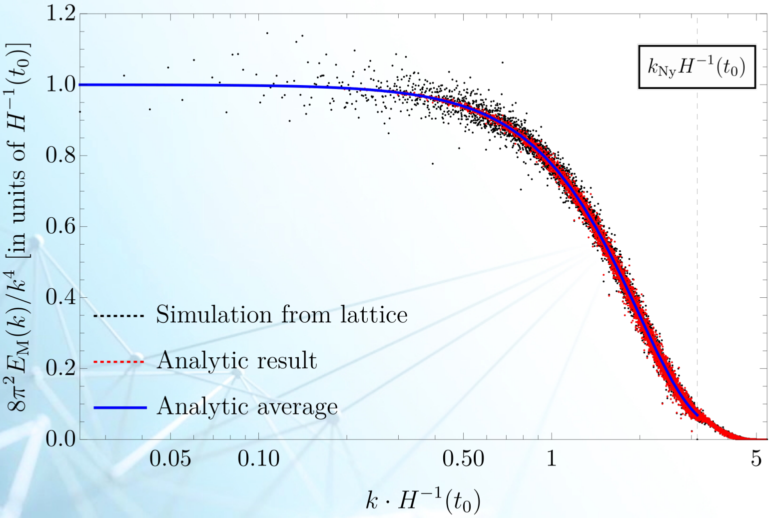
- The energy spectrum is the Hänkel transform of $\xi^{(\text{c})}_{\text{iso}}(r)$:

$$E_{\text{M}}(k) = \frac{k^2}{\pi}\int_0^\infty \mathrm{d}r\, r^2 \xi^{(\text{c})}_{\text{iso}}(r) j_0(kr)$$

$$\boxed{8\pi^2 \frac{E_{\text{M}}(k)}{k^4} = \frac{6}{k^2 \delta x}\left(1 - j_0(k\delta x)\right) \quad \rightarrow \quad E_{\text{M}}(k) \propto k^4}$$

Simulation vs Analytic solution



Continuous fields and spectra

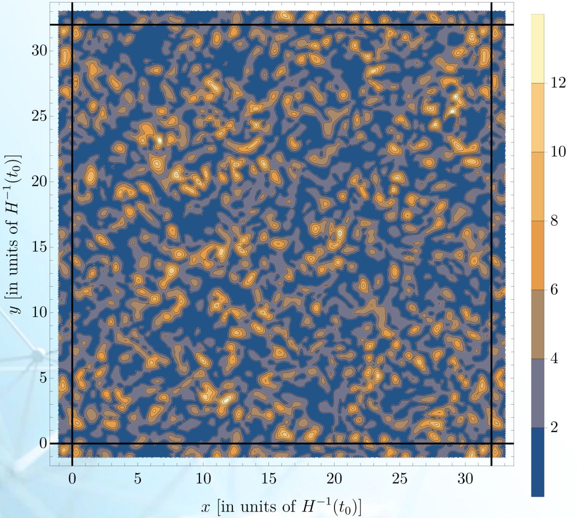
Going further: Interpolated Fields

- There remain some issues with the above lattice approach:
  - Crudeness of derivatives (link variables)
  - Insensitivity for scales below the lattice spacing
  - Regularity of the cubic lattice

Going further: Interpolated Fields

- There remain some issues with the above lattice approach:
  - Crudeness of derivatives (link variables)
  - Insensitivity for scales below the lattice spacing
  - Regularity of the cubic lattice
- We attempt to solve all of these with the following changes:
  - Irregular lattice setup
  - Continuous interpolation of the lattice field
    $\rightarrow$ Using "radial basis function (RBF)" interpolation due to irregularity
  - Sampling of the continuous field on a dense grid $\rightarrow$ DFT as before...
    $\rightarrow$ Technically, direct FT is possible but very slow

Example Slice of the Continuous Magnetic Field



Causal correlation functions

- Due to the interpolation $\longrightarrow$ nearby lattice sites are not independent
- Need "reasonable" analytic formulae to fit for the correlation function and the power spectrum

- Important detail: **Causal correlation functions** [Durrer et al., 2003]
  - ▷ Mathematics: Provided that $\xi(r) = 0$ for $r > \ell$ then $E_\mathrm{M}(k) \propto k^4$ for small k (actually, exponentially vanishing is enough)
  - ▷ Our above analytic calculation confirmed this explicitly
  - ▷ If $\xi(r)$ is exponentially small for large r then so is $E_\mathrm{M}(k)$ for large k
  - ▷ Empirical formula for the k^4 energy spectrum with an exponential cutoff:

$$\mathcal{E}(k; \{b_a\}, \sigma) \propto k^4 \left(1 + \sum_{a=1}^{n/2} b_a k^{2a}\right) \exp\left(-\frac{2k^2}{\sigma^2}\right)$$

Admissible Correlation Functions

- What specific form should a causal correlation function take to give $\mathcal{E}(k) \propto k^4$?

$$\mathcal{C}(r) = \sum_{n=0}^{n_{\max}} p_n r^n \exp(-ar^2)\,, \quad \text{with } p_{\text{odd}} = 0$$

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- Energy spectrum → Hänkel transform of $\mathcal{C}(r)$: [Fubini's theorem]

$$\mathcal{E}(k) = \frac{1}{\pi} \sum_{n=0}^{n_{\max}} p_n k^{-(n+1)} \int_0^\infty \mathrm{d}q\, q^{n+2} \exp(-aq^2/k^2) j_0(q)$$

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[${}_1\mathrm{F}_1(a;b;z)$ is the confluent hypergeometric function of the 1st kind]

$$\mathcal{E}(k) = \frac{1}{2\pi} \sum_{n=0}^{n_{\max}} p_n \frac{k^2}{a^{m_n}} \Gamma(m_n)\, {}_1\mathrm{F}_1\left(m_n; m_0; -\frac{k^2}{4a}\right), \quad \text{where } m_n = \frac{3+n}{2}$$

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$$\mathcal{E}(k) = \frac{1}{2\pi} \sum_{n=0}^{n_{\max}} p_n \frac{\Gamma(m_n)}{a^{m_n}} \sum_{j=0}^{n/2} \frac{(-1)^j}{(2j+1)!!} \frac{k^{2(j+1)}}{(2a)^j} \binom{n/2}{j} \exp\left(-\frac{k^2}{4a}\right)$$

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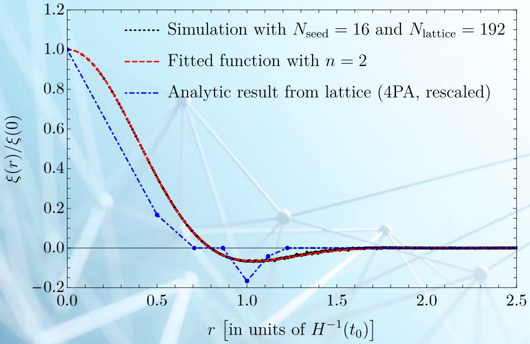
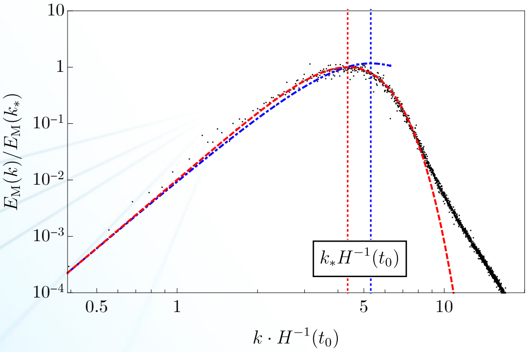
- The k^2 term ↔ the $j = 0$ part of the expression has to cancel:

$$\boxed{\sum_{n=0}^{n_{\max}} p_n a^{-n/2} \Gamma\left(\frac{3+n}{2}\right) = 0}$$

Spectrum of the Primordial Magnetic Field

- The spatial correlation function and the energy spectrum (normalized to the spectral peak at k_*):





Conclusions and Outlook

- **Conclusions:**
  - We investigated the spectrum of the magnetic field that could be produced around the time of the EWPT
  - We showed that the spectrum may be directly derived without the need of numerical simulations
  - We confirmed that the spectrum is "causal" and that it scales as $E_{\mathrm{M}} \propto k^4$
  - We derived a general formula for the isotropic spectra
  - We showed that the general formula fits the results of the "continuous" simulations
- **Outlook:**
  - The spectrum is evolved using MHD equations to compare to the magnetic field strength of $\mathcal{O}(10^{-16})$ G at $\lambda = 1$ Mpc scales
  - This simulation does not generate helical fields thus the obtained fields today are too weak
  - Introduce parity violation initially and get a non-zero helicity spectrum