Topological String Theory and Generalized Geometry

Zoltán Kökényesi

MTA Wigner Research Center for Physics, Budapest

based on 1805.11485

with A. Sinkovics and R. J. Szabo

ELTE, 10 Oct 2018
Outline

Introduction
- Topological string theory
- T-duality
- Generalized geometry

AKSZ sigma-models
- Courant-algebroids and Courant sigma-models
- Double field theory and the corresponding algebraic structures
- AKSZ sigma-models

A/B-models and generalized geometry
- Double field construction
- Relation to generalized geometry
- Topological S-duality from generalized complex structure
Topological string theory

- Where does it come from?

\[ \mathcal{N} = 2 \text{ sigma-model} \]
\[ & \text{\& coupled to gravity} \]
\[ \text{topological sigma-model} \]
\[ & \text{\& coupled to gravity} \]
\[ \rightarrow \text{bosonic string theory} \]
\[ \rightarrow \text{topological string theory} \]

- Procedure to get the topological sigma-model: \( \sim \) topological twisting

\[ \mathcal{N} = 2 \text{ sigma-model} \rightarrow \text{topological sigma-model} \]

- Two non-equivalent twists:

\( \sim \) A- and B-models

- Where does it appear in 'physical' string theory?

  a) IIA or IIB compactifications \( \sim \mathcal{N} = 2 \text{ superpotential in 4 dim} \)
  b) IIA compactification \( \sim \) entropy of BPS black hole
**Topological A-model**

- **A-twist:**
  \[ Q = Q_{++} + Q_{--} \]
  is a cohomological charge \( \sim \) topological theory.

- **A-model action**
  \[
  S_A = 2t \int_{\Sigma_2} d^2z \left( g_{ab} \partial z X^a \partial \bar{z} X^b + i g_{ab} \left( \chi_{\bar{z}}^a \nabla_z \psi^b + \chi_z^b \nabla_{\bar{z}} \psi^a \right) \right. \\
  \left. - R_{abcd} \chi_{\bar{z}}^a \chi_z^b \psi_c \psi_{\bar{d}} \right) + t \int_{\Sigma_2} d^2z \ X^*(k) 
  \]

  where
  \[
  X^a, X^\bar{a} : \Sigma_2 \to X \ \text{Calabi-Yau}, \\
  (\psi^a, \psi^{\bar{a}}) \ \text{fermionic scalars with ghost number 1}, \\
  (\chi_{\bar{z}}^a, \chi_z^a) \ \text{fermionic 1-forms with ghost number } -1, \\
  \]

- **Complex structure deformation \( \rightarrow \) BRST exact term**
  \( \sim \) Quantities depend only on the Kähler structure of \( X \) (\( \leftrightarrow \) Poisson str.)
Topological B-model

- B-twist:
  \[ Q = Q_{+-} + Q_{-+} \]
  is a cohomological charge \( \sim \) topological theory.

- B-model action
  \[
  S_B = t \int_{\Sigma_2} d^2z \left( g_{\bar{a}b} \left( \partial_z X^a \partial_{\bar{z}} X^{\bar{b}} + \partial_{\bar{z}} X^a \partial_z X^{\bar{b}} \right) - g_{\bar{a}b} \left( \rho^a_z \nabla_{\bar{z}} \eta^{\bar{b}} + \rho^a_{\bar{z}} \nabla_z \eta^b \right) \right.
  
  + \left. \rho^a_z \nabla_{\bar{z}} \chi^a - \rho^a_{\bar{z}} \nabla_z \chi_a - R^{a}_{b\bar{c}d} \rho^b_z \rho^d_{\bar{z}} \eta^{\bar{c}} \chi^a \right)
  \]

  where
  \[ X^a, X^{\bar{a}} : \Sigma_2 \to X \text{ Calabi-Yau}, \]
  \[(\eta^{\bar{a}}, \chi^a) \text{ fermionic scalars with ghost number 1,} \]
  \[(\rho^a_z, \rho^a_{\bar{z}}) \text{ fermionic 1-forms with ghost number } -1, \]

- Holomorphic and antiholomorphic fields are explicitly distinguished
  \( \sim \) Quantities depend only on the complex structure of \( X \)
T-duality

- Compactification on a 6 dim torus $X$
  \[ \sim 4 \text{ dim effective theory on } M_4 \]
- In this case there exists a symmetry transformation on the inner space $X$
  
  IIA string theory $\xleftrightarrow{T}^\dagger$ IIB string theory

$\Rightarrow$ The closed string spectrum

\[ M^2 = (N + \tilde{N} - 2) + p^2 \frac{l_s^2}{R^2} + w^2 \frac{R^2}{l_s^2} \]

is invariant under T-duality:

\[ R \xleftrightarrow{T} \frac{l_s^2}{R} \]

- string momenta ($p_\mu$) $\xleftrightarrow{T}$ winding numbers ($w_\mu$)

- Generalized tangent bundle:

\[ \left( p_\mu \, dx^\mu, w_\mu \, \partial_\mu \right) \in T^*X \oplus TX \xrightarrow{\Rightarrow} \text{Generalized geometry} \]
Generalized geometry

- ’Generalization’ is based on two premises:
  
  1) $TX$ is replaced by $TX \oplus T^*X$
  
  2) Lie bracket on $TX$ is replaced by the Courant-bracket

$$[A + \alpha, B + \beta]_C = [A, B]_{TX} + \mathcal{L}_A \beta - \mathcal{L}_B \alpha - \frac{1}{2} d(\iota_A \beta - \iota_B \alpha)$$

for $A, B \in TX$ and $\alpha, \beta \in T^*X$.

- T-duality is a covariant $O(d, d)$ symmetry group ($d = \dim X$), which leaves the symmetric pairing on $TX \oplus T^*X$ invariant

$$\langle A + \alpha, B + \beta \rangle = \frac{1}{2} (\iota_A \beta + \iota_B \alpha)$$

- Unified framework for symplectic geometry and complex geometry:

  $\leadsto$ within generalized complex geometry

Different choice of gen. complex str. $\xrightarrow{\sim}$ symplectic str. $\xrightarrow{\sim}$ complex str.
Fluxes in string theory

- After compactification, the theory is left with massless scalar fields with no potential (called moduli) $\sim$ instability.
- Solution: introducing fluxes ($H = dB$ or RR-fields).
  $\Rightarrow$ stabilizes the moduli & breaks supersymmetry.

- We have a chain of fluxes acting with T-duality:
  
  \[
  H_{\mu \nu \rho} \xlongleftarrow{T} F_{\mu \nu \rho} \xlongleftarrow{T} Q^{\mu \nu \rho} \xlongleftarrow{T} R^{\mu \nu \rho}
  \]

  torsion of the metric  non-comm. space  non-assoc. space

  We need all fluxes to stabilize the moduli!

- The fluxes back-react on the compactified geometry:

  Compact. without fluxes $\rightarrow$ \begin{align*}
  M_4 : \mathcal{N} = 2 \text{ SuSy} \\
  X : \text{Calabi-Yau}
  \end{align*}

  Compact. with fluxes $\rightarrow$ \begin{align*}
  M_4 : \mathcal{N} = 1 \text{ SuSy} (!) \\
  X : \text{Generalized Calabi-Yau}
  \end{align*}
Courant algebroid

- Algebraic construction to the Courant bracket is called Courant algebroid.
- Vector bundle $E \rightarrow X$ equipped with
  
  \[ \langle \cdot , \cdot \rangle : \Gamma(E) \times \Gamma(E) \rightarrow \mathbb{R} \quad \text{non-degen. bilin. form} \]
  
  \[ \rho : E \rightarrow TX \quad \text{anchor map} \]
  
  \[ \llbracket \cdot , \cdot \rrbracket : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E) \quad \text{Dorfman bracket} \]

  which together satisfy

  \[ \llbracket e_1, \llbracket e_2, e_3 \rrbracket \rrbracket = \llbracket \llbracket e_1, e_2 \rrbracket, e_3 \rrbracket + \llbracket e_2, \llbracket e_1, e_3 \rrbracket \rrbracket \]
  
  \[ \rho(e_1) \langle e_2, e_3 \rangle = \langle \llbracket e_1, e_2 \rrbracket, e_3 \rangle + \langle e_2, \llbracket e_1, e_3 \rrbracket \rangle \]
  
  \[ \rho(e_1) \langle e_2, e_3 \rangle = \langle e_1, \llbracket e_2, e_3 \rrbracket + \llbracket e_3, e_2 \rrbracket \rangle \]

- Courant bracket is the antisymmetrization of a particular Dorfman bracket.
- Fluxes appear as twist deformations of Courant algebroids.

  e.g. for H-flux: \[ \llbracket A + \alpha, B + \beta \rrbracket_H = [A,B]_{TM} + \mathcal{L}_A \beta - \iota_B d\alpha + \iota_A \iota_B H \]
QP-manifolds

- Theorem by Roytenberg
  
  Courant algebroid \( \leftrightarrow \) QP-manifold of degree 2

- \((\mathcal{M}, Q_\gamma, \omega)\) is a QP-manifold of degree \(n\) \((\mathcal{M}: \text{dg-manifold})\)
  
  \(\omega\): graded symplectic structure of degree \(n + 2\)
  
  \(\to \{ ., . \}\) graded Poisson bracket of degree \(-n\)

  \(Q_\gamma\): Hamiltonian cohomological vector field \(Q_\gamma = \{ \gamma, . \}\)

  \[Q_\gamma^2 = 0 \iff \{ \gamma, \gamma \} = 0\] master equation

- Courant algebroid operations \((n = 2)\) on degree 1 functions \(e_1, e_2\) are

  \[[e_1, e_2] = \{ \{ e_1, \gamma \}, e_2 \}\quad \langle e_1, e_2 \rangle = \{ e_1, e_2 \}\quad \rho(e) = \{ e, \{ \gamma, \cdot \} \}\]

- Standard Courant algebroid \((E = TX \oplus T^*X, \mathcal{M} = T^*[2]T[1]X)\)

  \[
  \omega = dX_i^0 \wedge dF_i + d\chi_i^0 \wedge d\psi_j^1 \quad \Rightarrow \quad \{ X^i, F_j \} = \delta_j^i \quad \& \quad \{ \chi_i, \psi^j \} = \delta_i^j
  \]

  \[\gamma = F_i \psi^i \quad \rightsquigarrow \quad \text{Courant bracket}\]
Courant sigma-models

- General QP-manifold of degree 2

\[ \omega = \sum_{0}^{1} \omega_i = dX^i \wedge dF_i + \frac{1}{2} k_{ab} \zeta^a \wedge \zeta^b \]

\[ \gamma = \rho^i(X) F_i \zeta^a + \frac{1}{3!} T_{abc}(X) \zeta^a \zeta^b \zeta^c \]

- It corresponds to a 3 dim topological sigma-model: Courant sigma-model

A coordinate \( \phi \) with degree \( |\phi| \rightarrow \) field \( \phi(\sigma) \) with ghost number \( |\phi| \)

\[ S_C = \int_{\Sigma_3} d^3 \sigma \left( F_i \wedge dX^i + \frac{1}{2} k_{ab} \zeta^a \wedge d\zeta^b + \rho^i(X) F_i \wedge \zeta^a \right. 
\]

\[ \left. + \frac{1}{3!} T_{abc}(X) \zeta^a \wedge \zeta^b \wedge \zeta^c \right) \]

Eq. of motion \( \iff \{\gamma, \gamma\} = 0 \iff \) Courant algebroid axioms

- BV-BRST quantization:

\( \implies \) AKSZ construction
AKSZ sigma-models

- **AKSZ construction**: a geometric method for constructing BV quantized topological sigma-models.

  [Alexandrov, Kontsevich, Schwartz, Zaboronsky]

  Well known examples are Poisson sigma-model, A/B-models, Chern-Simons theory, Courant sigma-model, etc.

- **Construction**:  
  
  **Source manifold**: \( T[1] \Sigma_d \) graded worldvolume with \( \dim \Sigma_d = d \)  
  
  **Target manifold**: \( (\mathcal{M}, Q_\gamma, \omega) \) QP-manifold of degree \( d - 1 \)

- **The space of fields** is the mapping space \( \mathcal{M} = \text{Map}(T[1] \Sigma_d, \mathcal{M}) \)

  \( \implies \) An arbitrary coordinate \( \phi \in \mathcal{M} \) of degree \( |\phi| \) corresponds to a superfield \( \phi \in \mathcal{M} \) of ghost number \( |\phi| \)

  \[
  \phi(\sigma, \theta) = \phi^{(0)}(\sigma) + \phi^{(1)}_\mu(\sigma) \theta^\mu + \ldots + \frac{1}{d!} \phi^{(d)}_{\mu_1 \ldots \mu_d} \theta^{\mu_1} \ldots \theta^{\mu_d}
  \]

  \( \hat{z} = (\sigma, \theta) \in T[1] \Sigma_d \) are even and odd coordinates respectively.
AKSZ sigma-models

- \( \mathcal{M} = \text{Map}(T[1]\Sigma_d, \mathcal{M}) \) is also a QP-manifold with cohomological vector field \( Q = Q_0 + Q_\gamma \), which in local 'coordinates' \( \phi^\hat{i}(\hat{z}) \) of \( \mathcal{M} \) is

\[
Q_0 = \int_{T[1]\Sigma_d} d^d\hat{z} D\phi^\hat{i}(\hat{z}) \frac{\delta}{\delta \phi^\hat{i}(\hat{z})}, \quad Q_\gamma = \int_{T[1]\Sigma_d} d^d\hat{z} Q^\hat{i}_\gamma(\phi(\hat{z})) \frac{\delta}{\delta \phi^\hat{i}(\hat{z})}
\]

where \( D := \theta^\mu \frac{\partial}{\partial \sigma^\mu} \), and the BV symplectic form is

\[
\omega = \int_{T[1]\Sigma_d} d^d\hat{z} \text{ev}^*(\omega), \quad \text{ev} : (\hat{z}, \phi) \mapsto \phi(\hat{z})
\]

- The AKSZ action is the Hamiltonian function of \( Q = (\mathcal{S}, \cdot)_{\text{BV}} \)

\[
\mathcal{S} = \mathcal{S}_{\text{kin}} + \mathcal{S}_{\text{int}} \quad (\omega = d\theta)
\]

It gives a solution to the master equation

\[
(\mathcal{S}, \mathcal{S})_{\text{BV}} = 0 \quad \longleftrightarrow \quad \{\gamma, \gamma\} = 0
\]

and defines the BV-BRST transformation

\[
Q = (\mathcal{S}, \cdot)_{\text{BV}}
\]
Example: Poisson sigma-model (or A-model)

- Target: $\mathcal{M} = T^*[1]X$ (degree $n = 1$); general Hamiltonian (with $|\gamma| = 2$):
  \[
  \omega = d\chi_i \wedge dX_i^0 \quad \text{and} \quad \gamma = \frac{1}{2} \pi^{ij}(X) \chi_i \chi_j
  \]

- The master equation makes $\pi^{ij}$ to define a Poisson structure
  \[
  \{\gamma, \gamma\} = 0 \quad \longleftrightarrow \quad (\pi^{[i|l} \partial_l \pi^{jk]} = 0)
  \]

- In general, $\gamma$ determines a derived bracket:
  \[
  \{\{f, \gamma\}, g\} = - \{f, g\}_\pi = - \pi(df \wedge dg) \quad \text{Poisson bracket on } C^\infty(X)
  \]

- BV bracket and AKSZ action are
  \[
  (\cdot, \cdot)_{BV} = \int_{T[1]\Sigma_2} d^2\hat{z} \left( \frac{\delta}{\delta X^i} \right) \wedge \left( \frac{\delta}{\delta \chi_i} \right)
  \]
  \[
  S_\pi^{(2)} = \int_{T[1]\Sigma_2} d^2\hat{z} \left( \chi_i DX^i + \frac{1}{2} \pi^{ij} \chi_i \chi_j \right) \quad \text{on-shell quant.} \leftarrow \int_{\Sigma_2} X^*(B)
  \]
Gauge fixing

- **Gauge fixing:**
  1) Assign 'fields' $\phi^a$ and 'antifields' $\phi_a^+$:

  \[ \omega = \int_{\Sigma} d^d \hat{z} \delta \phi_a^+(\hat{z}) \delta \phi^a(\hat{z}) \]

  paired in

  2) Gauge fix the antifields $\iff$ Choose a Lagrangian submanifold $\mathcal{L} \subset \mathcal{M}$ ($\omega|_{\mathcal{L}} = 0$)

  \[ \phi_a^+(\hat{z}) = (-1)^{|\Phi^a|(d+1)} \frac{\delta \Psi}{\delta \phi_a^a(\hat{z})} \Rightarrow \omega|_{\Psi} = 0 \]

- **Example: A-model from AKSZ Poisson sigma-model ($\pi \sim$ Kähler form)**

  \[ X^{(0)i} \rightarrow X^{a,\bar{a}} \quad \chi^{(0)}_i \rightarrow \psi^{a,\bar{a}} \quad X^{(1)}_\mu \rightarrow \chi^{a}_{\bar{z}}, \chi^{\bar{a}}_z \]

  Choosing an appropriate $\Psi$ gauge fixing fermion

  \[ \Rightarrow S_A = 2t \int_{\Sigma} d^2z \left( g_{\bar{a}b} \partial_{\bar{z}} X^a \partial_z X^\bar{b} + ig_{\bar{a}b} \left( \chi^{a}_{\bar{z}} \nabla_z \psi^\bar{b} + \chi^{\bar{b}}_z \nabla_{\bar{z}} \psi^a \right) \right. \]

  \[ \left. - R_{\bar{a}b\bar{c}d} \chi^{a}_{\bar{z}} \chi^{\bar{b}}_{z} \psi^c \psi^d \right) \]
Further examples

- **Complex structure sigma-model** or **B-model** \( (\mathcal{M} = T^*[1] T^* X \text{ doubled}) \)

\[
\omega = d\chi^i_1 \wedge dX^i_0 + d\tilde{\chi}^i_1 \wedge d\tilde{X}^i_0 \quad \text{and} \quad \gamma = J^i_j \chi^j_0 - \partial_j J^i_k \tilde{X}^i_0 \tilde{\chi}^j_0 \tilde{\chi}^k_0
\]

\[
\{\gamma, \gamma\} = 0 \iff \text{Integrability cond. for complex str. } J
\]

\[
S^{(2)}_J = \int_{T[1] \Sigma_2} (\chi^i DX^i - \tilde{X}^i D\tilde{\chi}^i + J^i_j \chi^j_0 + \partial_j J^i_k \tilde{X}^i_0 \tilde{\chi}^j_0 \tilde{\chi}^k_0)
\]

[Ikeda, Tokunaga]

- **Courant sigma-model**; \( \mathcal{M} \): general QP-manifold of degree 2

\[
\omega = dX^i_0 \wedge dF_i + \frac{1}{2} k_{ab} d\zeta^a_1 \wedge d\zeta^b_1
\]

\[
\gamma = \rho_a^i (X) F_i \zeta^a + \frac{1}{3!} T_{abc} (X) \zeta^a \zeta^b \zeta^c
\]

\[
\{\gamma, \gamma\} = 0 \iff \text{Courant algebroid axioms}
\]

\[
S^{(3)} = \int_{\Sigma_3} d^3 \hat{z} \left( F_i DX^i + \frac{1}{2} k_{ab} \zeta^a D\zeta^b + \rho_a^i (X) F_i \zeta^a + \frac{1}{3!} T_{abc} (X) \zeta^a \zeta^b \zeta^c \right)
\]
Double field theory (DFT)

- In string compactification: Conjugate canonical coordinates \((x, \tilde{x})\) to momenta \((p, \tilde{p})\) on \(TX \oplus T^*X\):
  \[
  \Rightarrow \text{Double field theory}
  \]

- A ’section condition’ is needed to reduce a theory to a \(d\) dim slice

- Algebraic structure: \(\sim\) DFT algebroid
  
  [Chatzistavrakidis, Jonke, Khoo, Szabo]

- It reduces to ordinary Courant algebroid after section condition

- Construction:
  1) Start with a doubled Courant algebroid \((X \sim T^*X)\)

  \[
  \omega = dX^I_0 \wedge dF_I + d\chi^I_1 \wedge d\psi^I_1
  \]

  2) Halve the degree 1 coordinates with a so called DFT projection:

  \[
  \chi_i, \tilde{\psi}_i \rightarrow p_i \quad \& \quad \psi_i, \tilde{\chi}^i \rightarrow q^i \quad \Rightarrow \quad \omega = dq^i_1 \wedge dp_i + \ldots
  \]

  \(\sim\) DFT algebroid, C-bracket of DFT
A/B-models and generalized geometry

[1805.11485]
Doubled Poisson sigma-model for A/B-models I.

- A/B-models with H-flux background and generalized complex structures:
  [Kapustin][Kapustin,Li][Pestun,Witten]
- AKSZ construction of A/B-model and generalized complex structure:
  [Zucchini][Pestun][Ikeda,Tokunaga][Stojevic]
- We give a DFT description for their AKSZ theory, which leads to a natural AKSZ theory corresponding to a generalized complex structure
- Poisson sigma-model on doubled targetspace $\mathcal{M} = T^*[1] T^*X$

\[
\omega = d\chi_I \wedge dX^I \quad \text{and} \quad \gamma = \frac{1}{2} \Omega^{ij}(X) \chi_I \chi_J
\]

with

\[
X^I = \begin{pmatrix} X^i \\ \tilde{X}_i \end{pmatrix} \quad \text{and} \quad \chi_I = \begin{pmatrix} \chi_i \\ \tilde{\chi}^i \end{pmatrix}
\]

\[
S^{(2)}_{\Omega} = \int_{T[1]\Sigma^2} d^2 \hat{z} \left( \chi_I D X^I + \frac{1}{2} \Omega^{ij}(X) \chi_I \chi_J \right).
\]
Doubled Poisson sigma-model for A/B-models II.

\[ S^{(2)}_{\Omega} = \int_{T[1]\Sigma_2} d^2 \hat{z} \left( \chi_i \, DX^i + \frac{1}{2} \Omega^{ij} \chi_i \chi_j \right). \]

**Observation I.:**

- It gives the A-model with
  \[ \Omega^{ij} = \begin{pmatrix} \pi^{ij} & 0 \\ 0 & 0 \end{pmatrix} \]

  \[ \Rightarrow S^{(2)}_{\pi} = \int_{T[1]\Sigma_2} d^2 \hat{z} \left( \chi_i \, DX^i + \tilde{\chi}^i D\tilde{X}_i^{GF} + \frac{1}{2} \pi^{ij} \chi_i \chi_j \right) \]

- It also gives the B-model with
  \[ \Omega^{ij} = \begin{pmatrix} 0 & J^i_j \\ -J^i_j & -2 \partial_i J^k_j \tilde{X}_k \end{pmatrix} \]

  \[ \Rightarrow S^{(2)}_{J} = \int_{T[1]\Sigma_2} d^2 \hat{z} \left( \chi_i \, DX^i + \tilde{\chi}^i D\tilde{X}^i + J^i_j \chi_i \tilde{\chi}^j - \partial_j J^i_k \tilde{X}_i \tilde{\chi}^j \tilde{\chi}^k \right) \]
Lift up to AKSZ membrane I.

Observation II.:

- Poisson sigma-model can be lifted up to the membrane level as a contravariant Courant sigma-model

\[ \{\gamma, \gamma\} = 0 \iff \left\{\begin{array}{l}
\pi \text{ Poisson str.} \\
[\pi, R]_S = 0 \quad \text{Schouten bracket on } \Lambda^\bullet TX
\end{array}\right. \]

AKSZ construction: \( \mathcal{M} = T^*[2] T[1] X \)

\[
\omega = \frac{1}{2} dF_i \wedge dX^i + d\chi_i \wedge d\psi^i \\
\gamma = \pi^{ij} F_i \chi_j - \frac{1}{2} \partial_i \pi^{jk} \psi^i \chi^j \chi^k + \frac{1}{3!} R^{ijk} \chi_i \chi_j \chi_k
\]

\[
S^{(3)}_{\pi,R} = \int_{T[1] \Sigma^3} d^3 \hat{z} \left( F_i DX^i - \chi_i D\psi^i + \pi^{ij} F_i \chi_j \\
- \frac{1}{2} \partial_i \pi^{jk} \psi^i \chi_j \chi_k + \frac{1}{3!} R^{ijk} \chi_i \chi_j \chi_k \right)
\]
Lift up to AKSZ membrane II.

\[ S^{(3)}_{\pi,R} = \int_{T[1] \Sigma_3} d^3 \hat{z} \left( F_i DX^i - \chi_i D \psi^i + \pi^{ij} F_i \chi_j \right. \]
\[ \left. - \frac{1}{2} \partial_i \pi^{jk} \psi^i \chi_j \chi_k + \frac{1}{3!} R^{ijk} \chi_i \chi_j \chi_k \right) \]

- For \( R = 0 \) it gives back the Poisson sigma-model on the boundary \( T[1] \partial \Sigma_3 \):

  \[ \text{Partial gauge: } F_i = D \chi_i \text{ and } \psi^i = -DX^i \]

  \[ \Rightarrow \omega_{gf} = \int_{T[1] \partial \Sigma_3} d^2 \hat{z} \delta X^i \delta \chi_i \quad \sim \quad \text{full gauge fix on the bulk,} \]

  \[ \sim \quad \text{AKSZ theory on the boundary} \]

  \[ S^{(3)}_{\pi,0} \xrightarrow{gf} \int_{T[1] \partial \Sigma_3} d^2 \hat{z} \left( \chi_i DX^i + \frac{1}{2} \pi^{ij} \chi_i \chi_j \right) \]
Lift up to AKSZ membrane III.

\[ S_{\pi,R}^{(3)} = \int_{T[1] \Sigma_3} d^3 \hat{z} \left( F_i DX^i - \chi_i D\psi^i + \pi^{ij} F_i \chi_j \right. \]
\[ \left. - \frac{1}{2} \partial_i \pi^{jk} \psi^i \chi_j \chi_k + \frac{1}{3!} R^{ijk} \chi_i \chi_j \chi_k \right) \]

- For \( R \neq 0 \) but with \( \pi = 0 \);
  - Partial gauge: \( F_i = D\chi_i \) and \( \psi^i = -DX^i \)

\[ S_{0,\text{gf}}^{(3)} = \int_{T[1] \partial \Sigma_3} d^2 \hat{z} \chi_i DX^i + \frac{1}{3!} \int_{T[1] \Sigma_3} d^3 \hat{z} R^{ijk} \chi_i \chi_j \chi_k \]

- It gives back the topological part of the string sigma-model on \( T[1] \partial \Sigma_3 \) that quantizes the non-geometric R-flux background

  [Mylonas, Schupp, Szabo]

- Quantization \( \leadsto \) non-associative star-product
We lift up our doubled Poisson sigma-model as a doubled contravariant Courant sigma-model on $\mathcal{M} = T^*[2] T[1] T^*X$:

$$S_{\Omega, R}^{(3)} = \int_{T[1] \Sigma^3} d^3 \hat{z} \left( F_I DX^I - \chi_I D\psi^I + \Omega^{IJ} F_I \chi_J 
- \frac{1}{2} \partial_I \Omega^{JK} \psi^I \chi_J \chi_K + \frac{1}{3!} R^{IJK} \chi_I \chi_J \chi_K \right)$$

On the boundary it gives the AKSZ constructions of both A- and B-models (partial gauge fix)

It allows the definition of fluxes for A/B-models

$$R^{IJK} \rightarrow H_{ijk}, F^i_{jk}, Q^{ij} k, R^{ijk}$$

The master equation gives them Bianchi identities

$$\{\gamma, \gamma\} = 0 \quad \Rightarrow \quad [\Omega, R]_S = 0$$
We apply the DFT projection
\[ \chi_i, \tilde{\psi}_i \rightarrow p_i \quad \& \quad \psi^i, \tilde{\chi}^i \rightarrow q^i \]
and set the dual coordinate to \( \tilde{X}_i = 0 \)
\[ \Rightarrow \quad \omega = \frac{1}{2} dF_i \wedge dX^i_0 + dq^i_1 \wedge dp_i \]

The resulting theory is not AKSZ (master equation does not hold)

\( \sim \quad \) We impose the master equation on \( \Omega \) (section condition):
\[ \Omega^{ij} \rightarrow \left( \begin{array}{cc} P^{ij} & J^i_j \\ -J^j_i & Q_{ij} \end{array} \right) \rightarrow \left( \begin{array}{cc} \pi^{ij} & J^i_j \\ -J^j_i & \omega_{ij} \end{array} \right) \]

We set \( \omega = 0 \) (we do not need it in order to describe the A/B-models later)
\[ \Rightarrow \quad \gamma_{\pi,j} = \pi^{ij} F_i p_j + J^i_j F_i q^j - \frac{1}{2} \partial_i \pi^{jk} q^i p_j p_k + \partial_i J^k_j q^i q^j p_k \]
Relation to generalized complex structure II.

\[ \gamma_{\pi,J} = \pi^{ij} F_i p_j + J^i_j F_i q^j - \frac{1}{2} \partial_i \pi^{jk} q^i p_j p_k + \partial_i J^k_j q^i q^j p_k \]

- The reduced double Poisson str. is

\[ \Omega^{IJ} \longrightarrow \mathbb{J}^{IJ} = \begin{pmatrix} J^i_j & \pi^{ij} \\ 0 & -J^i_j \end{pmatrix} \]

- The master equation gives the integrability conditions

\[ \pi^{[i|l} \partial_l \pi^{jk]} = 0 \]
\[ J^l_i \partial_l \pi^{jk} + 2 \pi^{il} \partial_{[i} J^{k]}_j + \pi^{kl} \partial_l J^j_i - J^l_i \partial_l \pi^{lk} = 0 \]
\[ J^l_i \partial_l J^{k]}_j - J^k_i \partial_{[i} J^{j]}_j = 0 \]

for the generalized complex str. \( \mathbb{J}^{IJ} \)!

- It defines a novel Courant algebroid corresponding a gen. complex str. with the Dorfman bracket and anchor

\[ \{ e_1 , e_2 \}_{\pi,J} = \{ \{ e_1 , \gamma_{\pi,J} \}, e_2 \} \quad \text{and} \quad \rho(e) = \{ e, \{ \gamma_{\pi,J}, \cdot \} \} \]
Relation to generalized complex structure III.

- It defines a AKSZ membrane model corresponding to the gen. complex str. $J$:

$$S_{\pi,J}^{(3)} = \int_{T[1] \Sigma_3} d^3 \hat{z} \left( F_i DX^i - p_i Dq^i + \pi^{ij} F_i p_j + J^{ij} F_i q^j - \frac{1}{2} \partial_i \pi^{jk} q^i p_j p_k + \partial_i J^{jk} q^i q^j p_k \right)$$

- It can contain flux terms as well (twists of the Courant algebroid)

$$\int_{T[1] \Sigma_3} d^3 \hat{z} \left( H_{ijk} q^i q^j q^k + F_{ijk} p_i q^j q^k + Q^{ij} p_i p_j q^k, + R^{ijk} p_i p_j p_k \right)$$

- If $J = 0$ it gives the A-model on the boundary (as before)

Partial gauge: $F_i = D\chi_i$ and $\psi^i = -DX^i$

$$S_{\pi,J}^{(3)} \xrightarrow{gf} \int_{T[1] \partial \Sigma_3} d^2 \hat{z} \left( \chi_i DX^i + \frac{1}{2} \pi^{ij} \chi_i \chi_j \right)$$
Relation to generalized complex structure III.

\[ S^{(3)}_{\pi,J} = \int_{T[1]\Sigma_3} d^3\hat{z} \left( F_i DX^i - p_i Dq^j + \pi^{ij} F_i p_j + J^i_j F_i q^j - \frac{1}{2} \partial_i \pi^{jk} q^i p_j p_k + \partial_i J^{k}{}_{j} q^i q^j p_k \right) \]

For \( \pi = 0 \) it also gives the B-model on the boundary:

- We cover \( \Sigma_3 \) as
  \[ \Sigma_3 = U \cup U' \]
  where \( U \) contains the boundary, but \( U' \) does not
  \[ \Sigma_3 \big|_U = \partial \Sigma_3 \times \mathbb{R}^+ \quad \text{and} \quad U' \subseteq \Sigma_3 \setminus \partial \Sigma_3 \]
  \[ \sim \quad \text{both } \omega \text{ and } S^{(3)}_{0,J} \text{ split to two parts on } U \text{ and } U' \]
- We use a gauge on \( U' \), which gives \( S^{(3)}_{0,J}|_{U'} = 0 \) (partial gauge)
- We will integrate out \( \mathbb{R}^+ \) to reduce it to the B-model on the boundary
Relation to generalized complex structure IV.

- We use the notation \((t \in \mathbb{R}^+)\)
  \[
  \phi = \hat{\phi} + \phi_t \theta^t
  \]

- We use further gauge fixing:
  \[
  X_i^t = 0 \quad \text{and} \quad q_i^t = 0
  \]

- We integrate out \(\hat{F}_i\) and \(\hat{p}_i\) \(\Rightarrow\) \(\partial_t \hat{X}^i = 0\) and \(\partial_t \hat{q}^i = 0\)

\[
S_{0,J|U,\text{gf}}^{(3)} = \oint_{T[1] \partial \Sigma_3} d^2 \hat{z} \int_{\mathbb{R}^+} dt \left( \hat{F}_i \partial_t \hat{X}^i \right) + \hat{p}_i \partial_t \hat{q}^i - (F_t)_i \hat{D} \hat{X}^i - (p_t)_i \hat{D} \hat{q}^i - J^i_j (F_t)_i \hat{q}^j + \partial_j J^k_j \hat{q}^i \hat{q}^j (p_t)_k\right).

- In the notations
  \[
  \chi_i = - \int_{\mathbb{R}^+} dt (F_t)_i, \quad X^i = \hat{X}^i
  \]
  \[
  \tilde{X}_i = - \int_{\mathbb{R}^+} dt (p_t)_i, \quad \tilde{\chi}^i = \hat{q}^i
  \]

- We get the B-model on the boundary
  \[
  S_{0,J}^{(3)} \rightarrow \oint_{T[1] \partial \Sigma_3} d^2 \hat{z} \left( \chi_i D X^i + \tilde{X}_i D \tilde{\chi}^i + J^i_j \chi_i \tilde{\chi}^j - \partial_j J^i_k \tilde{X}_i \tilde{\chi}^j \tilde{\chi}^k \right)
  \]
Topological S-duality from generalized complex structure I.

- Topological S-duality originates from S-duality of type IIB strings
- Exchanges the weak/strong coupling sector of A- and B-models

\[ g_A = \frac{1}{g_B} \quad \text{and} \quad k_A = \frac{k_B}{g_B} \]

- We take the rescaling transformation \((\lambda \in \mathbb{R})\)

\[ p_i \mapsto \lambda p_i \quad \text{and} \quad q^i \mapsto \frac{1}{\lambda} q^i \]

- It is a canonical transformation on the BV level (i.e. leaves \(\omega\) invariant)

\[ \gamma^\lambda_{\pi, J} = \lambda \pi^{ij} F_i p_j - \frac{\lambda}{2} \partial_i \pi^{jk} q^i p_j p_k + \frac{1}{\lambda} J^i_j F_i q^j + \frac{1}{\lambda} \partial_i J^k_j q^i q^j p_k \]

- Two limits give the Poisson or complex str. Courant algebroids:

\[ \frac{1}{\lambda} \gamma^\lambda_{0, J} \xleftarrow{\lambda \ll 1} \gamma^\lambda_{\pi, J} \xrightarrow{\lambda \gg 1} \lambda \gamma^\lambda_{\pi, 0} \]

- Gen. complex str. interpolates between the two Courant algebroids
Topological S-duality from generalized complex structure II.

- It can be lift up to the AKSZ level (Hamiltonian $\lambda \gamma_{\pi, J}$ and we introduce an overall coupling $1/\lambda$ then rescale with $\lambda$):

$$
S_{A/B}^{(3)} = \int_{T[1]\Sigma_3} d^3 \hat{z} \left( \frac{1}{\lambda} F_i DX^i - \frac{1}{\lambda} p_i Dq^i + \lambda \pi^{ij} F_i p_j \\
- \frac{\lambda}{2} \partial_i \pi^{jk} q^i p_j p_k + \frac{1}{\lambda} J^i_j F_i q^j + \frac{1}{\lambda} \partial_i J^k_j q^i q^j p_k \right)
$$

- We get the A-model for $\lambda \gg 1$

$$
S_{A/B}^{(3)} \xrightarrow{\lambda \gg 1} \frac{\lambda}{2} \int_{T[1]\partial\Sigma_3} d^2 \hat{z} \; \pi^{ij} p_i p_j \Rightarrow \lambda = \frac{1}{g_A}
$$

- And the B-model for $\lambda \ll 1$

$$
S_{A/B}^{(3)} \xrightarrow{\lambda \ll 1} \frac{1}{\lambda} \int_{T[1]\partial\Sigma_3} d^2 \hat{z} \left( \chi_i DX^i + \tilde{\chi}_i D\tilde{\chi}^i + J^i_j \chi_i \tilde{\chi}^j - \partial_j J^i_k \tilde{\chi}_i \tilde{\chi}^j \tilde{\chi}^k \right)
$$

$$
\lambda = g_B = \frac{1}{g_A} \Rightarrow \text{Topological S-duality!}
$$
Different reductions and connections between AKSZ string and membrane sigma-models related to A- and B-models
Outlook

- What physical quantities does our AKSZ membranes calculate? (In general, it is not easy to compute something in a membrane sigma-model).
- What is the relevance of geometric and non-geometric fluxes in topological string theory?
- Have our constructions any relevance in flux compactifications? (topological strings appear in compactifications without fluxes)
- Has it any relation to topological mirror symmetry?
- Further study of the new classes of Courant algebroid (also their twists).
- Does there exist a Courant algebroid for the general version of generalized complex structure?
- Our S-duality arises from the T-duality inspired generalized complex geometry. Whether there is a physical origin behind this relation or whether it is just a coincidence found in the topological field theories.
- Lift up our approach to topological M-theory and exceptional generalized geometry? (First step: [1802.04581] by ZK, Sinkovics and Szabo)
Thank you for your attention!