QFT on a rotating box at finite temperature

Sebestyen Nagy and Daniel Nogradi

Eotvos University, Budapest

Physics motivation

Infinite space-time volume QCD: only parameters quark masses

Other much studied parameters: T, L, v, B, μ, $\ldots$

Partition function $Z(T, L, B, v, \mu, \ldots)$, **observables** $\langle \mathcal{O} \rangle_{T,L,B,v,\mu,\ldots}$

$$\langle \mathcal{O} \rangle_T - \langle \mathcal{O} \rangle_0 \qquad \frac{\partial^n}{\partial T^n} \log Z(T) \qquad \langle \mathcal{O} \rangle_{L,B} - \langle \mathcal{O} \rangle_{L,0}$$

$$L \frac{dg(L)}{dL} \qquad \frac{\partial^n}{\partial \mu^n} \langle \mathcal{O} \rangle_\mu$$

All introduced in path integral formulation $\rightarrow$ 4D lattice

What about rotating QCD matter?

New parameter, ω angular velocity, $Z(\omega)$, $\langle\mathcal{O}\rangle_\omega$

STAR: $\omega \sim O(10MeV)$

How to do lattice simulations?

Even more basic: continuum path integral formulation?

Continuum path integral

Parameters α coupled to conserved charges Q

$$Z(T,\alpha) = \mathrm{Tr}\, e^{-\beta(H-\alpha Q)}$$

If $\alpha = 0$, path integral formulation as usual

$$H \to \mathcal{L}(\partial_\mu \phi(x), \phi(x)) \qquad\qquad \phi\left(t + \frac{1}{T}, \boldsymbol{x}\right) = \phi(t, \boldsymbol{x})$$

If $\alpha \neq 0$, two ways to proceed

Continuum path integral

$$Z(T,\alpha) = \mathrm{Tr}\, e^{-\beta(H-\alpha Q)}$$

- **Option 1, action modified,** $\mathcal{L}(\alpha) = \mathcal{L}(0) + O(\alpha) + O(\alpha^2)$ **same p.b.c.** $\phi(x)$
- **Option 2, action same,** $\mathcal{L}(\alpha) = \mathcal{L}(0)$ **but** α **in non-trivial temporal b.c.**

Depending on Q one is better/practical/useful the other less so

Example, moving frame Giusti, Meyer, 1011.2727, 1110.3136

Poincare invariant $H, \quad \alpha = v_k, \quad Q = P_k$

$$Z(T,v) = \mathrm{Tr}\, e^{-\beta(H - v_k P_k)}$$

Euclidean setup, $i v_k = T y_k$**, in trace we have** $e^{-i y_k P_k}$

Option 2, action remains same, modified b.c.

$$\phi\left(t + \frac{1}{T}, \mathbf{x}\right) = \phi(t, \mathbf{x} - \mathbf{y})$$

Continuum path integral

Well defined continuum path integral, finite T, finite spatial $\mathbb{T}^3$

Shifted boundary condition, can easily simulate, space-time $\mathbb{T}^4$

What about rotation, ω ?

Trying the same, Poincare invariant H, $\quad \alpha = \omega_k, \quad Q = J_k$

J_k generators of $SO(3)$ rotations, infinite spatial volume

Continuum path integral

Try the same

$$Z(T,\omega) = \mathrm{Tr}\, e^{-\beta(H-\omega_k J_k)}$$

First problem: rigid rotation in infinite volume not okay

Must have finite volume, spatial $\mathbb{T}^3$

Second problem: on $\mathbb{T}^3$ no $SO(3)$ symmetry, no J_k

Euclidean setup, $i\omega_k = T\vartheta_k$

No infinitesimal generators, but finite $U(\vartheta)$ unitary operator

$$Z(T,\vartheta) = \mathrm{Tr}\, e^{-\beta H} U(\vartheta)$$

Continuum path integral

$$Z(T,\vartheta) = \text{Tr}\, e^{-\beta H} U(\vartheta)$$

Turn it into path integral (without rotations: $\mathbb{T}^4$ space-time)

With $\vartheta \neq 0$, choose option 2

$\mathcal{L}(\vartheta) = \mathcal{L}(0)$ same action as before

Put ϑ into boundary condition

Rotated boundary conditions: periodic up to rotation

Rotated boundary conditions

Clearly not all ϑ allowed, only some discrete values

Note: "periodic up to rotation" means space-time is not $\mathbb{T}^4$

2 dimensional analogy: $\mathbb{T}^2$, Mobius-band, Klein-bottle

Still flat, compact, orientable, smooth 4-manifold

There are 26 topologically different non-trivial flat, compact, orientable 4-manifolds

Rotated boundary conditions

All 26 look like $\mathcal{M} = \mathbb{T}^4/G$ with finite $G \subset E(4) = SO(4) \ltimes \mathbb{R}^4$ Euclidean group

G one of

$$\mathbb{Z}_{2,3,4,6} \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad D_{6,8,12} \qquad A_4$$

Some abelian, some non-abelian

Shift around time is translation, followed by rotation, together in Euclidean group

Rotated boundary conditions

Fun fact: this was the original motivation :)

Shifted b.c.: periodic up to shift $\rightarrow$ shift element of Poincare $\rightarrow$ other elements of Poincare: rotations $\rightarrow$ periodic up to rotation?

What kind of b.c. are these??

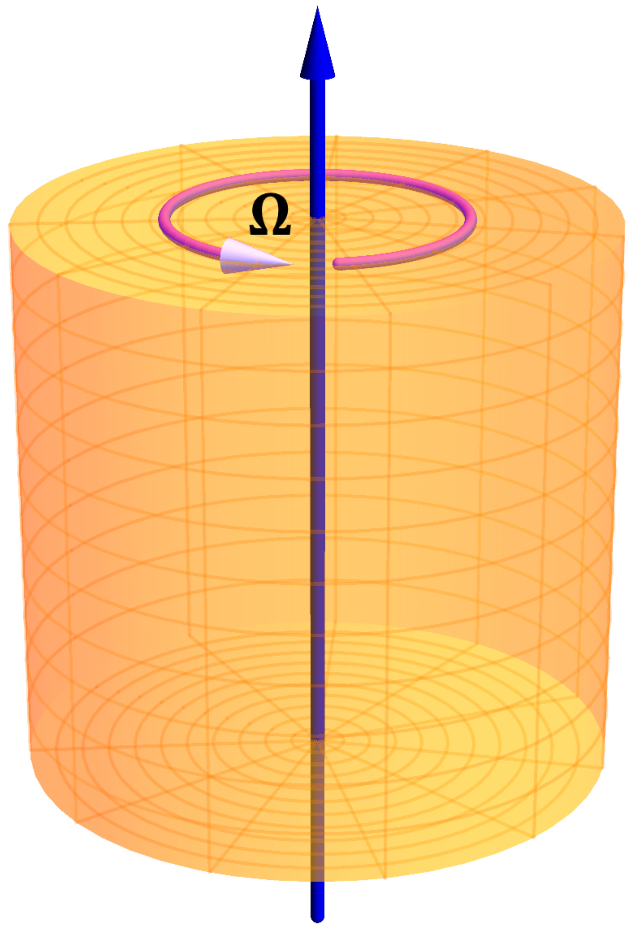
Very non-trivial to get all consistent such b.c. by hand

Classification up to 6D possible

Some of 26 do have rotating frame interpretation

What did people do before?

What did people do before?



Chernodub, Gongyo 1611.02598

What did people do before?

Finite cylindrical slab

Introduction rotation by going to rotating frame Chernodub, Gongyo 1611.02598

$$(t, x, y, z) = (t, r\cos\phi, r\sin\phi, z) \qquad \tilde{\phi} = \phi - \omega t$$

$$g_{\mu\nu}(x) = \begin{pmatrix} 1-(x^2+y^2)\omega^2 & y\omega & -x\omega & 0 \\ y\omega & -1 & 0 & 0 \\ -x\omega & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Lagrangian in curved space-time, $\mathcal{L}(\omega)$, option 1.

What did people do before?

Why is this not ideal?

Temporal b.c. the same as before, but b.c. on the cylinder?

Discretization awful, breaks $SO(2)$, curved space, continuum limit?, ...

Negative moment of inertia?

Our setup: flat space, spatial b.c. straightforward, easy to discretize

Minor changes relative to $\omega = 0$

Compact, flat, orientable 4-manifolds

Beyond topological classification, need to classify flat metrics on $\mathcal{M} = \mathbb{T}^4/G$

These come from flat metrics on $\mathbb{T}^4$

First classify flat metrics on $\mathbb{T}^4$ – very easy

See which are compatible with action of G

Flat metrics on $\mathbb{T}^4$

$\mathbb{T}^4 = \mathbb{R}^4/\Lambda$ **with lattice** $\Lambda \subset \mathbb{R}^4$ **defined by basis vectors** e^a

$$\Lambda = \left\{ \sum_{a=0}^{3} n_a e^a \mid n_a \in \mathbb{Z} \right\}$$

Standard flat metric on $\mathbb{R}^4$

$$\mathcal{M} = \mathbb{T}^4/G = \mathbb{R}^4/(G \ltimes \Lambda) \qquad\qquad G_\Lambda = G \ltimes \Lambda$$

Compact, flat, orientable 4-manifolds

Usual $\mathbb{T}^4$: $|e^0| = \frac{1}{T}$ and $|e^{1,2,3}| = L$ and orthogonal

Will need non-orthogonal $\mathbb{T}^4$ too

4 vectors into 4×4 matrix $e = (e^1|e^2|e^3|e^4)$

Flat metric on $\mathcal{M}$ fully determined by e and G, topology by G_Λ

$$vol(\mathcal{M}) = \frac{|\det(e)|}{|G|} \qquad \pi_1(\mathcal{M}) = G_\Lambda$$

Main task: for each G_Λ find most general e

Look at first $G = \mathbb{Z}_k$ with $k = 2, 3, 4, 6$ (non-abelian or $\mathbb{Z}_2 \times \mathbb{Z}_2$ also possible)

Rotation by $\vartheta = \frac{2\pi}{k}$

Choose metrics, $GL(4, \mathbb{Z})$ and $SO(4)$ freedom

$$e = \begin{pmatrix} L_0 & 0 \\ 0 & E \end{pmatrix}$$

With E a 3×3 matrix $E = \left(E^1 | E^2 | E^3\right)$, defining spatial $\mathbb{T}^3$

$G = \mathbb{Z}_k$ defined by generator $g = (A, b)$ where $A \in SO(4)$ and $b \in \mathbb{R}^4$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\pi/k) & -\sin(2\pi/k) & 0 \\ 0 & \sin(2\pi/k) & \cos(2\pi/k) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} L_0/k \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Clearly $g^k = 1$ and $T = \frac{k}{L_0}$

Two topologically different $k = 2, 3, 4$ and single $k = 6$

$G = \mathbb{Z}_{2,3,4,6}$ and $G = \mathbb{Z}'_{2,3,4}$

No other k possible

Boundary conditions

$$\phi\left(t+\frac{L_0}{k},x_1,x_2,x_3\right)=\phi(t,x_1',x_2',x_3) \qquad \phi\left(t,\boldsymbol{x}+E^a\right)=\phi(t,\boldsymbol{x})$$

$$x_1' = \quad x_1\cos(2\pi/k)+x_2\sin(2\pi/k)$$

$$x_2' = -x_1\sin(2\pi/k)+x_2\cos(2\pi/k)$$

Rotated in temporal direction, periodic spatially

Very easy to implement on lattice

Need to specify metrics E for each case, $G = \mathbb{Z}_{2,3,4,6}$ and $G = \mathbb{Z}'_{2,3,4}$

$$E(\mathbb{Z}_2) = \frac{L}{|z_2 z_3|^{1/3}} \begin{pmatrix} 1 & 0 & 0 \\ z_1 & z_2 & 0 \\ 0 & 0 & z_3 \end{pmatrix} \qquad E(\mathbb{Z}_3) = L \left(\frac{\sqrt{3}}{2|z|} \right)^{1/3} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & z \end{pmatrix}$$

$$E(\mathbb{Z}_4) = \frac{L}{|z|^{1/3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{pmatrix} \qquad E(\mathbb{Z}_6) = E(\mathbb{Z}_3)$$

z **parameters continuous, dimensionless, arbitrary,** $vol(\mathbb{T}^3) = |\det(E)| = L^3$

$$E(\mathbb{Z}_2') = \frac{L}{(2|z_2 z_3|)^{1/3}} \begin{pmatrix} z_1 & -z_1 & z_3 \\ z_2 & -z_2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad E(\mathbb{Z}_3') = L\frac{2^{1/3}}{|z|^{1/3}\sqrt{3}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \\ -z & -z & z \end{pmatrix}$$

$$E(\mathbb{Z}_4') = \frac{L}{|z|^{1/3}} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & z \end{pmatrix}$$

QFT on rotating boxes at finite temperature

Correspond to all possible rotations of a parallelopiped by $\vartheta = \frac{2\pi}{k}$

Note: orthogonal $\mathbb{T}^3$ **possible with** $k = 2, 3, 4$ **(not** $k = 6$**)**

For $k = 6$ **angles:** $(90^o, 60^o, 60^o)$

Summary

Defined QFT, $Z(\omega)$, on rotating box at finite T in path integral formulation

Action same as before, rotated boundary conditions

Imaginary ω not arbitrary but 4 discrete values

Space-time still flat, spatially periodic

Easy to discretize and do lattice, clear continuum limit

Easy to compare with $\omega = 0$

Can also introduce moving frame $y = (0, 0, y_3)$

Summary

Gauge theory straightforward

All 7 have spin structure $\rightarrow$ fermions

Opens door to lattice simulations

Can address negative moment of inertia?

Only 7 compact, flat, orientable manifolds, there are 19 more (16 spin)

Lots of other applications beyond rotating box

Thank you for your attention!

Outlook

Finite volume effects on $\mathbb{T}^3$

If orthogonal: $\sim e^{-mL}$

If not orthogonal: $\sim e^{-\alpha mL}$ with $\alpha > 1$ possible

Exponentially smaller finite volume effects

Largest $\alpha = 2^{1/6}$

$\alpha > 1$ can be realized with all $k = 2, 3, 4, 6$