Loop calculations with graphical functions

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Budapest, Nov. 26, 2024

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Setup

The graphical functions method works for

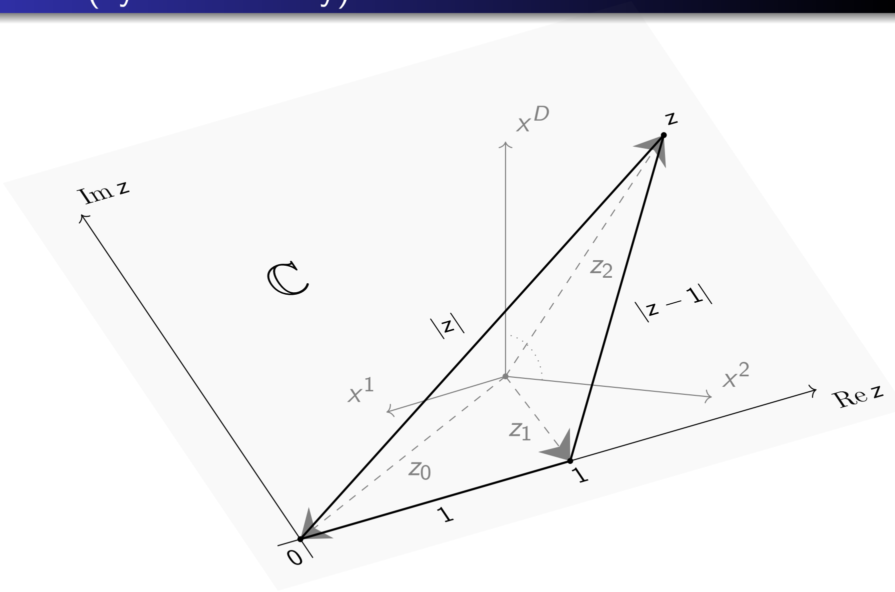
- massless,
- 2pt, 3pt, or convergent (conformal) 4pt amplitudes
- in even dimensions ≥ 4.

In this setup, high loop orders are possible.
Ideal playground: renormalization functions $\beta(g)$, $\gamma(g)$, $\gamma_m(g)$.

Idea

- Massless 2pt amplitudes are scalars (periods). Add a third point for more structure.
- Massless 3pt integrals (or 4pt conformal) are the simplest functions in QFT (two-scale).
- Construct a given Feynman integral by an increasing sequence of 3pt subgraphs.
- Use position space. Three points span a plane in $\mathbb{R}^D$. Consider this plane as $\mathbb{C}$.
- Study the 3pt integrals as functions on $\mathbb{C}$ using the theory of complex functions.
- Add edges by solving the Laplace equation.

Picture (by M. Borinsky)



Definition

Consider a Feynman graph G with three external vertices $z_0, z_1, z_2 \in \mathbb{R}^D$ and V_G^{int} internal vertices $x_1, \ldots, x_{V_G^{\mathrm{int}}} \in \mathbb{R}^D$. In D-dimensional position space an edge $e = y_1 y_2$ in G has the propagator

$$p_e(y_1, y_2) = \frac{1}{||y_1 - y_2||^{D-2}}.$$

The vertices y_1 and y_2 can be internal or external. We generalize the propagator by allowing edge weights $\nu_e \in \mathbb{R}$,

$$p_{e,\nu_e}(y_1, y_2) = \frac{1}{||y_1 - y_2||^{2\lambda\nu_e}},$$

where $\lambda = D/2 - 1$. The Feynman integral of the graph G is

$$A_G(z_0, z_1, z_2) = \int \frac{\mathrm{d}x_1}{\pi^{D/2}} \cdots \int \frac{\mathrm{d}x_{V_G^{\mathrm{int}}}}{\pi^{D/2}} \prod_{e \in E_G} p_{e,\nu_e}(x, z).$$

Definition

The graphical functions $f_G(z)$ is defined by

$$f_G(z) = A_G(z_0, z_1, z_2)$$

for the external vectors

$$z_0 = 0,\ z_1 = (1, 0, 0, \dots, 0)^T,\ z_2 = (\mathrm{Re}\, z, \mathrm{Im}\, z, 0, \dots, 0)^T.$$

For general z_0, z_1, z_2 one has the relation

$$A_G(z_0, z_1, z_2) = ||z_1 - z_0||^{-2\lambda N_G} f_G(z),$$

with invariants

$$\frac{||z_2 - z_0||^2}{||z_1 - z_0||^2} = z\overline{z}, \quad \frac{||z_2 - z_1||^2}{||z_1 - z_0||^2} = (z-1)(\overline{z}-1),$$

and the scaling weight (superficial degree of divergence)

$$N_G = \Big(\sum_{e \in E_G} \nu_e \Big) - \frac{(\lambda + 1) V_G^{\mathrm{int}}}{\lambda}.$$

General properties

- Reflection symmetry $f_G(z) = f_G(\overline{z})$.
- f_G is a real-analytic single-valued function on $\mathbb{C}\backslash\{0,1\}$ (with M. Golz, E. Panzer).
- There exist single-valued log-Laurent expansions for the ϵ^k coefficients of $f_G(z)$ at the singular points $s = 0, 1$ and at ∞.

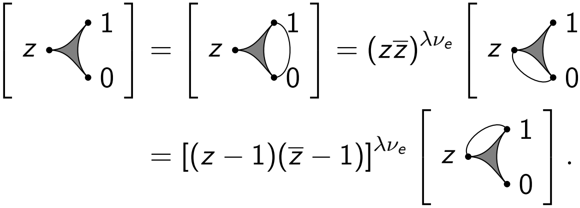
$$\sum_{\ell\geq 0} \sum_{m,n=M_s}^{\infty} c_{\ell,m,n}^{s,k}[\log(z-s)(\overline{z}-s)]^\ell (z-s)^m(\overline{z}-s)^n \quad \text{if } |z-s| < 1,$$

$$\sum_{\ell\geq 0} \sum_{m,n=-\infty}^{M_\infty} c_{\ell,m,n}^{\infty,k}(\log z\overline{z})^\ell z^m \overline{z}^n \quad \text{if } |z| > 1,$$

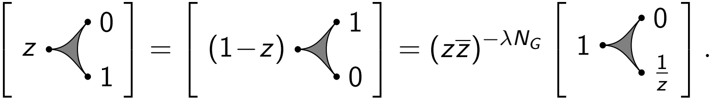
with $c_{\ell,m,n}^{\bullet,k} = c_{\ell,n,m}^{\bullet,k} \in \mathbb{R}$.

Construction

- Add edges between external vertices

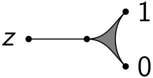


- Permute external vertices

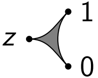


- Invert the effective Laplace operator $\Box_D$ for an isolated edge of weight 1 at vertex z,

$$\left(\Delta_n + \frac{\varepsilon/2}{z-\overline{z}}(\partial_z - \partial_{\overline{z}})\right)\left[\quad\right.$$



$$\left.\quad\right] = -\frac{1}{\Gamma(\lambda)}\left[\quad\right.$$



$$\left.\quad\right]$$

$$\text{with}\quad \Delta_n = \frac{1}{(z-\overline{z})^{n+1}}\partial_z\partial_{\overline{z}}(z-\overline{z})^{n+1} + \frac{n(n+1)}{(z-\overline{z})^2},$$

$$\text{where}\quad D = 2n + 4 - \epsilon.$$

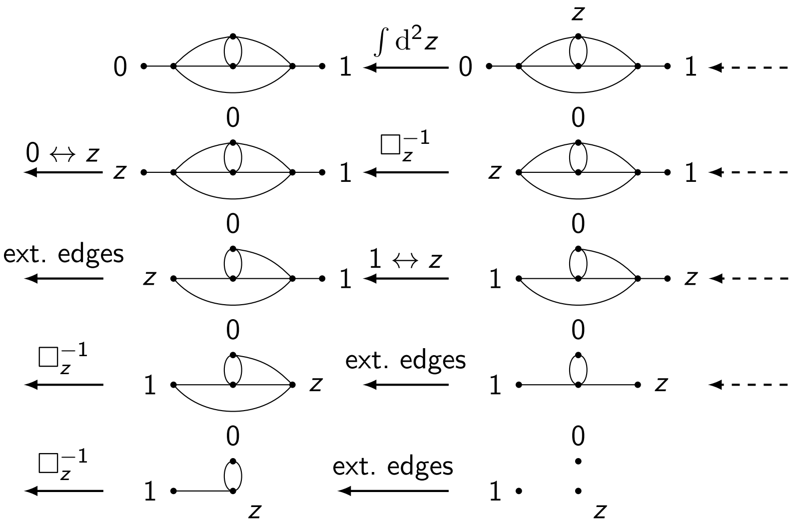
Vertex integration

- In the last step one may want to integrate over z to pass from a 3pt function to a 2pt function using

$$\frac{1}{(2\mathrm{i})^{2\lambda}\sqrt{\pi}\Gamma(\lambda+1/2)}\int_{\mathbb{C}} f_G(z)(z-\overline{z})^{2\lambda}\mathrm{d}^2z.$$

  In even integer dimensions one can use a residue theorem to do the integral.
  In non-integer dimensions we add an edge between 0 and z of weight -1, append an edge of weight 1 to z, and set $z=0$.

Picture (by M. Borinsky)



The five miracles of graphical functions

- For even integer D there exists a closed solution for the effective Laplace equation by taking single-valued primitives (with M. Borinsky). This is trivial in $D = 4$ dimensions.
- The solution is unique in the space of graphical functions.
- Generalized single-valued hyperlogarithms (GSVHs) are closed under solving the effective Laplace equation. The algorithm is efficient for GSVHs.
- The solution generalizes to non-integer dimensions $2n + 4 - \epsilon$.
- Spin $k \in \mathbb{Z}_{>0}$ in D dimensions (Yukawa-ϕ^4, QED, Yang-Mills) makes the effective Laplace equation a coupled system with triangular matrix whose diagonal is populated by (copies of) $\Box_D, \Box_{D+2}, \ldots, \Box_{D+2k}$.

GSVHs

Generalized single-valued hyperlogarithms (GSVHs) are iterated single-valued primitives of differential forms

$$\frac{\mathrm{d}z}{az\overline{z}+bz+c\overline{z}+d}, \qquad a,b,c,d\in\mathbb{C},$$

on the punctured (!) Riemann sphere $\mathbb{C}\backslash\{s_1,\ldots,s_n\}$.
Example (C. Duhr et al.):

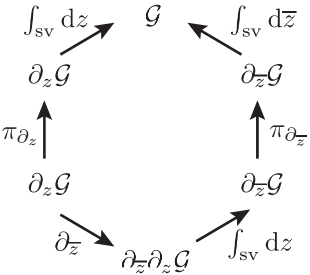
$$\int_{\mathrm{sv}}\frac{D(z)\,\mathrm{d}z}{z-\overline{z}},$$

where $D(z)$ is the Bloch-Wigner dilogarithm,

$$D(z)=\mathrm{Im}\,(\mathrm{Li}\,_2(z)+\log(1-z)\log|z|).$$

The commutative hexagon

GSVHs can be constructed with a commutative hexagon:



where $\mathcal{G}$ is the $\mathbb{C}$-algebra of GSVHs and π_{∂_z} ($\pi_{\partial_{\overline{z}}}$) kills (anti-)residues in $\partial_z \mathcal{G}$ ($\partial_{\overline{z}} \mathcal{G}$).

- Taylor coefficients of convergent graphical functions in non-integer dimensions are obtained by a straight forward expansion method.
- For singular graphical functions a sophisticated subtraction method is necessary to obtain the Laurent coefficients.
  Problem: inversion of the effective Laplace equation.
  Example: bottom line in the cat eye calculation,

$$\frac{1}{(z\overline{z})^{2\lambda}((z-1)(\overline{z}-1))^{\lambda}}.$$

After inverting the effective Laplace operator, the graphical function has a singular part which is annihilated by Δ_0,

$$\frac{1}{z-\overline{z}}\partial_z\partial_{\overline{z}}(z-\overline{z})\,\frac{2}{\epsilon z\overline{z}}=0.$$

Subtraction of subdivergences

Solution: Subtract (logarithmic) subdivergences:

$$\left(\frac{1}{(z\overline{z})^{2\lambda}((z-1)(\overline{z}-1))^{\lambda}}-\frac{1}{(z\overline{z})^{2\lambda}}\right)+\frac{1}{(z\overline{z})^{2\lambda}}.$$

- The first term is sufficiently regular at $z=0$: The effective Laplace equation can be inverted uniquely.
- The inversion of the second term is a convolution:

$$\frac{1}{\pi^{D/2}}\int_{\mathbb{R}^D}\frac{1}{||x||^{4\lambda}||x-z_2(z)||^{2\lambda}}\mathrm{d}x.$$

- The general situation is fully algorithmic.
- Quadratic subdivergences are mere 2pt insertions.
- No a priori analysis or extra orders in ϵ necessary.

The graphical function toolbox

There exists a large toolbox for calculating low order Laurent coefficients of (singular) graphical functions.

- Completion: conformal symmetry.
- Uniqueness: (approximate) star-triangle identities.
- Approximation: replace a subgraph with a sum of simpler graphs with the same low order ϵ expansion.
- Rerouting: subtraction of subdivergences with simpler graphs to reduce the pole order in ϵ (F. Brown, D. Kreimer).
- Integration by parts (in particular spin > 0 or dimension ≥ 6).
- Special identities: Twist, planar duals. . .
- Parametric integration: F. Brown, HyperInt (E. Panzer), Á. Kardos, O.S.
- . . .

Comparison with classical techniques

- Momentum space techniques are more general (masses, Npt functions).
- Momentum space techniques can also be applied to graphical functions (master integrals).
- The theory of graphical functions performs integrations.
- The large set of constructible graphs is always computable with graphical functions (to sensible orders in ϵ).
- It is not necessary to solve large systems of linear equations.
- One always obtains a reduction of complexity by integrating out some vertices of the Feynman graph.

Results

- Calculation of many primitive ϕ^4 periods up to 11 loops (and primitive ϕ^3 periods up to 9 loops) which lead to the discovery of the connection between motivic Galois theory and QFT (the coaction principle, the cosmic Galois group).
- ϕ^4 theory (4 dim.): 8 loops field anomalous dimension γ.
  7 loops β, mass anomalous dimension γ_m, self-energy Σ.
- ϕ^3 theory (6 dim.): 6 loops field anomalous dimension γ, β, mass anomalous dimension γ_m.
  5 loops self-energy Σ.

Six loops ϕ^3

$$
\begin{aligned}
\beta_6^{\phi^3} &= \frac{245045}{144}\zeta(9) + 37\zeta(3)^3 + \frac{3357}{40}\zeta(5,3) - \frac{11}{3}\zeta(5)\zeta(3) \\
&- \frac{81733}{2016000}\pi^8 - \frac{456443}{1152}\zeta(7) + \frac{99}{800}\pi^4\zeta(3) - \frac{2425}{384}\zeta(3)^2 \\
&+ \frac{176425}{2612736}\pi^6 - \frac{24878747}{34560}\zeta(5) + \frac{42654751}{74649600}\pi^4 \\
&- \frac{85523425}{186624}\zeta(3) - \frac{173655397121}{3224862720} \\
&= -241.455497609497\ldots
\end{aligned}
$$

$\zeta(5,3) = \sum_{k_1 > k_2 \geq 1} \frac{1}{k_1^5 k_2^3}$ (May 19, 2023).

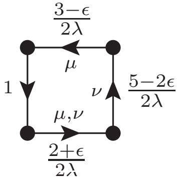
General theory (with S. Theil)

In a fermionic theory, a graphical function becomes a vector whose components represent the spin structure. E.g.

$$f_G(z) = f_0(z) + f_1(z)\, \not{z}_1\, \not{z}_2\,,$$

where z_1 corresponds to 1 and z_2 corresponds to the complex number z. We obtain matrix identities in triangular form with dimension shifts by 2.

Each Feynman period is represented by an unlabeled vacuum graph. Any choice of two vertices 0 and 1 give the same Feynman integral. E.g.

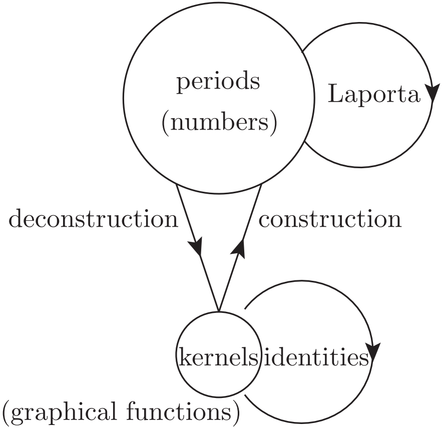


The theory becomes more complex but it fully generalizes.

Reduction of complexity

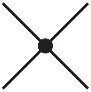
- A sizable subset of Feynman periods can be calculated immediately.
- One can increase the number of known Feynman periods by calculating kernel graphical functions.
- One can use IBP identities to determine unknown Feynman periods or unknown kernel graphical functions.
- A combination of both techniques is used for the calculation of all six loop primitive graphs in ϕ^3 theory:
  M. Borinsky, O. Schnetz, *Recursive computation of Feynman periods*, JHEP No. 08, 291 (2022).

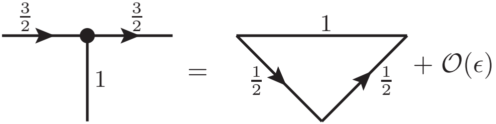
(De-)construction



Yukawa-ϕ^4 theory

The most accessible fermionic theory is Yukawa-ϕ^4 theory in four dimensions (Gross-Neveu-Yukawa model) which has a spin 0 boson with a 4pt interaction (Higgs) and a spin 1/2 fermion with a threepoint coupling to the boson.





Yukawa-ϕ^4 theory has uniqueness (depicted) and completion (conformal symmetry in convergent integrals). These are powerful tools for calculating non-constructible periods.
First goal: Calculate the periods of primitive graphs up to five loops (and beyond).

HyperlogProcedures

- `HyperlogProcedures` is a Maple package that performs calculations using graphical functions and GSVHs.
- It is also a toolbox to handle multiple zeta values (MZVs) including extensions to second (Euler sums), third, fourth, and sixth roots of unity.
- A large number of manipulations for hyperlogarithms (Goncharov polylogs) are implemented in `HyperlogProcedures`.
- `HyperlogProcedures` has the results for the renormalization functions in ϕ^4 and ϕ^3 with a large number of extra data.
- `HyperlogProcedures` is available for free download from my homepage. `https://www.math.fau.de/person/oliver-schnetz/`