Hadronic vacuum polarization contribution to the muon magnetic moment from lattice QCD

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[Editors’ Suggestion])
Lepton magnetic moments and BSM physics
Interaction with an external EM field: Dirac eqn

Dirac eqn w/ minimal coupling:

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ \vec{\alpha} \cdot \left( c \frac{\hbar}{i} \vec{\nabla} - e_\ell \vec{A} \right) + \beta c^2 m_\ell + e_\ell A_0 \right] \psi$$

nonrelativistic limit ↓ (Pauli eqn)

$$i\hbar \frac{\partial \phi}{\partial t} = \left[ \left( \frac{\hbar}{i} \vec{\nabla} - \frac{e_\ell}{c} \vec{A} \right)^2 - \frac{e_\ell \hbar}{2m_\ell} \vec{\sigma} \cdot \vec{B} + e_\ell A_0 \right] \phi$$

with

$$\vec{\mu}_\ell = g_\ell \left( \frac{e_\ell}{2m_\ell} \right) \vec{S}, \quad \vec{S} = \hbar \vec{\sigma}$$

and

$$g_\ell \mid_{\text{Dirac}} = 2$$
Assuming Poincaré invariance and current conservation ($q^\mu J_\mu = 0$ with $q \equiv p' - p$):

$$
\langle \ell(p')|J_\mu(0)|\ell(p)\rangle = \bar{u}(p') \left[ \gamma_\mu F_1(q^2) + \frac{i}{2m_\ell} \sigma_{\mu\nu} q^\nu F_2(q^2) - \gamma_5 \sigma_{\mu\nu} q^\nu F_3(q^2) + \gamma_5 (q^2 \gamma_\mu - 2m_\ell q_\mu) F_4(q^2) \right] u(p)
$$

- $F_1(q^2) \rightarrow$ Dirac form factor: $F_1(0) = 1$
- $F_2(q^2) \rightarrow$ Pauli form factor, magnetic dipole moment: $F_2(0) = a_\ell = \frac{g_\ell}{2}$
- $F_3(q^2) \rightarrow$ $\not{\mathcal{P}}$, $\not{T}$, electric dipole moment: $F_3(0) = d_\ell / e_\ell$
- $F_4(q^2) \rightarrow$ $\not{\mathcal{P}}$, anapole moment: $\vec{\sigma} \cdot (\vec{\nabla} \times \vec{B})$

- $F_2(q^2)$ & $F_{3,4}(q^2)$ come from loops but UV finite once theory’s couplings are renormalized (in a renormalizable theory)

- $a_\ell$ dimensionless
  - corrections including only $\ell$ and $\gamma$ are mass independent, i.e. universal
  - contributions from particles w/ $M \gg m_\ell$ are $\propto (m_\ell / M)^2 p \times \ln^q (m_\ell^2 / M^2)$
  - contributions from particles w/ $m \ll m_\ell$ are e.g. $\propto \ln^2 (m_\ell^2 / m^2)$
Why are $a_\ell$ special?

- Loop induced $\Rightarrow$ sensitive to new dofs
- CP and flavor conserving, chirality flipping $\Rightarrow$ complementary to other measurements: EDMs, $b \rightarrow s\ell^+\ell^-$, $\mu \rightarrow e\gamma$, $B \rightarrow D^{(*)}\ell\nu_\ell$, EW precision observables, LHC direct searches, ... 
- In EW theory, only source of chirality flips is $y_\ell \bar{\ell}_L H \ell_R$
  
  $$m_\ell = y_\ell \langle H \rangle, \quad a_{\ell}^{\text{weak}} \propto \frac{\alpha}{4\pi} \left( \frac{m_\ell}{M_W} \right)^2$$

- BSM can be very different
  
  $$a_{\ell}^{N\Phi} \propto \left( \frac{\Delta_{N\Phi}^m m_\ell}{m_\ell} \right) \left( \frac{m_\ell}{M_{N\Phi}} \right)^2$$

Laurent Lellouch
ELTE, Budapest, 24 April 2019
Why is $a_\mu$ special?

$m_e : m_\mu : m_\tau = 0.0005 : 0.106 : 1.777 \text{ GeV} \quad \tau_e : \tau_\mu : \tau_\tau = \infty : 2 \cdot 10^{-6} : 3 \cdot 10^{-15} \text{ s}$

- $a_\mu$ is $(m_\mu / m_e)^2 \sim 4 \times 10^4$ times more sensitive to new $\Phi$ than $a_e$

- $a_\tau$ is even more sensitive to new $\Phi$, but is too shortly lived

(Miller et al '12)
Big question

If not, what is the new $\Phi$ and can it be seen elsewhere?
Experimental measurement of $a_\mu$
Measurement principle for $a_\mu$

\[ \vec{\mu}_\mu = 2(1 + a_\mu) \frac{Qe}{2m_\mu} \vec{S}, \quad \vec{d}_\mu = \eta_\mu \frac{Qe}{2m_\mu c} \vec{S} \]

\[ \vec{\omega}_{a\eta} = \vec{\omega}_a + \vec{\omega}_\eta = -\frac{Qe}{m_\mu} \left[ a_\mu \vec{B} + \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \frac{\vec{\beta} \times \vec{E}}{c} \right] - \eta_\mu \frac{Qe}{2m_\mu} \left[ \frac{\vec{E}}{c} + \vec{\beta} \times \vec{B} \right] \]

- Experiment measures very precisely $\vec{B}$ with $|\vec{B}| \gg |\vec{E}|/c$ &
  \[ \Delta \omega \equiv \omega_S - \omega_C \simeq \sqrt{\omega_a^2 + \omega_\eta^2} \simeq \omega_a \]
  since $d_\mu = 0.1(9) \times 10^{-19} \text{e} \cdot \text{cm}$ (Benett et al '09)

- Consider either magic $\gamma = 29.3$ (CERN/BNL/Fermilab) or $\vec{E} = 0$ (J-PARC)

  \[ \rightarrow \Delta \omega \simeq -a_\mu B \frac{Qe}{m_\mu} \]
Two new experiments plan to reduce error on $a_\mu$ to $\sim 0.14$ ppm

- New $g-2$ (E989) @ Fermilab: has started taking data fall 2017
- $g-2$/EDM (E34) @ J-PARC: should start taking data $\geq 2021$
Standard model calculation of $a_\mu$

\[ a^{\text{SM}}_\mu = a^{\text{QED}}_\mu + a^{\text{had}}_\mu + a^{\text{weak}}_\mu \]

\[ = O\left(\frac{\alpha}{\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{g_2}{4\pi}\right)^2 \left(\frac{m_\mu}{M_W}\right)^2\right) \]

\[ = O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right) \]
QED contributions to $a_\ell$

Loops with only photons and leptons

$$a_{\text{QED}}^{\ell} = C_{\ell}^{(2)} \left( \frac{\alpha}{\pi} \right) + C_{\ell}^{(4)} \left( \frac{\alpha}{\pi} \right)^2 + C_{\ell}^{(6)} \left( \frac{\alpha}{\pi} \right)^3 + C_{\ell}^{(8)} \left( \frac{\alpha}{\pi} \right)^4 + C_{\ell}^{(10)} \left( \frac{\alpha}{\pi} \right)^5 + \cdots$$

$$C_{\ell}^{(2n)} = A_1^{(2n)} + A_2^{(2n)} (m_\ell / m_{\ell'}) + A_3^{(2n)} (m_\ell / m_{\ell'}, m_\ell / m_{\ell''})$$

- $A_1^{(2)}$, $A_1^{(4)}$, $A_1^{(6)}$, $A_2^{(4)}$, $A_2^{(6)}$, $A_3^{(6)}$ known analytically (Schwinger ’48; Sommerfield ’57, ’58; Petermann ’57; ...)

- $O((\alpha/\pi)^3)$: 72 diagrams (Laporta et al ’91, ’93, ’95, ’96; Kinoshita ’95)

- $O((\alpha/\pi)^4; (\alpha/\pi)^5)$: 891;12,672 diagrams (Laporta ’95; Aguilar et al ’08; Aoyama, Kinoshita, Nio ’96–’18)
  - Automated generation of diagrams
  - Numerical evaluation of loop integrals
  - Only some diagrams are known analytically
5-loop QED diagrams

(Aoyama et al '15)
QED contribution to $a_\mu$

$$a_\mu^{\text{QED}}(Rb) = 1165847189.51(9)m(19)\alpha^4(7)\alpha^5(77)\alpha(Rb) \times 10^{-12}$$

$$a_\mu^{\text{QED}}(a_e) = 1165847188.41(7)m(17)\alpha^4(6)\alpha^5(28)\alpha(a_e) \times 10^{-12}$$

(Aoyama et al '12, '18)

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} = 737.2(6.3) \times 10^{-10}$$

$$? = a_\mu^{\text{weak}} + a_\mu^{\text{had}}$$
Weak contributions to $a_\mu$: $Z$, $W$, $H$, etc. loops

$$a_{\text{weak},(1)}^\mu = O \left( \frac{\sqrt{2} G_F m_\mu^2}{16\pi^2} \right)$$

$$= 19.480(1) \times 10^{-10}$$

(Gnendiger et al '15 and refs therein)

$$a_{\text{weak},(2)}^\mu = O \left( \frac{\sqrt{2} G_F m_\mu^2 \alpha}{16\pi^2 \pi} \right)$$

$$= -4.12(60) \times 10^{-10}$$

(Gnendiger et al '15 and refs therein)

$$a_{\mu}^\text{weak} = 15.36(10) \times 10^{-10}$$
Hadronic contributions to $a_\mu$: quark and gluon loops

$$a_{\mu}^{\text{exp}} - a_{\mu}^{\text{QED}} - a_{\mu}^{\text{weak}} = 721.8(6.3) \times 10^{-10} \equiv a_{\mu}^{\text{had}}$$

Clearly right order of magnitude:

$$a_{\mu}^{\text{had}} = O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_{\mu}}{M_\rho}\right)^2\right) = O\left(10^{-7}\right)$$

Write

$$a_{\mu}^{\text{had}} = a_{\mu}^{\text{LO-HVP}} + a_{\mu}^{\text{NLO-HVP}} + a_{\mu}^{\text{HLbyL}} + O\left(\left(\frac{\alpha}{\pi}\right)^4\right)$$
Hadronic contributions to $a_\mu$: diagrams

\[ \rightarrow a_\mu^{\text{LO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^2\right) \]

\[ \rightarrow a_\mu^{\text{NLO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right) \]

\[ \rightarrow a_\mu^{\text{HLbyL}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right) \]
LO-HVP from $e^+ e^- \rightarrow \text{had}$

Use (Bouchiat et al 61) optical theorem (unitarity)

\[ \text{Im}[\text{hadrons}] \propto |\text{hadrons}|^2 \Rightarrow \text{Im} \Sigma(s) = -\frac{R(s)}{12\pi}, \quad R(s) \equiv \frac{\sigma(e^+ e^- \rightarrow \text{had})}{4\pi\alpha(s)^2/(3s)} \]

and a dispersion relation w/ data for $R(s)$ (CMD-2&3, SND, BES, KLOE '08,'10&'12, BABAR '09, etc.)

\[
\hat{\Sigma}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s + Q^2)} \frac{1}{\pi} \text{Im} \Sigma(s) = \frac{Q^2}{12\pi^2} \int_0^\infty ds \frac{1}{s(s + Q^2)} R(s)
\]

Three recent determinations:

\[
a_{\mu}^{\text{LO-HVP}} = 693.27(2.46) \times 10^{-10} \quad [3.5\%] \quad \text{(KNT '18)}
\]
\[
= 693.1(3.4) \times 10^{-10} \quad [4.9\%] \quad \text{(DHMZ '17)}
\]
\[
= 688.1(4.1) \times 10^{-10} \quad [6.0\%] \quad \text{(Jegerlehner '17)}
\]

Higher orders:

\[
a_{\mu}^{\text{NLO-HVP}} = -9.87(0.09) \times 10^{-10} \quad \text{(Kurz et al '14)}
\]
\[
a_{\mu}^{\text{NNLO-HVP}} = 1.24(0.01) \times 10^{-10} \quad \text{(Kurz et al '14)}
\]
Standard model prediction and comparison to experiment
## SM prediction vs experiment

<table>
<thead>
<tr>
<th>SM contribution</th>
<th>$a_\mu^{\text{contrib.}} \times 10^{11}$</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>QED [5 loops]</strong></td>
<td>116584718.841 ± 0.034</td>
<td>[Aoyama et al '18]</td>
</tr>
<tr>
<td><strong>HVP LO</strong></td>
<td>6933 ± 25</td>
<td>[KNT '18]</td>
</tr>
<tr>
<td></td>
<td>6931 ± 34</td>
<td>[DHMZ '17]</td>
</tr>
<tr>
<td></td>
<td>6881 ± 41</td>
<td>[Jegerlehner '17]</td>
</tr>
<tr>
<td><strong>HVP NLO</strong></td>
<td>−98.7 ± 0.9</td>
<td>[Kurz et al '14]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[Kurz et al '14, Jegerlehner '16]</td>
</tr>
<tr>
<td><strong>HVP NNLO</strong></td>
<td>12.4 ± 0.1</td>
<td>[Kurz et al '14]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[Jegerlehner '16]</td>
</tr>
<tr>
<td><strong>HLbyL</strong></td>
<td>105 ± 26</td>
<td>[Prades et al '09]</td>
</tr>
<tr>
<td></td>
<td>54 ± 14 ± ??</td>
<td>[RBC '16]</td>
</tr>
<tr>
<td><strong>Weak (2 loops)</strong></td>
<td>153.6 ± 1.0</td>
<td>[Gnendiger et al '15]</td>
</tr>
<tr>
<td><strong>SM Tot</strong> [0.31 ppm]</td>
<td>116591824 ± 36</td>
<td>[w/ KNT '18]</td>
</tr>
<tr>
<td></td>
<td>116591822 ± 43</td>
<td>[w/ DHMZ '17]</td>
</tr>
<tr>
<td></td>
<td>116591772 ± 49</td>
<td>[Jegerlehner '17]</td>
</tr>
<tr>
<td><strong>Exp [0.54 ppm]</strong></td>
<td>116592091 ± 63</td>
<td>[Bennett et al '06]</td>
</tr>
<tr>
<td><strong>Exp − SM</strong></td>
<td>267 ± 72</td>
<td>[KNT '18]</td>
</tr>
<tr>
<td></td>
<td>269 ± 76</td>
<td>[DHMZ '17]</td>
</tr>
<tr>
<td></td>
<td>319 ± 80</td>
<td>[Jegerlehner '17]</td>
</tr>
</tbody>
</table>
Today $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} \simeq 3.5 \div 4.0 \sigma$

Fermilab E989 began fall 2017 and aims for 0.14 ppm

J-PARC E34 to begin $\geq$ 2021 and aims for similar precision

If both central values stay the same:

- E989 alone (exp. error /4) $\rightarrow \sim 6 \div 7 \sigma$
- E989 + new HLbyL (w/ error $\sim 10\%$) $\rightarrow \sim 7 \div 9 \sigma$
- E989 + new HLbyL + new HVP (w/ error /2) $\rightarrow \sim 7 \div 12 \sigma$

Must have a completely independent first principles determination of the 2 contributions w/ leading theory errors before concluding the presence of BSM physics

$\Rightarrow$ Lattice QCD

Discrepancy is large: 2× electroweak contribution

No new physics scenario, i.e. (experiment) − (other contributions)

- $a_\mu^{\text{LO-HVP}}|_{\text{no-N}\Phi} = 720.0(6.8) \times 10^{-10}$, i.e. 4% larger than DHMZ ’17
- $a_\mu^{HLbyL}|_{\text{no-N}\Phi} = 37.9(7.1) \times 10^{-10}$, i.e. 3.6× (Prades et al ’09)
A brief introduction to lattice QCD
What is lattice QCD (LQCD)?

To describe ordinary matter, QCD requires $\geq 10^4$ numbers at every point of spacetime $\rightarrow \infty$ number of numbers in our continuous spacetime$\rightarrow$ must temporarily “simplify” the theory to be able to calculate (*regularization*)$\Rightarrow$ Lattice gauge theory $\rightarrow$ mathematically sound definition of NP QCD:

- UV (& IR) cutoff $\rightarrow$ well defined path integral in Euclidean spacetime:

$$
\langle O \rangle = \int D U D \bar{\psi} D \psi \ e^{-S_G - \int \bar{\psi} D[M] \psi} \ O[U, \psi, \bar{\psi}]
$$

$$
= \int D U e^{-S_G} \ \det(D[M]) \ O[U]_{\text{Wick}}
$$

- $D U e^{-S_G} \ \det(D[M]) \geq 0 \ & \ \text{finite \ # \ of \ dofs}$
  $\rightarrow$ evaluate numerically using stochastic methods

LQCD is QCD when $m_q \rightarrow m_q^{\text{ph}}, \ a \rightarrow 0$ (after renormalization), $L \rightarrow \infty$ (and stats $\rightarrow \infty$)

HUGE conceptual and numerical ($O(10^9)$ dofs) challenge
Challenges of a full lattice calculation

To make contact with experiment need:

- **A valid approximation to the SM**
  - at least $u, d, s$ in the sea w/ $m_u = m_d \ll m_s$ ($N_f=2+1$)
  - better also include $c$ ($N_f=2+1+1$) & $m_u \leq m_d$ ($N_f=4 \times 1$) & EM ($N_f=4 \times 1 + \text{QED}$)

- **$u$ & $d$ w/ masses well w/in $SU(2)$ chiral regime** : $\sigma_\chi \sim (M_\pi/4\pi F_\pi)^2$
  - $M_\pi \sim 135 \text{ MeV}$ or many $M_\pi \leq 400 \text{ MeV}$ w/ $M_\pi^{\text{min}} < 200 \text{ MeV}$ for $M_\pi \rightarrow 135 \text{ MeV}$

- **$a \rightarrow 0$** : $\sigma_a \sim (a\Lambda_{\text{QCD}})^n, (am_q)^n, (a|\bar{p}|)^n$ w/ $a^{-1} \sim 2 \div 4 \text{ fm}$
  - at least 3 $a$'s $\leq 0.1 \text{ fm}$ for $a \rightarrow 0$

- **$L \rightarrow \infty$** : $\sigma_L \sim (M_\pi/4\pi F_\pi)^2 \times e^{-LM_\pi}$ for stable hadron pties, $\sim 1/L^n$ for resonances, QED, ...
  - many $L$ w/ $(LM_\pi)^{\text{max}} \sim 4$ for stable hadrons & better otherwise to allow for $L \rightarrow \infty$

- **These requirements $\Rightarrow O(10^9)$ dofs** that have to be integrated over

- **Renormalization** : best done nonperturbatively

- **A signal** : $\sigma_{\text{stat}} \sim 1/\sqrt{N_{\text{meas}}}$, reduce w/ $N_{\text{meas}} \rightarrow \infty$
Lattice QCD calculation of $a_{\mu}^{HVP}$
Consider in Euclidean spacetime \(^{(\text{Blum '02})}\)

\[
\Pi_{\mu\nu}(Q) = \int d^4 x \, e^{iQ \cdot x} \langle J_\mu(x)J_\nu(0) \rangle
\]

w/ \(J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots\)

Then \(^{(\text{Lautrup et al '69, Blum '02})}\)

\[
a^{\text{LO-HVP}}_\ell = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{dQ^2}{m_\ell^2} \, w(Q^2/m_\ell^2) \hat{\Pi}(Q^2)
\]

w/ \(\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0)\right]\)

Integrand peaked for \(Q \sim (m_\ell/2) \sim 50\text{ MeV}\) for \(\mu\)

However, \(Q_{\text{min}} \equiv \frac{2\pi}{T} \sim 135\text{ MeV}\) for lattice w/ \(T = \frac{3}{2}L \sim 9\text{ fm}\)

(HVP from Jegerlehner, “\text{alphaQEDc17}” (2017))
Low-$Q^2$ challenges in finite volume (FV)

A. Must subtract $\Pi_{\mu\nu}(Q = 0) \neq 0$ in FV that contaminates $\Pi(Q^2) \sim \Pi_{\mu\nu}(Q)/Q^2$ for $Q^2 \to 0$ w/ very large FV effects

B. On-shell renormalization requires $\Pi(0)$ which is problematic (see above)

C. Need $\hat{\Pi}(Q^2)$ interpolation due to $Q_{\text{min}} = 2\pi/T \sim 135 \text{ MeV} > \frac{m_\mu}{2} \sim 50 \text{ MeV}$ for $T \sim 9 \text{ fm}$

\[ \downarrow \]

- Compute on $T \times L^3$ lattice

\[ C_L(t) = \frac{a^3}{3} \sum_{i=1}^{L} \sum_{\vec{x}} \langle J_i(x) J_i(0) \rangle \]

- Decompose ($C_L^{i=1} = \frac{9}{10} C_L^{ud}$)

\[ C_L(t) = C_L^{ud}(t) + C_L^s(t) + C_L^c(t) + C_L^{\text{disc}}(t) \]

\[ = C_L^{i=1}(t) + C_L^{i=0}(t) \]

- Define (Bernecker et al ’11, BMWc ’13, Feng et al ’13, Lehner ’14, …) (ad A, B, C)

\[ \hat{\Pi}_L^f(Q^2) \equiv \Pi_L^f(Q^2) - \Pi_L^f(0) = \frac{1}{3} \sum_{i=1}^{3} \frac{\Pi_{ii,L}(0) - \Pi_{ii,L}(Q)}{Q^2} - \Pi_L^f(0) = 2a \sum_{t=0}^{T/2} \text{Re} \left[ \frac{e^{iQt} - 1}{Q^2} + \frac{t^2}{2} \right] \text{Re} C_L^f(t) \]
Our lattice definition of \( a_{\ell, f}^{\text{LO-HVP}} \)

Combining everything, get \( a_{\ell, f}^{\text{LO-HVP}} \) from \( C_L^f(t) \):

\[
a_{\ell, f}^{\text{LO-HVP}}(Q^2 \leq Q_{\text{max}}^2) = \lim_{a \to 0, L \to \infty, T \to \infty} \left( \frac{\alpha}{\pi} \right)^2 \left( \frac{a}{m_\ell^2} \right)^{T/2} \sum_{t=0}^{T/2} W(tm_\ell, Q_{\text{max}}^2/m_\ell^2) \Re C_L^f(t)
\]

where

\[
W(\tau, x_{\text{max}}) = \int_0^{x_{\text{max}}} dx \, w(x) \left( \tau^2 - \frac{4}{x} \sin^2 \frac{\tau \sqrt{x}}{2} \right)
\]

\[(144 \times 96^3, a \sim 0.064 \text{ fm}, M_\pi \sim 135 \text{ MeV})\]
Simulation challenges

D. $\pi\pi$ contribution very important $\rightarrow$ have physically light $\pi$

E. Two types of contributions

- quark-connected (qc)
- quark-disconnected (qd)

where qd contributions are SU(3)$_f$ and Zweig suppressed but very challenging

F. $\langle J_{\mu}^{ud}(x)J_{\nu}^{ud}(0)\rangle_{qc}$ & disc. have very poor signal at large $\sqrt{x^2}$ + need high-precision results

$\rightarrow$ very high statistics + many algorithmic improvements + rigorous bounds

$\rightarrow$ 9M / 39M conn./disc. measurements

G. Must control $\langle J_{\mu}(x)J_{\nu}(0)\rangle$ at $\sqrt{x^2} \gtrsim 2/m_{\mu}$ $\rightarrow L = 6.1 \div 6.6 \text{ fm}, T = 8.6 \div 11.3 \text{ fm}$

H. Need controlled continuum limit $\rightarrow$ have 6 a's: 0.134 $\rightarrow$ 0.064 fm
Continuum limit of $a_{\mu,f}^{\text{LO-HVP}}(Q^2 \leq 4 \text{ GeV}^2)$

- With 6 $a$'s, have full control over continuum limit
- Get good $\chi^2$/dof w/ extrapolation linear in $a^2$ and interpolations, linear in $M^2_\pi$ and $M^2_K$
- Strong continuum extrapolation for $a_{\mu,\text{ud/disc}}^{\text{LO-HVP}}$ due to taste violations and for $a_{\mu,c}^{\text{LO-HVP}}$ due to large $m_c$
- Continuum extrapolation systematic from all results and by cutting results with $a \geq 0.134, 0.111, 0.095 \text{ fm}$
- Obtain other $a_{\mu,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}})$ and $\hat{\Pi}(Q^2_{\text{max}})$, $Q^2_{\text{max}} = 1, \cdots, 5 \text{ GeV}^2$, in entirely analogous fashion
More challenges

I. Need $\hat{\Pi}(Q^2)$ for $Q^2 \in [0, +\infty[$, but $\frac{\pi}{a} \sim 9.7 \text{ GeV}$ for $a \sim 0.064 \text{ fm}$

→ match onto perturbation theory

$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}}) + \gamma_{\ell}(Q_{\text{max}}) \hat{\Pi}^{f}(Q_{\text{max}}^2) + \Delta_{\text{pert}}^{f} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\text{max}})$$

using $O(\alpha_s^4)$ results from Harlander et al '03

J. Include $c$ quark for higher precision and good matching onto perturbation theory → done

K. Even in our large volumes w/ $L \sim 6.1 \text{ fm}$ & $T \geq 8.7 \text{ fm}$, finite-volume (FV) effects can be significant

→ correct using 1-loop SU(2) $\chi$PT (Aubin et al '16)

L. Our $N_f = 2 + 1 + 1$ calculation has $m_u = m_d$ and $\alpha = 0$

⇒ missing effects compared to HVP from dispersion relations that are relevant at %-level precision

→ use phenomenology (F.Jegerlehner & M. Benayoun, private communication)
Systematic errors and results for $a^{\text{LO-HVP}}_{\mu}$

- Stat. error: jackknife
- $a \rightarrow 0$: from 4 (3) cuts on $a$ for conn. (disc.)
- Bounds: from $t_c = 3.000(2.600) \pm 0.134$ fm vs $t_c = 2.866(2.466) \pm 0.134$ fm for conn. (disc.)
- PT match: from $Q^2_{\text{max}} = 2$ GeV$^2$ vs $Q^2_{\text{max}} = 5$ GeV$^2$
- $\delta a \simeq 0.4\% \Rightarrow \delta a a^{\text{LO-HVP}}_{\mu} \simeq 0.8\%$
- FV:
  \[ a^{\text{LO-HVP}}_{\mu}(\infty) - a^{\text{LO-HVP}}_{\mu}(L=6 \text{ fm}) = (13.5 \pm 13.5) \times 10^{-10} \]
  from $\chi$PT
- IB: $\Delta_{\text{IB}} a^{\text{LO-HVP}}_{\mu} = (7.8 \pm 5.1) \times 10^{-10}$ from pheno.

\[
\begin{array}{l|c}
\text{Contrib.} & a^{\text{LO-HVP}}_{\mu} \times 10^{10} \\
\hline
l = 1 & 583(7)(7)(0)(0)(5)(14) \\
\text{Total} & 711(8)(8)(0)(0)(6)(13)(5) \\
\end{array}
\]

Error on total:
- Stat. = 1.1\%
- LQCD syst. = 1.2\%
- FV = 2.3\%
- IB = 0.8\%
- Total = 2.7\%

Compare w/ upper bound (Bell et al '69) using $\Pi_1$ from BMWc, PRD96 = 792(24)
Comparison and outlook
### Comparison

#### Plot

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Value (×10(^{-10}))</th>
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<td>ETM 14</td>
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<tr>
<td>HPQCD 17</td>
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<td>BMWc 17</td>
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<tr>
<td>RBC/UKQCD 18</td>
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</tr>
</tbody>
</table>

**“No New Physics” scenario:**

\[ (720 \pm 7) \times 10^{-10} \]

- **BMWc ’17** consistent w/ “No new physics” scenario & pheno.
- Total uncertainty of 2.7% is \( \sim 6 \times \) pheno. error
- **BMWc ’17** first result to indicate possibly larger result

\[ \rightarrow \] since, all results compatible w/ that value
all calculations are now compatible with BMWc ’17 \(ud\) result

BMWc ’17 is only calculation performed directly at physical quark masses w/ 6 \(a\)'s to fully control continuum extrapolation

most collaborations now agree on \(s\) and \(c\) contributions

\(a_{\mu,LO-HVP}^{\text{LO-HVP}}\) already known with high enough precision for FNAL E989
Where we are

- Calculation of all relevant contributions to $a_{\mu}^{\text{LO-HVP}}$ directly at physical $m_{ud}$

$$a_{\mu}^{\text{LO-HVP}} = 711.0(7.5)(17.5) \times 10^{-10} \quad [2.7\%]$$

- Also have slope and curvature of $\hat{\Pi}(Q^2)$ at $Q^2 = 0$ (PRD96 '17)

- Fully controlled continuum limit and matching to perturbation theory

- Only model/pheno. assumptions for FV, QED and $m_u \neq m_d$ corrections

- Consistent with “no new physics” and dispersive methods, but error $\sim 6 \times$ larger; some tension with HPQCD 16 on $a_{\mu,ud}^{\text{LO-HVP}}$, partially resolved in FHM

- Total error is 2.7%, dominated by poorly controlled FV effects

- Need $\sim 0.2\%$ to match upcoming experiments!

- With same methods, compute (see also ETM '16)

$$a_e^{\text{LO-HVP}} = 189.3(2.6)(5.6) \times 10^{-14}[3.2\%] \leftrightarrow 184.6(1.2) \times 10^{-14}[0.7\%] \quad (\text{Jegerlehner '15})$$

$$a_{\tau}^{\text{LO-HVP}} = 341.3(0.8)(3.2) \times 10^{-8}[1.0\%] \leftrightarrow 338(4) \times 10^{-8}[1.2\%] \quad (\text{Eidelman et al. '07})$$
What next?

Need to reduce our error by 10!

→ Increase statistics by $\times50 \div 100$ (need new methods)

→ Understand and control FV effects much better

→ Compute QED and $m_d \neq m_u$ corrections (see RBC/UKQCD ’17-’18, ETM ’17)

→ Need high precision scale setting

→ Detailed comparison to phenomenology to understand where we agree and why if we don’t

→ Combine LQCD and phenomenology to improve overall uncertainty (RBC/UKQCD ’18), only if the two agree statistically with comparable errors