Exactly Solvable Quantum Mechanical Potentials

Seminar Presentation

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Outlook

1. Generating Exactly Solvable Potentials
2. Potentials with Jacobi polynomials
3. Basics of SUSYQM
4. SUSY partners in the P_I and the extended potential classes
5. Classification of exactly solvable potentials

Generating Exactly Solvable Potentials

- The general form of the differential equations of hypergeometric functions:

$$\frac{d^2F}{dz^2} + Q(z)\frac{dF}{dz} + R(z)F(z) = 0$$

- The wavefunction:

$$\psi(x) = f(x)F\big(z(x)\big) \quad \Longrightarrow$$

Motivation:
$F\big(z(x)\big)$ orthog. ↔ $\psi_n\ (x)$ orthog.

Example:
$$\int_{-1}^{1} dy(1-y)^{\alpha}(1+y)^{\beta} P_n^{(\alpha,\beta)}(y)\, P_m^{(\alpha,\beta)}(y) \sim \delta_{n,m}$$

- From the Schrödinger-equation:

$$E - V(x) = \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + (z'(x))^2\left(R(z(x)) - \frac{1}{2}\frac{dQ}{dz} - \frac{1}{4}Q^2(z(x))\right)$$

$$\psi(x) \sim (z'(x))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int^{z(x)} Q(z)dz\right) F(z(x))$$

- **How can we identify the $z(x)$ function?**
  - **Bhattacharjie and Sudarshan**: **Constant E** on the left-hand side → one of the terms on the right-hand side has to be constant (first two terms give meaningless results)→ **PI, PII and PIII potential classes**
  - **Generalized method:**

$$\boxed{\left(\frac{dz}{dx}\right)^2 \Phi(z) = C}$$

Generating Exactly Solvable Potentials (Jac.-p.)

- **Jacobi polynomials (orthogonal polynomials)**:

$$Q(y) = [(\beta-\alpha)-(\alpha+\beta+2)y]/(1-y^2)$$
$$R(y) = n(n+\alpha+\beta+1)/(1-y^2)$$

$$\begin{aligned} E_n - V(x) = &\frac{y'''(x)}{2y'(x)} - \frac{3}{4}\left(\frac{y''(x)}{y'(x)}\right)^2 \\ &+\frac{(y'(x))^2}{1-y^2(x)}\left(n+\frac{\alpha+\beta}{2}\right)\left(n+\frac{\alpha+\beta}{2}+1\right) \\ &+\frac{(y'(x))^2}{(1-y^2(x))^2}\left[1-\left(\frac{\alpha+\beta}{2}\right)^2-\left(\frac{\alpha-\beta}{2}\right)^2\right] \\ &-\frac{2y(x)(y'(x))^2}{(1-y^2(x))^2}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right). \end{aligned}$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \Phi(z) \equiv \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \frac{\phi(y)}{(1-y^2(x))^2} \equiv \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \frac{p_{\mathrm{I}}(1-y^2)+p_{\mathrm{II}}+p_{\mathrm{III}}y}{(1-y^2(x))^2} = C$$

$$\begin{aligned} \psi(x) &\simeq (y'(x))^{-\frac{1}{2}}(1+y(x))^{\frac{\beta+1}{2}}(1-y(x))^{\frac{\alpha+1}{2}}P_n^{(\alpha,\beta)}(y(x)) \\ &\simeq (\phi(y(x)))^{\frac{1}{4}}(1+y(x))^{\frac{\beta}{2}}(1-y(x))^{\frac{\alpha}{2}}P_n^{(\alpha,\beta)}(y(x)) . \end{aligned}$$

Generating Exactly Solvable Potentials (Jac.-p.)

- **General form of the potential:**

$$V(x) = -\frac{y'''(x)}{2y'(x)} + \frac{3}{4}\left(\frac{y''(x)}{y'(x)}\right)^2 + \frac{C}{\phi(y)}\left[s_I(1-y^2(x)) + s_{II} + s_{III}y(x)\right]$$

- We obtain an **algebraic system of equations** $\left(\omega = \frac{\alpha+\beta}{2}, \rho = \frac{\alpha-\backslash\text{beta}}{2}\right)$:

$$\begin{aligned}
\left(n+\frac{1}{2}+\omega\right)^2 - \frac{1}{4} + s_I - p_I\frac{E_n}{C} &= 0 \\
\left(1-\omega^2-\rho^2\right) + s_{II} - p_{II}\frac{E_n}{C} &= 0 \\
-2\omega\rho + s_{III} - p_{III}\frac{E_n}{C} &= 0
\end{aligned}$$

Potentials with Jacobi polynomials: PI potential class

- **Parameters:**

$$p_I = \pm 1,\ p_{II} = p_{III} = 0$$

- We obtain the differential equation:

$$\left(\frac{dz}{dx}\right)^2 \frac{1}{1-z^2} = C$$

$$V = -\frac{1}{2}\frac{z'''(x)}{z'(x)} + \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{1}{4} - \frac{C}{(1-z^2(x))}(1-\omega^2-\rho^2) - C\frac{z(x)}{(1-z^2(x))}2\omega\rho$$

$$E_n = C\left(n + \frac{1}{2} + \omega\right)^2$$

$$\psi_n(\omega,\rho;x) = C_n^{(\alpha,\beta)}(z'(x))^{-\frac{1}{2}}(1+z(x))^{\frac{\omega+\rho+1}{2}}(1-z(x))^{\frac{\omega-\rho+1}{2}}P_n^{(\omega+\rho,\omega-\rho)}(z(x))$$

$z(x)$	$cosh(ax)$	$sin(ax)$	$i\ sin(ax)$
Potential	**Gen. Pöschl-Teller**	**Scarf I**	**Scarf II**
Domain	$[0,\infty[$	$\left[-\frac{\pi}{a},\frac{\pi}{a}\right]$	$]-\infty,\infty[$

Potentials with Jacobi polynomials: PII potential class

- **Parameters:**

$$p_{II} = \pm 1, p_I = p_{III} = 0$$

- We obtain the differential equation:

$$\boxed{(z'(x))^2 \frac{1}{(1-z^2(x))^2} = C}$$

- **n-independent energy, n-dependent potential→ need a sufficient paramater transformation!**

$$\chi(\alpha,\beta,n) = \frac{\alpha+\beta}{2} + n$$
$$\Omega(\alpha,\beta,n) = (\chi - n)\frac{\alpha-\beta}{2}$$

$$\rightarrow \boxed{\begin{aligned} E - V(x) = \frac{1}{2}\left[-2C(1-z^2(x)) + 4Cz^2(x)\right] - 3Cz^2(x) + \\ +C(1-z^2(x))(\chi)(\chi+1) + C\left[1 - (\chi-n)^2 - \left(\frac{\Omega}{\chi-n}\right)^2\right] - 2Cz(x)\Omega \end{aligned}}$$

$z(x)$	$i\,tan(ax)$	tanh(ax)	$cosh(ax)$
Potential	**Rosen-Morse I**	**Rosen-Morse II**	**Eckart**
Domain	$\left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$	$]-\infty,\infty[$	$[0,\infty[$

Potentials with Jacobi polynomials: Other potentials

Géza Lévai. PT symmetry in natanzon-class potentials. *International Journal of Theoretical Physics*, 54(8):2724–2736, 2015.

	C	p_I	p_{II}	p_{III}
Scarf II	-1	1	0	0
Gen. Pöschl–Teller	-1	1	0	0
Scarf I	1	-1	0	0
Rosen–Morse II	1	0	1	0
Eckart	1	0	1	0
Rosen–Morse I	-1	0	1	0
PIII	1	0	0	1
Ref. [29]	-1	1	δ	0
Ginocchio	$\frac{\lambda^4}{\lambda^2-1}$	1	$(\lambda^2-1)^{-1}$	0
Gen. Ginocchio	$4\lambda^4$	$\frac{\gamma^2-1}{4\lambda^4}$	$(2\lambda^2)^{-1}$	$(2\lambda^2)^{-1}$
DKV	1	-1	1	0
Ref. [41]	-1	1	-1	-1
Ref. [42]	1	-1	γ^2+1	2γ

Exactly solvable (P_I, P_{II}): Scarf II, Gen. Pöschl–Teller, Scarf I, Rosen–Morse II, Eckart, Rosen–Morse I

Implicit $z(x)$ function, explicit energy: PIII, Ginocchio, Gen. Ginocchio, Ref. [41]

Implicit energy, explicit $z(x)$ function: DKV

Not all exactly solvable

Potentials with Jacobi polynomials: Searching for potentials with explicit energy term

- **Based on the system of equations, we can ask the question whether we can obtain explicit energy expressions in this framework:**
- In the $p_I \neq 0$ case:

$$E_n = C\frac{1}{p_I}(n+\omega+\frac{1}{2})^2 - C\frac{1}{p_I}\frac{1}{4} + C\frac{1}{p_I}S_I$$

$$S_{II} = \frac{p_{II}}{p_I}\left[(n+\omega+\frac{1}{2})^2 - \frac{1}{4} + S_I\right] - 1 + \omega^2 + \rho^2$$

$$S_{III} = \frac{p_{III}}{p_I}\left[(n+\omega+\frac{1}{2})^2 - \frac{1}{4} + S_I\right] + 2\omega\rho$$

$$\Downarrow$$

$$S_{II}(\omega,\rho,n) = \frac{p_{II}}{p_I}\left[(n+\omega+\frac{1}{2})^2 - \frac{1}{4} + S_I\right] - 1 + \omega^2 + \frac{\left[S_{III} - \frac{p_{III}}{p_I}\left[(n+\omega+\frac{1}{2})^2 - \frac{1}{4} + S_I\right]\right]^2}{4\omega^2}$$

- <u>If we have an explicit expression for $\omega(S_I, S_{II}, n)$, we obtain explicit expression for E_n</u>

Potentials with Jacobi polynomials: Searching for potentials with explicit energy term

- S_{II} is quartic in ω → solvable quartic equation:

$$a\tilde{\omega}^2 + b\tilde{\omega} + c = 0$$

$$\tilde{\omega} = A\omega^2 + B\omega + D$$

$$b^2 - 4ac \geq 0$$

$$B^2 - 4A(D - \tilde{\omega}) \geq 0$$

⇓

$$aA^2\omega^4 + 2aAB\omega^3 + (aB^2 + 2aAD + bA)\omega^2 + (2aBD + bB)\omega + (aD^2 + bD + c) = 0$$

- **Matching the terms gives us an other equation-system**
  - **What are the variables, what are the fix parameters?**
  - **Setting the p_i parameters ⇒ gives back the explicit energy expressions**
  - **Setting the parameters of ω ⇒ given ω with different p_i, non-trivial isospectral potential pairs (SUSYQM???)**

Basics of SUSYQM

- The most wildy used model of supersymmetric quantum mechanics is the **N=2 SUSYQM**, which define the following ***sl(1/1)*** superalgebra between the so called supersymmetric Hamiltonian $\mathcal{H}$ and the supersymmetric charge operators $\boldsymbol{Q}$ and $\boldsymbol{Q}^{\dagger}$:

$$\{Q, Q^{\dagger}\} = \mathcal{H} \ , \qquad Q^2 = (Q^{\dagger})^2 = 0 \ , \qquad [Q, \mathcal{H}] = [Q^{\dagger}, \mathcal{H}] = 0 \ .$$

- The realization of this this superalgebra usually given in terms of 2x2 matrices:

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} , \qquad Q^{\dagger} = \begin{pmatrix} 0 & A^{\dagger} \\ 0 & 0 \end{pmatrix} ,$$

- Therefore, the supersymmetric Hamiltonian has the form:

$$\mathcal{H} = \begin{pmatrix} A^{\dagger}A & 0 \\ 0 & AA^{\dagger} \end{pmatrix} \equiv \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}$$

SUSYQM $\neq / \nsubseteq / \notin$ SUSY
Just „stole” the terminology
based on the math. construction

Basics of SUSYQM

- In the literature $\boldsymbol{H}_{-}$ and $\boldsymbol{H}_{+}$ operators are referred as the **„bosonic”** and **„fermionic” Hamiltonian**, and called **supersymmetric partners**.
- Therefore the general basis states have two components, representing the **„bosonic”** and **„fermionic” sectors**, and these sectors are connected by the charge operators:

$$Q\begin{pmatrix}\psi^{(-)}\\ 0\end{pmatrix}=\begin{pmatrix}0\\ A\psi^{(-)}\end{pmatrix}, \qquad Q^{\dagger}\begin{pmatrix}0\\ \psi^{(+)}\end{pmatrix}=\begin{pmatrix}A^{\dagger}\psi^{(+)}\\ 0\end{pmatrix},$$

- We can recover the one-dimensional Schrödinger equation for the supersymmetric partners with the following:

$$\boxed{A=\frac{\mathrm{d}}{\mathrm{d}x}+W(x) \quad \text{and} \quad A^{\dagger}=-\frac{\mathrm{d}}{\mathrm{d}x}+W(x).}$$

- Therefore the Hamiltonains recover the canonical form:

$$\boxed{H_{\pm}\psi^{(\pm)}(x)=\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V_{\pm}(x)\right]\psi^{(\pm)}(x)=E^{(\pm)}\psi^{(\pm)}(x),}$$

Basics of SUSYQM

- The partnerpotentials can be expressed with the „superpotetial” $\boldsymbol{W(x)}$:

$$V_{\pm}(x) = W^2(x) \pm \frac{\mathrm{d}}{\mathrm{d}x} W(x).$$

- Based on the underlying symmetry, **the two Hamiltonian has the same energy spectra**:

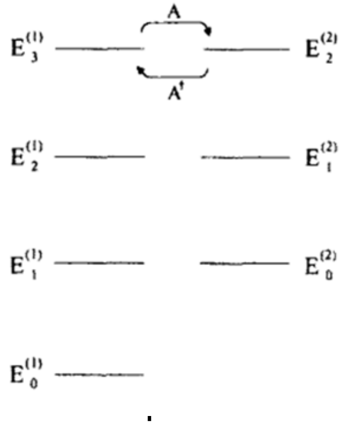
$$H_+ \left[A\psi^{(-)}(x) \right] = AA^{\dagger} \left[A\psi^{(-)}(x) \right] = AH_- \psi^{(-)}(x) = E^{(-)} A\psi^{(-)}(x)$$

- And the **eigenfunctions are connected** by:

$$A\psi^{(-)}(x) = \left[E^{(-)} \right]^{1/2} \psi^{(+)}(x)$$

$$A^{\dagger}\psi^{(+)}(x) = \left[E^{(-)} \right]^{1/2} \psi^{(-)}(x)$$

- **In the case, when $\boldsymbol{A\,\psi^{-}(x) = 0}$, the zero-energy level will miss from the spect**



$$E_{n+1}^{(-)} = E_n^{(+)} \quad (n = 0, 1, 2, \ldots) \quad \text{with } E_0^{(-)} = 0.$$

$$W(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_0^{(-)}(x)$$

Mathematical description:
Jean-Gaston Darboux <u>1882</u>

Basics of SUSYQM

- One can generate SUSY partner potentials even from non-physical, nodeless solution of H_-:

$$H_-\chi(x) \equiv A^\dagger A\chi(x) = \epsilon\chi(x)$$

$$V_-(x) = \frac{\chi''(x)}{\chi(x)} + \epsilon = \left(-\frac{\chi'(x)}{\chi(X)}\right)^2 - \left(-\frac{\chi'(x)}{\chi(x)}\right)' + \epsilon$$

- We can define a corresponding superpotential:

$$\tilde{W}(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\ln\chi(x)$$

- And the two partner potetntials:

$$V_\pm(x) = \tilde{W}^2(x) \pm \frac{\mathrm{d}}{\mathrm{d}x}\tilde{W}(x) + \epsilon$$

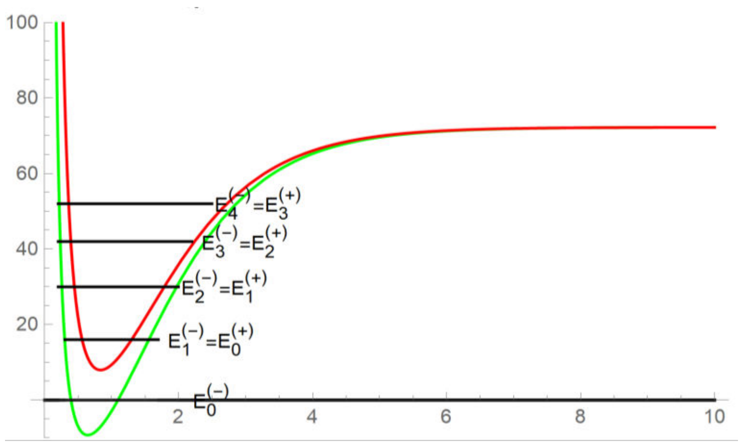
$\epsilon = 0$	$\epsilon \neq 0$
$\chi(x) = \psi_0^{(-)}(x)$	$\chi(x) \neq \psi_0^{(-)}(x)$
$A\chi(x) = 0 \leftrightarrow Q\Psi = 0$	$A\chi(x) \neq 0 \leftrightarrow Q\Psi \neq \mathrm{q}\Psi$
„SUSY is Unbroken”	**„SUSY is Broken”**

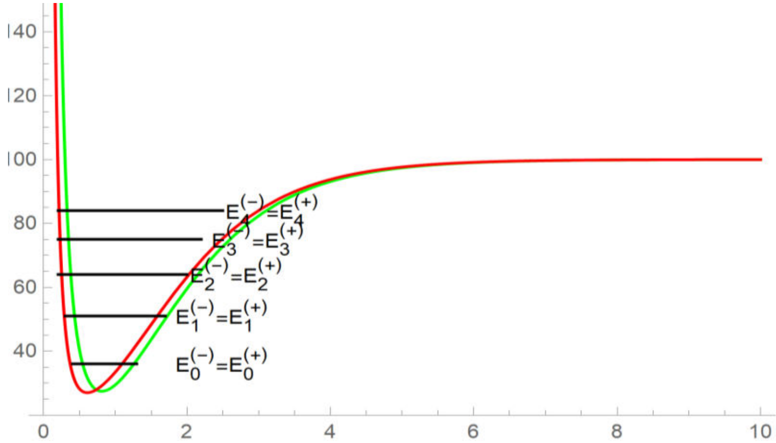
Basics of SUSYQM

- Four possible transformations:

T_i	T_1	T_2	T_3	T_4
ϵ	$\epsilon = E_0^{(-)}$	$\epsilon < E_0^{(-)}$	$\epsilon < E_0^{(-)}$	$\epsilon < E_0^{(-)}$
$\lim_{x \to x_-} \chi(x)$	convergent	divergent	convergent	divergent
$\lim_{x \to x_+} \chi(x)$	convergent	divergent	divergent	convergent
V_+ spectrum modification	Deletes ground state	Adds new ground state	none	none

- **Unbroken SUSY → Shift in the V_+ spectra**
- **Broken SUSY → Same energy spectrums**





SUSY partners in the P_I and the extended potential classes

- **Particular form of $\chi(x)$:**

$$\chi(x) = (1 - z(x))^t (1 + z(x))^s \frac{(p + z(x))^k}{(q + z(x))^j}$$

- The super- and partnerpotentials depend on the parameters
- Matching $V_-(x)$ with the general PI type potentials $V(\alpha, \beta, x)$ or their rationally extended version $\tilde{V}(\alpha, \beta, x)$

Tibor Soltész, Levente Ferenc Pethő, and Géza Lévai. Unified supersymmetric description of shape-invariant potentials within and beyond the natanzon class. *Symmetry*, 16(2), 2024.

Shape-invariance of P_I ←

Shape-invariance of the extended P_I ←

j, k	$V_-(x)$	t	s	p	q	$V_+(x)$	ϵ	$E_0^{(-)} - \epsilon$
$0, 0$	$V(\alpha,\beta,x)$							
		$\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$			$V(\alpha+1,\beta+1,x)$	$C\left(\frac{\alpha+\beta+1}{2}\right)^2$	0
		$-\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$			$V(\alpha-1,\beta-1,x)$	$C\left(\frac{-\alpha-\beta+1}{2}\right)^2$	$C(\alpha+\beta)$
		$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$			$V(\alpha+1,\beta-1,x)$	$C\left(\frac{\alpha-\beta+1}{2}\right)^2$	$C(\alpha+1)\beta$
		$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$			$V(\alpha-1,\beta+1,x)$	$C\left(\frac{-\alpha+\beta+1}{2}\right)^2$	$C\alpha(\beta+1)$
$0, 1$	$V(\alpha,\beta,x)$							
		$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta}{\alpha-\beta+2}$		$\hat{V}(\alpha+1,\beta-1,x)$	$C\left(\frac{\alpha-\beta+3}{2}\right)^2$	$C(\alpha+2)(\beta-1)$
		$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta}{\alpha-\beta-2}$		$\hat{V}(\alpha-1,\beta+1,x)$	$C\left(\frac{-\alpha+\beta+3}{2}\right)^2$	$C(\alpha-1)(\beta+2)$
$1, 0$	$\hat{V}(\alpha,\beta,x)$							
		$\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$		$\frac{\alpha+\beta}{\alpha-\beta}$	$V(\alpha+1,\beta-1,x)$	$C\left(\frac{\alpha-\beta-1}{2}\right)^2$	$C\alpha(\beta+1)$
		$-\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$		$\frac{\alpha+\beta}{\alpha-\beta}$	$V(\alpha-1,\beta+1,x)$	$C\left(\frac{-\alpha+\beta-1}{2}\right)^2$	$C\beta(\alpha+1)$
$1, 1$	$\hat{V}(\alpha,\beta,x)$							
		$\frac{\alpha}{2}+\frac{1}{4}$	$\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta+2}{\alpha-\beta}$	$\frac{\alpha+\beta}{\alpha-\beta}$	$\hat{V}(\alpha+1,\beta+1,x)$	$C\left(\frac{\alpha+\beta+1}{2}\right)^2$	0
		$-\frac{\alpha}{2}+\frac{1}{4}$	$-\frac{\beta}{2}+\frac{1}{4}$	$\frac{\alpha+\beta-2}{\alpha-\beta}$	$\frac{\alpha+\beta}{\alpha-\beta}$	$\hat{V}(\alpha-1,\beta-1,x)$	$C\left(\frac{-\alpha-\beta+1}{2}\right)^2$	$C(\alpha+\beta)$

SUSY partners in the P_I and the extended potential classes

- P_I potential class:

$$V(\alpha,\beta,x) = \frac{C}{1-z^2(x)}\left[\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha-\beta}{2}\right)^2 - \frac{1}{4}\right] + \frac{2Cz(x)}{1-z^2(x)}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right)$$

$$\psi_n(\alpha,\beta;x) = C_n^{(\alpha,\beta)}(1-z(x))^{\frac{\alpha}{2}+\frac{1}{4}}(1+z(x))^{\frac{\beta}{2}+\frac{1}{4}}P_n^{(\alpha,\beta)}(z(x))$$

$$E_n = C\left(n+\frac{\alpha+\beta+1}{2}\right)^2$$

- Rationally extended version:

X_1 type exceptional Jacobi polynomials:

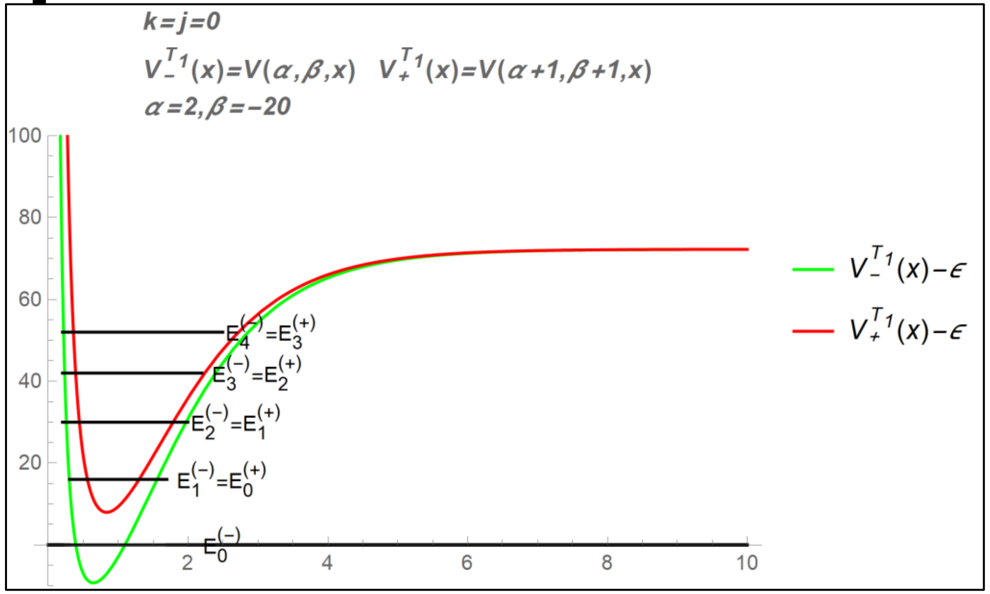
$$\hat{P}_n^{(\alpha,\beta)}(z) = \left[-\frac{(\alpha-\beta)z+\alpha+\beta}{2(\alpha-\beta)} - \frac{\alpha+\beta}{(\alpha-\beta)(\alpha+\beta+2n-2)}\right]P_{n-1}^{(\alpha,\beta)}(z) - \\ -\frac{1}{\alpha+\beta+2n-2}P_{n-2}^{(\alpha,\beta)}(z)$$

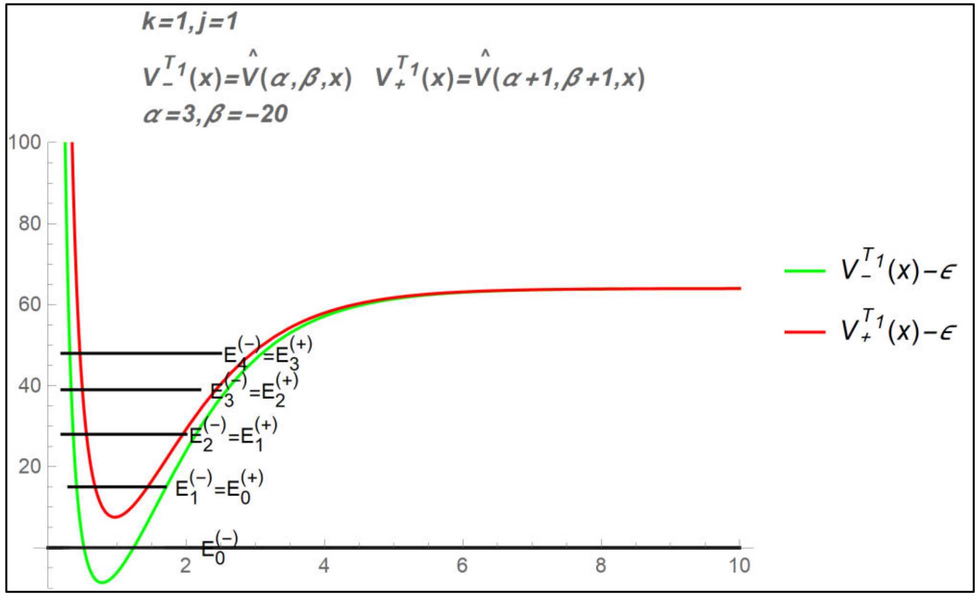
$$\hat{V}(\alpha,\beta,x) = V(\alpha,\beta;x) + \frac{2C(\alpha+\beta)}{(\alpha-\beta)z(x)+\alpha+\beta} + \frac{2C\left[(\alpha-\beta)^2-(\alpha+\beta)^2\right]}{[(\alpha-\beta)z(x)+\alpha+\beta]^2}$$

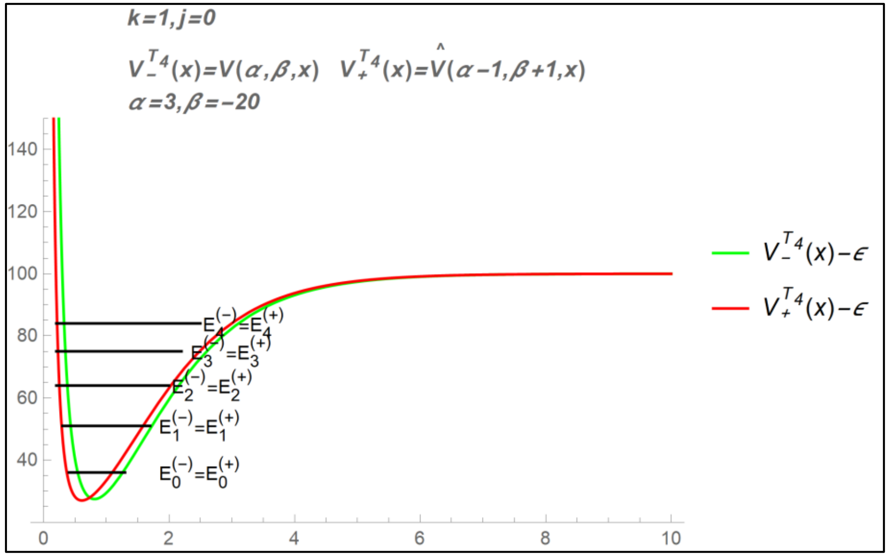
$$\hat{E}_n = C\left(n-1+\frac{\alpha+\beta+1}{2}\right)^2$$

$$\hat{\psi}_n(\alpha,\beta;x) = \hat{C}_n^{(\alpha,\beta)}(1-z(x))^{\frac{\alpha}{2}+\frac{1}{4}}(1+z(x))^{\frac{\beta}{2}+\frac{1}{4}}[(\alpha-\beta)z(x)+\alpha+\beta]^{-1}\hat{P}_n^{(\alpha,\beta)}(z(x))$$

SUSY partners in the P_I and the extended potential classes

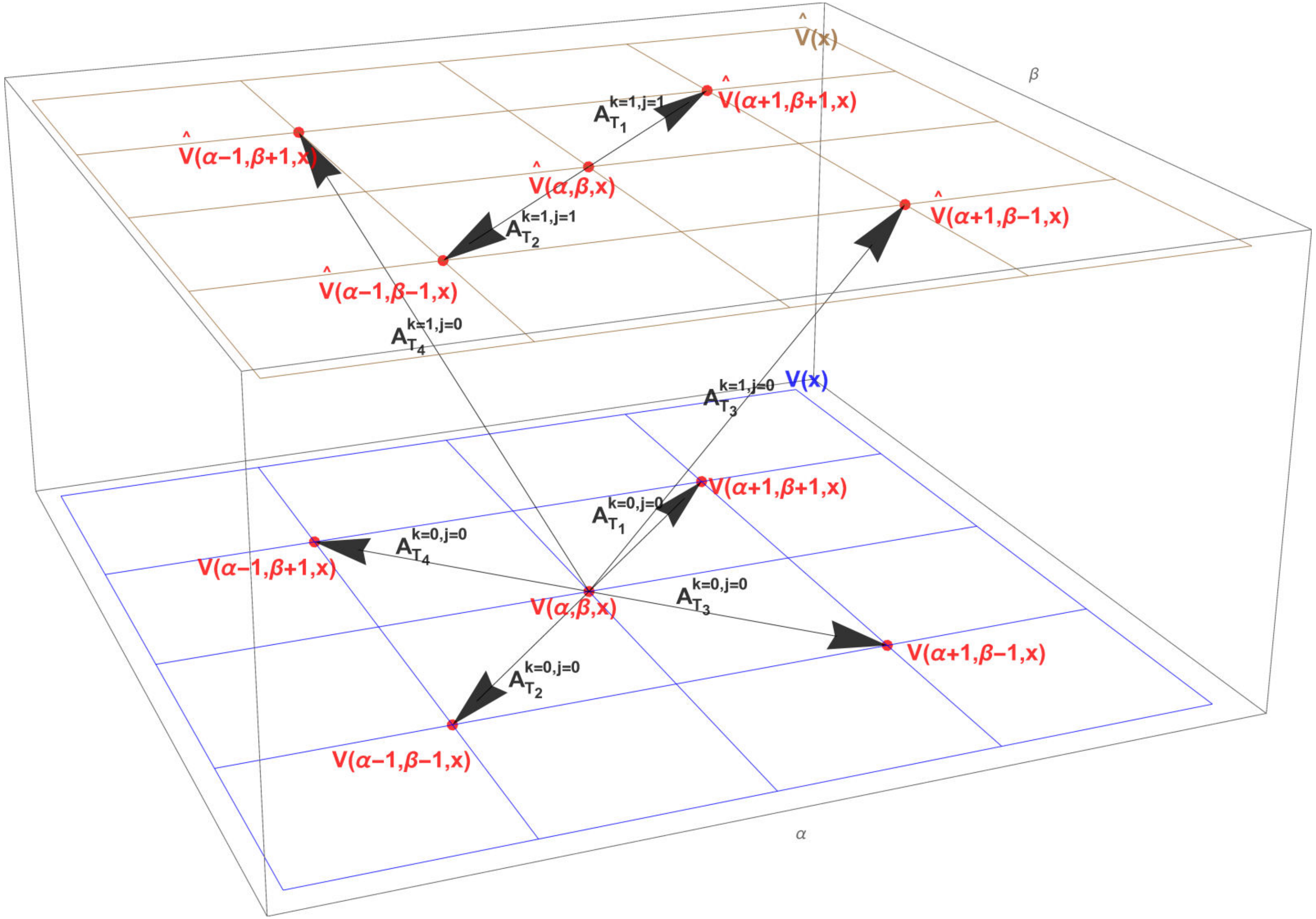






SUSY partners in the P_I and the extended potential classes

- **Result:** A network of SUSQM transformations **inside and between** shape invariant potentials



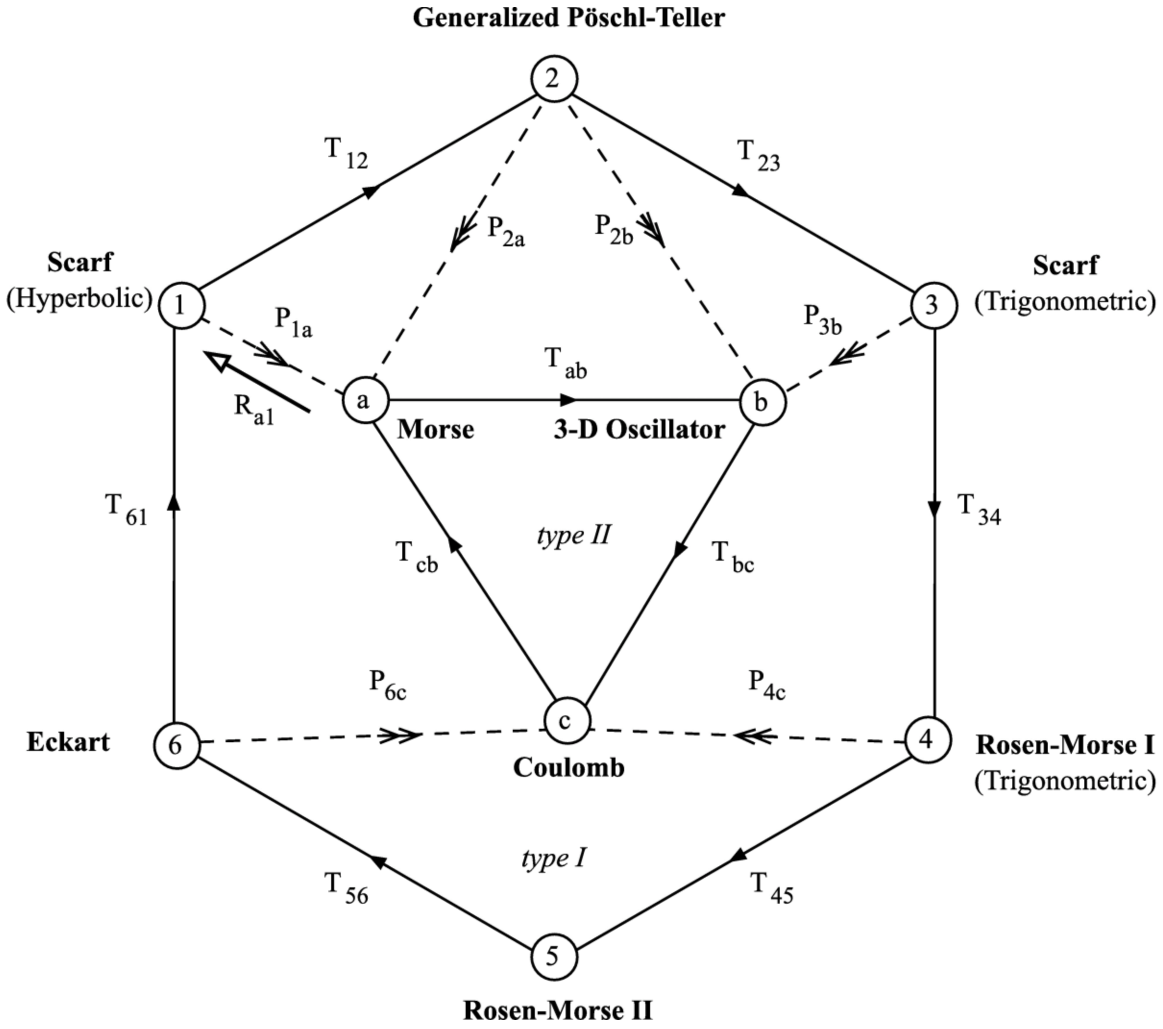
Classification of „exactly solvable" potentials

- **The classification scheme naturally emerges from the two framworks:**
  - **Natanzon-class potentials**: 6 parameters, generated from hypergeometric functions (Jacobi-polynomial→ P_i pot. classes)
  - **Confluent Natanzon-class potentials**: 6 parameters, generated from hypergeometric functions (generalised Laguerre-polynomial, Hermite-polynomials)
  - **Shape-invariant potentials**: Subclasses of the Natanzon- and confluent Natanzon-class potentials, closed for SUSYQM transformations
  - **Further potentials correspond to exceptional orthogonal polynomials (can be originated form the confluent Heun-function)**

- **Other investigations:**
  - **Jacobi polynomials** $\leftrightarrow$ **generalised Laguerre-polynomials: radial harmonic oscillator** $(\mathrm{L}_{\boldsymbol{I}})$**, Coulomb potential** $(\mathrm{L}_{\mathrm{II}})$**, Morse potential** $(\mathrm{L}_{\mathrm{III}})$
  - **Mapping other sectors with SUSYQM**

Classification of „exactly solvable” potentials: Further relations

Jeffry V. Mallow, Asim Gangopadhyaya, Jonathan Bougie, and Constantin Rasinariu. ***Inter-relations between additive shape invariant superpotentials.*** *Physics Letters A,384(6):126129, 2020*



Thank you for your attention!