Mass-coupling relation in multi-scale integrable models

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• Introduction: mass-coupling relations in quantum field theories

• Motivation: AdS/CFT correspondence and gluon scattering

• mass-coupling relation in 7 easy steps

• numerical checks

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Mass-coupling relation in quantum field theories

basic problem in QFT: \[ \mathcal{L}(\lambda, \Phi) \implies A(m, p) \]

QCD: \[ \Lambda_{\text{MS}} \implies m_{\text{nucl}} \quad \frac{m_{\text{nucl}}}{\Lambda_{\text{MS}}} = 6.2... \quad (2\pi ?) \]

Mass-coupling relation calculable in integrable 1 + 1 dimensional models.

\(O(3)\) \(\sigma\)-model: \[ \frac{m}{\Lambda_{\text{MS}}} = \frac{8}{e} \]

Hasenfratz, Maggiore, Niedermayer '90

\(O(3)\) model: Lagrangian model (AF) and integrable: exact S-matrix

trick: put system in external magnetic field
Lagrangian side: \[ \mathcal{F}(h) = -\frac{h^2}{g_0^2} + \ldots \implies -\frac{h^2}{\alpha_{\text{lagr}}(h/\Lambda)} + \ldots \]

Integrable side (from TBA): \[ \mathcal{F}(h) = -\frac{h^2}{\alpha_{\text{integ}}(h/m)} + \ldots \]

Mass-coupling relation (MC rel) known in many integrable models:

(O(\mathcal{N}) \sigma models, Sine-Gordon model, etc.)

All these models: 1-scale model (spectrum follows from S-matrix)

No mass-coupling relation for genuine multi-scale models!

this talk: MC-rel for the simplest multi-scale model, the homogeneous Sine-Gordon model (HSG) with 2 mass scales.
AdS/CFT correspondence and gluon scattering

Why $1+1$ dimensional integrable models?

Modern version of string theory: AdS/CFT correspondence

$$\mathcal{N} = 4 \text{ SYM} \equiv AdS_5 \times S_5 \text{ (super) string}$$

Maldacena '97

Further unexpected connections to integrable models:

$$\mathcal{A}(p, g) \equiv \mathcal{F}(m, T) \text{ (large coupling)}$$

Alday, Maldacena '07
\[ p \implies (m, T) \text{ known} \]

Special points in gluon momentum space \[\implies\] HSG model

\[ 2(2r + 1) \text{ gluons: } (HSG)_r \text{ model } (r \geq 2) \]

Use large \( T \) expansion from \( HSG_r \): need mass-coupling relation
Mass-coupling relations in 7 easy steps

Method:

Lagrangian description (UV side) \( \{ \lambda \} \)

Integrable description (IR side) \( \{ m \} \)

Find (sufficiently many) physical quantities which can be calculated on both sides
Lagrangian (UV) description
CFT:

\[
\frac{\text{SU}(3)^2}{[\text{U}(1)]^2} \quad \text{coset model} \quad c = \frac{6}{5}
\]

perturbing fields:

\[\text{SU}(3) \text{ weight 0 adjoint fields (2 fields) with conformal weight 3/5}\]

Minimal model representation:

\[
\frac{\text{SU}(3)^2}{[\text{U}(1)]^2} = \text{Ising} \otimes \text{TCI} \quad c = \frac{1}{2} + \frac{7}{10} = \frac{6}{5}
\]
Ising model: free fermion $\psi(z)$

Tricritical Ising model: superconformal

Superconformal generator $G(z)$ and super-multiplet $(A(z), B(z))$ with weight $1/10$

Weight 3/5 fields:

$\phi_1 = \psi \otimes A \quad \phi_2 = I \otimes B$

Perturbing scalar fields:

$\Phi_{ij}(z, \bar{z}) = \phi_i(z) \bar{\phi}_j(\bar{z})$

Perturbed CFT lagrangian:

$$\mathcal{L}_{UV} = \mathcal{L}_{\text{coset}} - \lambda_i \bar{\lambda}_j \Phi_{ij}$$

Only 3 independent couplings (scale invariance)
Free energy:

\[ F = - \lim_{V \to \infty} \frac{1}{V} \ln Z = \frac{1}{2} \langle \Theta \rangle \]

Free energy Ward identities:

\[ \partial_i F = -\langle \psi_i \rangle \quad \psi_i = -\bar{\lambda}_j \Phi_{ij} \quad \partial_i = \frac{\partial}{\partial \lambda_i} \]

\[ \bar{\partial}_j F = -\langle \bar{\psi}_j \rangle \quad \bar{\psi}_j = -\lambda_i \Phi_{ij} \quad \bar{\partial}_j = \frac{\partial}{\partial \bar{\lambda}_j} \]

Further identities:

\[ \partial_i \bar{\partial}_j F = -\langle \Phi_{ij} \rangle - \int d^2 x \langle \psi_i(x) \bar{\psi}_j(0) \rangle_c \]
Bootstrap (IR) description
2 types of fermionic particles with masses $m_1, m_2$

Lightcone momenta: $P_a^\pm(\theta) = m_a e^{\pm \theta}$

S-matrix: $S_{11}(\theta) = S_{22}(\theta) = -1$

$$S_{12}(\theta - \sigma) = -S_{21}(\theta + \sigma) = \tanh(\theta/2 - i\pi/4)$$

Form factors: $\langle 0 | X | \theta_1, \ldots, \theta_n \rangle_{a_1 \ldots a_n} = \mathcal{F}^X_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_n)$

$$\mathcal{F}^X_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_n) = Q^X_{a_1 \ldots a_n}(x_1, \ldots, x_n) \prod_{j < k} \mathcal{F}^{\text{min}}_{a_j a_k}(\theta_j, \theta_k)$$

$$x_i = e^{\theta_i}$$
Minimal form factors:

\[ \mathcal{F}_{11}^{\text{min}}(\theta, \theta') = \mathcal{F}_{22}^{\text{min}}(\theta, \theta') = -\frac{\sinh(\theta/2 - \theta'/2)}{2\pi(x+x')} \]

Trace of energy-momentum tensor (\( \Theta \))

\[ Q_{a_1 \ldots a_n}^{\Theta}(\{x_i\}, m_1, m_2, \sigma) = P^2 q_{a_1 \ldots a_n}(\{x_i\}, \sigma) \]

Mass dependence only through total momentum squared!

\[ P^2 = P^+ P^- \quad P^\pm = m_1 P^\pm_{(1)} + m_2 P^\pm_{(2)} \quad P^\pm_{(a)} = \sum_{a-\text{type}} e^{\pm \theta_j} \]
More scalar operators:

\[ X_{ab} : \quad Q_{a_1...a_n}^{X_{ab}} = P^+_{(a)} P^-_{(b)} q_{a_1...a_n} \]

Simple mass dependence of \( \Theta \):

\[ \Theta = m_1^2 X_{11} + m_2^2 X_{22} + m_1 m_2 (X_{12} + X_{21}) \]

VEV from TBA: \( \langle \Theta \rangle = m_1 m_2 \cosh \sigma \)

All VEVs known

\[ \langle X_{11} \rangle = \langle X_{22} \rangle = 0 \quad \langle X_{12} + X_{21} \rangle = \cosh \sigma \]

Tensor operators:

\[ Y_{ab}^{\mu \nu} : \quad \sim P_{(a)}^\mu P_{(b)}^\nu q_{a_1...a_n} \]
Step 1: Operator identification
The 4 perturbing operators $\Phi_{ij}$ and the 4 scalars found by their FFs must be related

$$\Phi_{ij} = \mathcal{N}^{ab}_{ij}(\lambda, \bar{\lambda})X_{ab}$$

First constraint on the coefficients

$$\Theta = -\frac{4}{5}\lambda_i\bar{\lambda}_j\Phi_{ij} = m_1^2X_{11} + m_2^2X_{22} + m_1 m_2(X_{12} + X_{21})$$
Step 2: Form factor perturbation theory
Diagonal form factors:

\[ f_{aa}^X = \mathcal{F}_{aa}^X(i\pi, 0) \]

\[ f_{abab}^X(\theta) = \mathcal{F}_{abab}^X(\theta + i\pi, i\pi, \theta, 0) \]

Change of masses:

\[ \partial_i m_a^2 = -4\pi f_{aa}^{\psi_i} \quad \bar{\partial}_j m_a^2 = -4\pi f_{aa}^{\bar{\psi}_j} \]
Change of S-matrix:

\[ 8\pi^2 i f_{abab}^{\psi_i}(\theta) = 2m_a m_b \sinh \theta \partial_i S_{ab}(\theta) \]

\[-[\partial_i m_a^2 + \partial_i m_b^2 + 2 \cosh \theta \partial_i (m_a m_b)] \frac{\partial S_{ab}(\theta)}{\partial \theta} \]

From known form factors (+ first constraint)

\[ 2\psi_i = -2X_{11} \partial_i \ln m_1 - 2X_{22} \partial_i \ln m_2 \]

\[-X_{12} \partial_i \ln (m_1 m_2 e^{-\sigma}) - X_{21} \partial_i \ln (m_1 m_2 e^{\sigma}) \]
Step 3: Conserved tensor currents
CFT: any chiral field $\Lambda(z)$ conserved

$$\bar{\partial}\Lambda(z) = 0$$

pCFT: leading perturbative correction

$$\bar{\partial}\Lambda(z, \bar{z}) = -\lambda_i \bar{\lambda}_j \oint_{(z)} \frac{dw}{2i} \Lambda(z) \phi_{ij}(w, \bar{z})$$

Counting argument (Zamolodchikov): leading perturbative formula actually exact!

If:

$$\bar{\partial}\Lambda(z, \bar{z}) = \partial(\ldots)$$

Takes form of a conservation law!
Energy-momentum always conserved:

\[ L = L_{\text{Ising}} + L_{\text{TCI}} \]

\[ \bar{\partial}L = \pi (1 - h) \partial (\lambda_i \psi_i) = \frac{2\pi}{5} \lambda_i \partial \psi_i \]

Second conserved tensor:

\[ J^- = L_{\text{Ising}} + \alpha \psi \otimes G_{\text{TCI}} \]

Conserved if \[ 4\alpha = \sqrt{5} \lambda_1 / \lambda_2 \]:

\[ \bar{\partial}J^- = \partial J^+ \quad J^+ = v_i \psi_i \quad v_1 = \pi \lambda_1 / 2, \quad 6v_2 = \pi \lambda_1^2 / \lambda_2 \]

\[ \lambda_i \psi_i, \quad v_i \psi_i \quad \text{are both} \quad ^+ \quad \text{components of conserved tensors} \]
Conservation law:

$$\partial \psi_i = \bar{\partial} \tau_i$$

Constraint on form factors:

$$\mathcal{F}^{\psi_i} \sim P^+ \quad \mathcal{F}^{\bar{\psi}_j} \sim P^-$$

Chiral factorization of coefficients:

$$\partial_i \ln \left(\frac{m_1}{m_2} e^{-\sigma}\right) = 0 \quad \bar{\partial}_j \ln \left(\frac{m_1}{m_2} e^{\sigma}\right) = 0$$

Chiral masses:

$$2\mu_a = m_a e^{\sigma_a} \quad 2\bar{\mu}_a = m_a e^{-\sigma_a} \quad \sigma_1 - \sigma_2 = \sigma$$

$$\mu_1 / \mu_2 \quad \text{only depends on} \quad \lambda_1 / \lambda_2$$

$$\bar{\mu}_1 / \bar{\mu}_2 \quad \text{only depends on} \quad \bar{\lambda}_1 / \bar{\lambda}_2$$
Step 4: Tensor current algebra
\( \mathcal{Q} \) conserved charge corresponding to second conserved tensor pCFT:

\[
[\mathcal{Q}, \bar{J}^{-}] = \frac{5}{2} v_i \bar{v}_j \partial \Phi_{ij}
\]

Calculating the commutator in IR language leads to

\[
\mathcal{N}_{ij}^{ab} = -\frac{5}{4} (\partial_i \ln m_b) (\bar{\partial}_j \ln m_a)
\]
Step 5: Generalized $\Theta$ sum rule
\[ \begin{align*}
\text{c sum rule (Cardy)} & \quad \int d^2x \, x^2 \langle \Theta(x) \Theta(0) \rangle_c = \frac{4\pi}{3} c \\
\text{Θ sum rule} & \quad \int d^2x \langle \Theta(x) \Psi(0) \rangle_c = -2\Delta \langle \Psi \rangle \\
\text{Generalization for arbitrary conserved tensor current:} & \quad \partial_\mu Y^{\mu\nu} = 0
\end{align*} \]
Trace part:

\[ \int d^2x \langle Y^+(x) \Psi(0) \rangle_c = -\pi \xi \]

Tensor part:

\[ \langle Y^-(x) \Psi(0) \rangle \sim \frac{\xi}{Z^2} + \ldots \]

Present problem:

\[ \int d^2x \langle J^+(x) \bar{\psi}_j(0) \rangle_c = v_i \int d^2x \langle \psi_i(x) \bar{\psi}_j(0) \rangle_c \]

\[ J^-(z) \bar{\psi}_j(0, 0) \sim -\frac{3}{2\pi z^2} v_i \phi_i(0) \bar{\phi}_j(0) \]
Generalized $\Theta$ gives

$$\int d^2x \langle \psi_i(x) \bar{\psi}_j(0) \rangle_c = \frac{3}{2} \langle \Phi_{ij} \rangle$$

Free energy Ward identity:

$$\partial_i \bar{\partial}_j \mathcal{F} = -\frac{5}{2} \langle \Phi_{ij} \rangle$$

Complete factorization:

$$\mu_a = \mathcal{M}_a(\lambda_1, \lambda_2) \quad \bar{\mu}_a = \mathcal{M}_a(\bar{\lambda}_1, \bar{\lambda}_2)$$

3 $\Longrightarrow$ 3 mapping problem reduced to 2 copies of 2 $\Longrightarrow$ 2 mapping problem
Step 6: Differential equation
Generalized \( \Theta \) sum rule again:

\[
J^-(z) \phi_i(w) \sim -\frac{M_{ij} \phi_j(w)}{(z-w)^2} + \ldots
\]

\[
M_{11} = 1 \quad M_{22} = 0 \quad M_{12} = M_{21} = \frac{\lambda_1}{2\lambda_2}
\]

Free energy Ward identity applied to \( \langle \Phi_{k,j} \rangle \)

\[
v_i \partial_i \langle \Phi_{k,j} \rangle = v_i \int d^2x \langle \psi_i(x) \Phi_{k,j}(0) \rangle_c = \int d^2x \langle J^+(x) \Phi_{k,j}(0) \rangle_c = \pi M_{ki} \langle \Phi_{i,j} \rangle
\]
Identity on VEVs:

\[ \nu_i \partial_i N_{k,j}^{ab} \langle X_{ab} \rangle = \pi M_{k,i} N_{i,j}^{ab} \langle X_{ab} \rangle \]

Dimensional analysis:

\[ 2\mu_a = \lambda_1^{5/2} q_a(\lambda_1/\lambda_2) \]

Differential equation:

\[ \eta^2 \left(1 - \frac{\eta^2}{3}\right) q''_a(\eta) + \eta \left(4 - \frac{2\eta^2}{3}\right) q'_a(\eta) + \frac{5}{4} q_a(\eta) = 0 \]
Step 7: Solution
Hypergeometric differential equation

\[ F(z) = 2F_1 \left( -\frac{1}{2}, \frac{3}{2}, 3; z \right) \]

Solution

\[ \mu_1(\lambda_1, \lambda_2) = B\lambda_1^2(\lambda_1 + \sqrt{3}\lambda_2)^{1/2} F \left( \frac{2\lambda_1}{\lambda_1 + \sqrt{3}\lambda_2} \right) \]

\[ \mu_2(\lambda_1, \lambda_2) = \frac{B}{4}(\lambda_1 - \sqrt{3}\lambda_2)^2(\lambda_1 + \sqrt{3}\lambda_2)^{1/2} F \left( \frac{\sqrt{3}\lambda_2 - \lambda_1}{\lambda_1 + \sqrt{3}\lambda_2} \right) \]

Normalization from special case \( \lambda_1 = 0 \) (single scale model)

\[ B = \frac{5\pi}{16} 3^{-1/4} \kappa_{\text{RSOS}}^{-5/4} \]

\[ \kappa_{\text{RSOS}} = \frac{1}{2(12\pi)^{1/5}} \left( \frac{\Gamma(2/5)\Gamma(4/5)}{\Gamma(3/5)\Gamma(1/5)} \right)^{1/2} \]
Numerical checks of the mass-coupling relation
Figure 1: The red and blue surfaces represent the analytical formula $\mu_1(\lambda_1, \lambda_2)$ and $\mu_2(\lambda_1, \lambda_2)$. Red and blue points represent data coming from numerical solution of the TBA equations.
Figure 2: Constant $\mu_1$ and constant $\mu_2$ lines and data points from the numerical solution of the TBA equations.
Thank you!