# Notes on weak interactions 

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Figure 1: Generic exchange of an intermediate vector boson between weakly interacting particles.

## 1 Introduction

Weak interactions are one of the four fundamental interactions in Nature. Weak interactions are responsible for a wide variety of phenomena, including $\beta$-decays of nuclei and other hadronic decays (pions, kaons, hyperons), decays of elementary particles (muons and taus), and reactions of astrophysical relevance involving neutrinos. They are also fully responsible for parity-violating effects, including in atomic spectra. All elementary particles (quarks and leptons) interact weakly, in an essentially universal manner.

Weak interactions are the least symmetric of interactions, and violate a large number of conservation laws, namely $P, C, C P, T \rrbracket^{1}$ and most flavour symmetries. The only symmetries fully respected are Poincaré, $C P T$, baryon and lepton number 2 From a modern perspective, they are described in a unified fashion together with electromagnetism in terms of a spontaneously broken gauge theory of the group $\mathrm{SU}(2) \times \mathrm{U}(1)$. In the resulting theory, the electromagnetic part of the interactions corresponds to the exchange of massless, electrically neutral photons between electrically charged particles. The weak part corresponds instead to the exchange of massive bosons between particles: these are the charged $W^{ \pm}$bosons mediating the charged weak interactions ( $m_{W} \simeq 80 \mathrm{GeV}$ ), and the $Z^{0}$ boson mediating the neutral weak interactions ( $m_{Z} \simeq 90 \mathrm{GeV}$ ). These are also known collectively as intermediate vector bosons (see Fig. (1).

In this section we provide a brief historical introduction and a discussion of the most important aspects of the weak interactions. In the subsequent sections we will follow mostly a phenomenological approach, avoiding the intricacies of the electroweak theory. A more detailed discussion of this subject is postponed till after most of the phenomenology has been dealt with.

### 1.1 Historical notes

We start with a brief history of weak interactions, based on Chapter 21 of Ref. [2] and on Chapter 3 of Ref. [3], where one can also find lists of the original references. Further sources are the biographical pages Ref. [4, and Ref. [5] on the neutrino hypothesis.

[^0]From radioactivity to the neutrino hypothesis The history of weak interactions begins with the discovery of radioactivity by Henri Becquerel in 1896. He discovered that a uranium salt (potassium uranyl sulfate) emitted some invisible radiation that could expose a photographic plate, even if this was wrapped in opaque paper. In 1898 this phenomenon was observed also with thorium by Schmidt and, independently, by Marie Skłodowska Curie, who named it "radioactivity", and new radioactive materials, polonium and radium, were discovered by Pierre and Marie Curie with Gustave Bémont. Work on finding out what was the origin of the rays emitted by radioactive material led to major breakthroughs, and ultimately to the establishment of the whole new branch of particle physics. In 1899 Ernest Rutherford distinguished $\alpha, \beta$ and $\gamma$ rays, corresponding to radiation with increasing penetrating power. In 1900 Becquerel measured mass and charge-to-mass ratios of the $\beta$ rays and showed that they were electrons, that had been discovered only three years earlier in 1897 by J. J. Thomson $\sqrt[3]{3}$ In 1911 Rutherford established the existence of the atomic nucleus, and in 1913 Bohr suggested that $\beta$-rays originated there.

In 1914 Chadwick showed that $\beta$-rays had a continuous energy spectrum. This was in contradiction with the quantum-mechanical idea that nuclear energy levels were discrete, and became a major puzzle. The possibility that a continuous spectrum was due to the effect of interactions on their way out of the nucleus was definitively excluded in 1927 by Ellis and Wooster. Studying the $\beta$-decay ${ }_{83}^{210} \mathrm{Bi} \rightarrow{ }_{84}^{210} \mathrm{Po}$ (in modern language), they found that the energy release of the reaction, measured in a calorimeter, did not equal the maximal possible energy of the $\beta$-rays, but rather their average energy. One way out of this problem was to assume non-conservation of energy (or rather, conservation only on average, in a statistical sense: this was the position of Bohr). Another, for the times possibly more extreme solution was suggested by Pauli in 1930: a new type of spin- $\frac{1}{2}$ particle was emitted in the decay process, which went undetected and carried away the missing energy. This was the neutrino hypothesis, although Pauli initially called the new particle "neutron" and thought it was a very light constituent of the nucleus. In 1932 Chadwick discovered the actual neutron, although this was thought by most to be a composite object, made of a proton and an electron - except by Majorana, who called it "neutral proton". In 1933-34 Fermi proposed his groundbreaking theory of $\beta$-decay based on the reaction ${ }^{4} n \rightarrow p e^{-} \bar{\nu}_{e}$, postulating that an electron and a neutrino were created at their emission when a neutron turns into a proton (regarded at this point as elementary particles), without being previously present in the nucleus (the same suggestion was made by Francis Perrin in 1933). This theory achieved a good description of experimental results.

Fermi theory of $\beta$-decay The theoretical background inspiring Fermi for his proposal were QED, the prototype of any successful quantum field theory so far, and Heisenberg's proposal that proton and neutron were two different states of the same particle. QED suggests that the interaction couples two vectors, which in the case of electromagnetism are the electric current and the photon field. Replacing the proton electric current with a neutron-proton current, and the gauge field with a neutrino-electron current, Fermi wrote the interaction part of the Hamiltonian as

$$
\begin{equation*}
H_{\text {Fermi }}^{\mathrm{int}}=G \int d^{3} x\left(\bar{p}(x) \gamma^{\mu} n(x)\right)\left(\bar{e}(x) \gamma_{\mu} \nu(x)\right)+\text { h.c. }, \tag{1.1}
\end{equation*}
$$

[^1]where $p(x), n(x), e(x)$ and $\nu(x)$ are the fermionic fields corresponding to the various particles, "h.c." stands for "Hermitian conjugate", and $G$ is a constant (now named "Fermi constant") with dimensions of inverse mass squared. Comparing predictions from his theory with experiments, Fermi concluded that the neutrino had to be massless or very light, and that $G \simeq 0.3 \cdot 10^{-5} \mathrm{GeV}^{-2}$ (this should be confronted with the modern value $G \simeq 1.1 \cdot 10^{-5} \mathrm{GeV}^{-2}$ ). While the neutrino hypothesis proved to be phenomenologically successful, neutrinos remained elusive, and could be directly detected only in 1956 by Reines and Cowan, studying the inverse reaction $\bar{\nu}_{e}+p \rightarrow n+e^{+}$, as proposed by Pontecorvo in 1946. An experiment by Davis in 1959 showed that neutrinos and antineutrinos were different particles ${ }^{5}$

Theoretical progress was achieved by Gamow (1936) who generalised Fermi's Hamiltonian Eq. (1.1), a necessary step to allow for the description of more general $\beta$-decay processes. Sticking to four-fermion, non-derivative couplings one finds that the most general Hamiltonian is of the form

$$
\begin{align*}
H_{\beta}^{\mathrm{int}} & =-\int d^{3} x \mathscr{L}_{\beta}^{\mathrm{int}}(x), \\
\mathscr{L}_{\beta}^{\mathrm{int}}(x) & =-\sum_{j=1}^{5} g_{j}\left(\bar{p}(x) M_{j} n(x)\right)\left(\bar{e}(x) M^{j} \nu(x)\right)+g_{j}^{\prime}\left(\bar{p}(x) M_{j} n(x)\right)\left(\bar{e}(x) M^{j} \gamma^{5} \nu(x)\right)+\text { h.c. }, \tag{1.2}
\end{align*}
$$

where $M^{j}=1, \gamma^{5}, \gamma^{\mu}, \gamma^{\mu} \gamma^{5}, \sigma^{\mu \nu}$ and $g_{j}, g_{j}^{\prime}$ are (generally complex) coupling constants. ${ }^{6}$ The requirement of $T$ invariance imposes that $g_{j}, g_{j}^{\prime}$ be real. The requirement of $P$ invariance imposes that all the $g_{j}^{\prime}$ are zero. These seemed perfectly reasonable requirements at that moment.

The muon and universality of the weak interactions While theory underwent these developments, from the experimental side a new particle emerged, the muon. Discovered in cosmic rays in 1936 by Anderson and Neddermayer and initially mistaken for Yukawa's meson, it gained its modern status in 1947, after that Powell, Occhialini and Lattes showed that another particle, the pion, also present in the cosmic rays, was actually Yukawa's meson. The muon, in fact, did not interact strongly, and was essentially a heavier relative of the electron, that decayed weakly via $\mu^{-} \rightarrow e^{-} \bar{\nu}_{e} \nu_{\mu}$. The existence of two types of neutrino was later demonstrated experimentally by Lederman and collaborators in 1962. In 1947 Pontecorvo suggested that weak interactions coupled muons and electrons to hadrons in the same way ( $\mu-e$ universality), and in 1948 Puppi inferred the approximate equality of couplings in muon decay and in $\beta$-decays. This suggested universality of weak interactions, i.e., that they affected equally leptons and nuclei.

Parity violations Still from the experimental side, the 1950s and 1960s saw the discovery of a large number of new hadrons, that often showed weak decays, sometimes involving leptons in the final state (semileptonic processes) and sometimes not (nonleptonic processes). This led to ask if a new term should be added to the Lagrangian for each new hadron, a clearly annoying kind of situation. Before we discuss this issue, though, it is interesting to focus on one particular puzzle that led to one of the most important breakthroughs in 20th-century physics. This was

[^2]the so-called $\theta-\tau$ puzzle: the scalar particles then named $\theta$ and $\tau$ displayed the following decay modes,
\[

$$
\begin{equation*}
\theta^{+} \rightarrow \pi^{+} \pi^{+} \pi^{-}, \quad \tau^{+} \rightarrow \pi^{+} \pi^{0} \tag{1.3}
\end{equation*}
$$

\]

which suggested that they had parity -1 and +1 , respectively. Very surprisingly, these two particles had the same mass and lifetime, a rather unexpected coincidence. In 1956 Lee and Yang proposed that the two particles were in fact the same particle (now known as the $K^{+}$), and that weak interactions did not conserve parity. They showed that previous experiments could not disprove parity violations in weak processes $]^{7}$ and suggested new experiments to test their proposal. Such experiments were performed in 1957 by Wu and collaborators, and by Garwin and collaborators, confirming the violation of parity in weak interactions.
$V-A$ structure of the interaction Violation of parity was a rather shocking result, but it led to finally understand the correct form of the weak Lagrangian, clarifying the so-called $V-A$ structure of the interaction: for $\beta$-decays,

$$
\begin{equation*}
\mathscr{L}_{\beta}^{\mathrm{int}}=-\frac{G_{\beta}}{\sqrt{2}}\left(\bar{p}(x) \gamma^{\alpha}\left(1-\frac{g_{V}}{g_{A}} \gamma^{5}\right) n(x)\right)\left(\bar{e}(x) \gamma_{\alpha}\left(1-\gamma^{5}\right) \nu_{e}(x)\right)+\text { h.c. } \tag{1.4}
\end{equation*}
$$

while for muon decays

$$
\begin{equation*}
\mathscr{L}_{\mu}^{\mathrm{int}}=-\frac{G_{\mu}}{\sqrt{2}}\left(\bar{\mu}(x) \gamma^{\alpha}\left(1-\gamma^{5}\right) \nu_{\mu}(x)\right)\left(\bar{e}(x) \gamma_{\alpha}\left(1-\gamma^{5}\right) \nu_{e}(x)\right)+\text { h.c. }, \tag{1.5}
\end{equation*}
$$

where $G_{\beta}$ and $G_{\mu}$ have dimensions of inverse mass squared, and $g_{V} / g_{A}$ is a real dimensionless constant. This was understood in 1956-57 thanks to works by Salam, Landau, and Lee and Yang; and by Feynman and Gell-Mann, Sudarshan and Marshak, and Theis. Of course, starting from the general form Eq. 1.2 , the breaking of parity implies that the couplings $g_{j}^{\prime}$ need not vanish. The two-component neutrino hypothesis, stating that neutrinos have definite helicity (Salam, Landau, and Lee and Yang), reduces the number of couplings back to five. This hypothesis implies that only a specific definite-handedness part of the neutrino fields enters the Lagrangian. This was confirmed experimentally by Goldhaber and collaborators in 1958, showing that neutrinos were left-handed particles. Extending this assumption to all the fields (Feynman and Gell-Mann, Sudarshan and Marshak) immediately entails Eq. (1.4) and (1.5). As anticipated above, very similar couplings were found in the two cases, with $G_{\beta} / G_{\mu} \simeq 0.98$. The fact that the same coupling works for a pointlike particle like the muon and an extended one like the nucleon is reminiscent of what happens with the electric charge, which is the same for a positron and a proton. This led to the fruitful conserved vector current (CVC) hypothesis, i.e., that the hadronic current was a conserved current (Gershtein, 1956, and Feynman, 1958).

From hadronic currents to the quark model We now return on the proliferation of hadrons, and how to achieve their description. It was known (Feynman, 1958) that one did not have to add a new term for each hadron, but that only a few hadronic currents with the appropriate quantum numbers sufficed. On the other hand, these had to be postulated, since no fundamental description was available for hadrons. This changed in 1964 with the quark

[^3]hypothesis (Zweig, 1964; Gell-Mann, 1964), i.e., the assumption that hadrons were bound states of quarks, and that the fundamental objects appearing in the weak Lagrangian were the quark currents. Nuclear $\beta$-decay and charged-pion decay would then be two manifestations of the same decay process of the $d$ quark, $d \rightarrow u e^{-} \bar{\nu}_{e}$, in the first case appearing in the combination $d d u$ of a neutron in a nucleus, and in the other case appearing in the combination $d \bar{u}$ of a $\pi^{+}$. This would require a quark current of the form
\[

$$
\begin{equation*}
\bar{u} \gamma^{\alpha}\left(1-\gamma^{5}\right) d \tag{1.6}
\end{equation*}
$$

\]

However, this could not possibly be the whole story, since it would not allow strangeness-changing processes, like the $K^{+}$decays of Eq. (1.3), or $K^{+} \rightarrow \mu^{+} \nu_{\mu}$. The solution proposed by Cabibbo (in 1963, so still in terms of hadronic currents) was to modify Eq. (1.6) to

$$
\begin{equation*}
\bar{u} \gamma^{\alpha}\left(1-\gamma^{5}\right) d \longrightarrow \bar{u} \gamma^{\alpha}\left(1-\gamma^{5}\right) d^{\prime}, \quad d^{\prime}=\cos \theta_{C} d+\sin \theta_{C} s \tag{1.7}
\end{equation*}
$$

This solved several standing issues at once: it made possible to explain strangeness-changing processes, and to explain the difference between $G_{\mu}$ and $G_{\beta}$ while essentially retaining universality of the charged current, if the latter was expressed in terms of the "rotated" quark field $d^{\prime}$. The angle $\theta_{C}$ is known as the Cabibbo angle. From experimental results on $\beta$-decays and muon decays one finds $\cos \theta_{C}=G_{\beta} / G_{\mu} \simeq 0.98$, and from the semileptonic decays of the $K^{+}$one finds instead $\sin \theta_{C} \simeq 0.21$, which are consistent with each other.

Neutral currents and the charm quark Although Cabibbo's proposal described very successfully all the semileptonic processes known until 1973 (nonleptonic processes are more complicated to describe, as they involve a yet underdeveloped knowledge of hadrons), further theoretical work did not stop, and finally found experimental confirmation in 1973 and 1974. Between 1963 and 1968 a unified theory of electroweak interactions was developed by Glashow, Weinberg, and Salam (see below). This predicted, among other things, the existence of electrically neutral currents, besides the well known charged ones, responsible for a new type of weak interaction. In 1973 processes mediated by these neutral currents were observed experimentally by the Gargamelle experiment (Hasert et al., 1973-74), in particular antineutrino-electron scattering $\bar{\nu}_{e} e^{-} \rightarrow \bar{\nu}_{e} e^{-}$, and elastic (anti)neutrino scattering on nuclei, i.e., on quarks. In 1974 the $J / \psi$ resonance was observed, and quickly recognised as evidence of a fourth type of quark, the charm c. Such a particle had been proposed by Glashow, Iliopoulos and Maiani in 1970 (GIM mechanism) to explain the experimentally observed suppression of certain weak processes. These discoveries made a convincing case for the electroweak unified theory, and for the microscopic theory of strong interactions that had emerged from the quark model, namely Quantum Chromodynamics, or QCD (Gell-Mann, Fritzsch, Leutwyler, 1973), and established what is now known as the Standard Model of particle physics.

The unified electroweak theory The ideas underlying the electroweak theory date back to Yukawa, who in 1935 suggested that, similarly to QED, weak interactions could be mediated by the exchange of some intermediate boson $]^{8}$ instead of coupling directly four fermions. Such boson would be very massive, contrary to the photon that is massless, resulting in an interaction

[^4]of very short range, compared to the infinite range of the Coulomb interaction. In fact, in the nonrelativistic limit the effect of such exchanges is described by the Coulomb and Yukawa potentials, respectively,
\[

$$
\begin{equation*}
V_{\text {Coulomb }}(\vec{r})=\frac{e^{2}}{4 \pi r}, \quad V_{\text {Yukawa }}(\vec{r})=\frac{g^{2}}{4 \pi r} e^{-m_{W} r} \tag{1.8}
\end{equation*}
$$

\]

where $g$ is a coupling constant and $m_{W}$ the mass of the intermediate boson. While in the massless limit $m_{W} \rightarrow 0$ the Yukawa potential reduces to Coulomb potential, in the large mass limit $m_{W} \rightarrow \infty$ one finds instead 9

$$
\begin{equation*}
V_{\text {Yukawa }}(r) \xrightarrow[m_{W} \rightarrow \infty]{ } \frac{g^{2}}{m_{W}^{2}} \delta^{(3)}(\vec{r}), \tag{1.10}
\end{equation*}
$$

i.e., it reduces to a point-like interaction with coupling $G=\frac{g^{2}}{m_{W}^{2}}$. Equivalently, from the relativistic point of view the exchange of a massive boson brings a factor $g^{2} /\left(m_{W}^{2}-p^{2}\right)$ in the scattering amplitude, where $p$ is the momentum carried by the boson. In the limit of very large mass, this reduces to the same constant $G$. Notice that if one assumes that the weak coupling $g$ and the electric charge $e$ are of the same order, $g^{2} \simeq e^{2}$, one finds

$$
\begin{equation*}
m_{W}^{2}=\frac{g^{2}}{G} \simeq \frac{e^{2}}{G}=\frac{4 \pi \alpha}{G} \simeq(90 \mathrm{GeV})^{2} \tag{1.11}
\end{equation*}
$$

that compares well with the modern measurements $m_{W} \simeq 80 \mathrm{GeV}$. The main reason to go beyond the four-fermion theory is its bad behaviour at high energy, which can be foreseen by the mass dimension of the Fermi coupling. Introducing a massive intermediate boson does not solve completely the problem, and a further trick is needed, namely the generation of the boson masses via spontaneous symmetry breaking, the so-called Higgs mechanism (Higgs, 1964; Brout and Englert, 1964; Guralnik, Hagen and Kibble; 1964). The original development of the unified electroweak theory is due to Glashow (1961), Weinberg (1967), and Salam (1968). When everything is put together one obtains a well-behaved theory, that has so far been very successful in describing experiments. In this theory weak interactions are mediated by three massive vector bosons, the $W^{ \pm}$and the $Z^{0}$ : these were experimentally observed in 1983 by the UA1 and UA2 collaborations at CERN. This theory also predicts the existence of a massive scalar particle, a leftover from spontaneous symmetry breaking: this is the Higgs boson $H$, observed in 2012 by the ATLAS and CMS collaborations at CERN.

### 1.2 Overview

After this historical excursus, we give here an overview of the theory in its pre-electroweak form (but in modern language). This corresponds to the low-energy limit of the electroweak theory, in which $W, Z$-boson exchanges are replaced by a four-fermion local interaction. This

$$
\begin{align*}
& { }^{9} \text { To see this, notice that for any function } f(\vec{r}) \\
& \qquad \int d^{3} r V_{\text {Yukawa }}(r) f(\vec{r})= \\
& \qquad \underset{m_{W} \rightarrow \infty}{\rightarrow \pi m_{W}^{2}} \int d^{3} r \frac{g^{-m_{W} r}}{r} f(\vec{r})=\frac{g^{2}}{4 \pi m_{W}^{2}} \int(\overrightarrow{0}) \int d \Omega \int_{0}^{\infty} x \frac{e^{-x}}{x} f\left(\frac{\vec{x}}{m_{W}}\right)  \tag{1.9}\\
& \qquad e^{-x}=\frac{g^{2}}{m_{W}^{2}} f(\overrightarrow{0}) .
\end{align*}
$$

is due to the fact that at low energies the square of the momentum flowing in internal boson lines is much smaller than the square of the masses of the intermediate vector bosons, and can therefore be neglected compared to them. The $W$ and $Z$ propagators are then replaced by constants, corresponding graphically to the corresponding internal lines shrinking to a point. This approximation already provides an excellent tool to do quantitative calculations in many cases of interest, while avoiding the technicalities of the full theory. Furthermore, it allows one to see clearly how matter particles are coupled by the weak interactions, without distractions from the intricacies of gauge theories. I mostly follow Ref. [6].

Low-energy Lagrangian In the low-energy limit, the weak Lagrangian reads

$$
\begin{equation*}
\mathscr{L}_{W}^{\mathrm{int}}=\mathscr{L}_{W, \mathrm{ch}}^{\mathrm{int}}+\mathscr{L}_{W, 0}^{\text {int }} \tag{1.12}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathscr{L}_{W, \mathrm{ch}}^{\mathrm{int}}=-\frac{G}{\sqrt{2}} J^{\alpha \dagger} J_{\alpha}, \quad \mathscr{L}_{W, 0}^{\mathrm{int}}=-\frac{G}{\sqrt{2}} J_{0}^{\alpha} J_{0 \alpha} \tag{1.13}
\end{equation*}
$$

where "ch" and " 0 " refer to the charged and neutral interaction, respectively. The currents $J^{\alpha}$ and $J_{0}^{\alpha}$ are the charged and neutral currents, respectively. The charged current is further decomposed into a leptonic and a hadronic part,

$$
\begin{equation*}
J^{\alpha}=J_{l}^{\alpha}+J_{h}^{\alpha}, \tag{1.14}
\end{equation*}
$$

with

$$
\begin{align*}
& J_{l}^{\alpha}=\bar{e} \mathcal{O}_{L}^{\alpha} \nu_{e}+\bar{\mu} \mathcal{O}_{L}^{\alpha} \nu_{\mu}+\bar{\tau} \mathcal{O}_{L}^{\alpha} \nu_{\tau}=\sum_{\ell=e, \mu, \tau} \bar{\ell} \mathcal{O}_{L}^{\alpha} \nu_{\ell}  \tag{1.15}\\
& J_{h}^{\alpha}=\bar{d}^{\prime} \mathcal{O}_{L}^{\alpha} u+\bar{s}^{\prime} \mathcal{O}_{L}^{\alpha} c+\bar{b}^{\prime} \mathcal{O}_{L}^{\alpha} t
\end{align*}
$$

where

$$
\begin{equation*}
\mathcal{O}_{L}^{\alpha}=\gamma^{\alpha}\left(1-\gamma^{5}\right) \tag{1.16}
\end{equation*}
$$

$\ell$ and $\nu_{\ell}$ are the fields of the charged leptons and of the neutrinos, $u, c, t$ are the fields of the positively charged quarks and

$$
\left(\begin{array}{c}
d^{\prime}  \tag{1.17}\\
s^{\prime} \\
b^{\prime}
\end{array}\right)=\left(\begin{array}{lll}
V_{u d} & V_{u s} & V_{u b} \\
V_{c d} & V_{c s} & V_{c b} \\
V_{t d} & V_{t s} & V_{t b}
\end{array}\right)\left(\begin{array}{c}
d \\
s \\
b
\end{array}\right)=V_{\mathrm{CKM}}\left(\begin{array}{c}
d \\
s \\
b
\end{array}\right),
$$

with $d, s, b$ the fields of the negatively charged quarks. The Dirac adjoint fields are denoted with $\bar{\psi} \equiv \psi^{\dagger} \gamma^{0}$. Here $\gamma^{\mu}$ and $\gamma^{5}$ are the usual gamma matrices, which read (in Dirac basis)

$$
\gamma^{0}=\left(\begin{array}{cc}
\mathbf{1} & \mathbf{0}  \tag{1.18}\\
\mathbf{0} & -\mathbf{1}
\end{array}\right), \quad \gamma^{i}=\left(\begin{array}{cc}
\mathbf{0} & \sigma^{i} \\
-\sigma^{i} & \mathbf{0}
\end{array}\right), i=1,2,3, \quad \gamma^{5}=i \gamma^{0} \gamma^{1} \gamma^{2} \gamma^{3}=\left(\begin{array}{cc}
\mathbf{0} & \mathbf{1} \\
\mathbf{1} & \mathbf{0}
\end{array}\right),
$$

with $\sigma^{i}$ the usual Pauli matrices. The unitary matrix $V_{\mathrm{CKM}}$ is the Cabibbo-Kobayashi-Maskawa matrix, and defines the "rotated" negative-charge quark fields that interact according to the universal charged interaction ${ }^{10}$ Notice that such rotated fields are linear combinations of fields of different definite mass, and as such they are not definite-mass fields. Stated differently, mass

[^5]

Figure 2: Four-fermion charged-current interaction vertex. Fermionic lines have been labelled for the case of the pair of leptonic currents $\left(\bar{\nu}_{\mu} \mathcal{O}_{L}^{\alpha} \mu\right)\left(\bar{e} \mathcal{O}_{L \alpha} \nu_{e}\right)$.
eigenstates of quarks, which are eigenstates of flavour as defined via strong interactions, are not the eigenstates of flavour as defined via the weak interactions. In the two-family approximation, when bottom and top quarks are neglected, or more precisely when the mixing of bottom with down and strange can be neglected, the CKM matrix can be reduced to

$$
V_{\mathrm{CKM}} \rightarrow V_{\mathrm{C}}=\left(\begin{array}{cc}
\cos \theta_{C} & \sin \theta_{C}  \tag{1.19}\\
-\sin \theta_{C} & \cos \theta_{C}
\end{array}\right),
$$

which we may call Cabibbo matrix. The typical vertex of the charged current is shown in Fig. 2. The neutral current reads instead

$$
\begin{equation*}
J_{0}^{\alpha}=\sum_{f} g_{f}^{L} \bar{f} \mathcal{O}_{L}^{\alpha} f+g_{f}^{R} \bar{f} \mathcal{O}_{R}^{\alpha} f \tag{1.20}
\end{equation*}
$$

where $f=e, \mu, \tau, \nu_{e}, \nu_{\mu}, \nu_{\tau}, u, d, c, s, t, b$ runs over the fermion species,

$$
\begin{equation*}
\mathcal{O}_{R}^{\alpha}=\gamma^{\alpha}\left(1+\gamma^{5}\right) \tag{1.21}
\end{equation*}
$$

and the coupling $g_{f}^{L, R}$ are

$$
g_{f}^{L}=\left\{\begin{array}{ll}
\frac{1}{2}, & f=\nu_{e}, \nu_{\mu}, \nu_{\tau},  \tag{1.22}\\
-\frac{1}{2}+\xi, & f=e, \mu, \tau, \\
\frac{1}{2}-\frac{2}{3} \xi, & f=u, c, t, \\
-\frac{1}{2}+\frac{1}{3} \xi, & f=d, s, b,
\end{array} \quad g_{f}^{R}= \begin{cases}0, & f=\nu_{e}, \nu_{\mu}, \nu_{\tau} \\
\xi, & f=e, \mu, \tau \\
-\frac{2}{3} \xi, & f=u, c, t \\
\frac{1}{3} \xi, & f=d, s, b\end{cases}\right.
$$

where $\xi=\sin ^{2} \theta_{W}$ and $\theta_{W}$ is called the weak, or Weinberg ${ }^{11}$ angle. The sub/superscripts $L, R$ refer to chirality, as we explain below. Notice the absence of flavour-changing neutral currents: neutral currents do not change flavour, and flavour-changing currents also change electric charge.

In the expressions above Lorentz indices have been dropped. For leptons, one has in full notation

$$
\begin{equation*}
\bar{\ell} \mathcal{O}_{L}^{\alpha} \nu=(\bar{\ell})_{\lambda}\left(\mathcal{O}_{L}^{\alpha}\right)_{\lambda \lambda^{\prime}}(\nu)_{\lambda^{\prime}} . \tag{1.23}
\end{equation*}
$$

Quark fields have a further colour index, which is contracted trivially: in full notation

$$
\begin{equation*}
\bar{q}_{1} \mathcal{O}_{L}^{\alpha} q_{2}=\left(\bar{q}_{1}\right)_{\lambda}^{i}\left(\mathcal{O}_{L}^{\alpha}\right)_{\lambda \lambda^{\prime}} \delta_{i i^{\prime}}\left(q_{2}\right)_{\lambda^{\prime}}^{i^{\prime}} \tag{1.24}
\end{equation*}
$$

Here and in the following, summation over repeated indices is understood.

[^6]Massive fermion fields It is useful to recall here the explicit expression for a free massive fermion field,

$$
\begin{equation*}
\psi(x)=\int d \Omega_{p} \sum_{s= \pm \frac{1}{2}}\left\{b_{s}(\vec{p}) u_{s}(\vec{p}) e^{-i p \cdot x}+d_{s}(\vec{p})^{\dagger} v_{s}(\vec{p}) e^{i p \cdot x}\right\}, \tag{1.25}
\end{equation*}
$$

where $b_{s}(\vec{p})$ and $d_{s}(\vec{p})$ are the fermion and antifermion annihilation operators, respectively, that remove a fermion or antifermion of momentum $\vec{p}$ and spin component $s$ in some chosen direction from the state on which they are applied,

$$
\begin{equation*}
\left\{b_{s}(\vec{p}), b_{s^{\prime}}\left(\vec{p}^{\prime}\right)^{\dagger}\right\}=\left\{d_{s}(\vec{p}), d_{s^{\prime}}\left(\vec{p}^{\prime}\right)^{\dagger}\right\}=\delta_{s s^{\prime}}(2 \pi)^{3} 2 p^{0} \delta^{(3)}\left(\vec{p}-\vec{p}^{\prime}\right), \tag{1.26}
\end{equation*}
$$

all other anticommutators vanishing. With this normalisation, the particle states $|\vec{p}, s\rangle=$ $b_{s}(\vec{p})^{\dagger}|0\rangle$, with $|0\rangle$ the vacuum state normalised to $\langle 0 \mid 0\rangle=1$, obey the relativistic normalisation condition,

$$
\begin{equation*}
\left\langle\vec{p} \mid \vec{p}^{\prime}\right\rangle=(2 \pi)^{3} 2 p^{0} \delta^{(3)}\left(\vec{p}-\vec{p}^{\prime}\right) . \tag{1.27}
\end{equation*}
$$

The bispinors $u_{s}(\vec{p})$ and $v_{s}(\vec{p})$ are the positive-energy and negative-energy solutions of the Dirac equation, respectively, which obey

$$
\begin{equation*}
(\not p-m) u_{s}(\vec{p})=0, \quad(\not p+m) v_{s}(\vec{p})=0, \tag{1.28}
\end{equation*}
$$

with $m$ the fermion mass, and are normalised according to

$$
\begin{equation*}
\bar{u}_{s^{\prime}}(\vec{p}) u_{s}(\vec{p})=2 m \delta_{s^{\prime} s}, \quad \bar{v}_{s^{\prime}}(\vec{p}) v_{s}(\vec{p})=-2 m \delta_{s^{\prime} s} . \tag{1.29}
\end{equation*}
$$

In the formulas above, we used the notation $A=A_{\mu} \gamma^{\mu}$, where $A^{\mu}$ is any four-vector, and again $\bar{u}=u^{\dagger} \gamma^{0}$ for the Dirac adjoint of a bispinor. The explicit expressions read

$$
\begin{equation*}
u_{s}(\vec{p})=\sqrt{p^{0}+m}\binom{\varphi_{s}}{\frac{\vec{p} \cdot \vec{\sigma}}{p^{0}+m} \varphi_{s}}, \quad v_{s}(\vec{p})=\sqrt{p^{0}+m}\binom{\frac{\vec{p} \cdot \vec{\sigma}}{p^{0}+m} \tilde{\varphi}_{s}}{\tilde{\varphi}_{s}}, \tag{1.30}
\end{equation*}
$$

where $\varphi_{s^{\prime}}^{\dagger} \varphi_{s}=\tilde{\varphi}_{s^{\prime}}^{\dagger} \tilde{\varphi}_{s}=\delta_{s^{\prime} s}$, with $\vec{\eta} \cdot \frac{\vec{\sigma}}{2} \varphi_{s}=s \varphi_{s}$ for some unit vector $\vec{\eta}$, and $\tilde{\varphi}_{s}=-i \sigma_{2} \varphi_{s}^{*}$ in order for the field $\psi(x)$ to have simple Lorentz transformation properties ${ }^{12}$ Notice also the following relations,

$$
\begin{align*}
\bar{u}_{s^{\prime}}(\vec{p}) v_{s}(\vec{p}) & =\bar{v}_{s^{\prime}}(\vec{p}) u_{s}(\vec{p})=0, \\
\bar{u}_{s^{\prime}}(\vec{p}) \gamma^{\mu} u_{s}(\vec{p}) & =\bar{v}_{s^{\prime}}(\vec{p}) \gamma^{\mu} v_{s}(\vec{p})=2 p^{\mu} \delta_{s^{\prime} s} . \tag{1.31}
\end{align*}
$$

Finally, $d \Omega_{p}$ denotes the invariant integration measure in momentum space,

$$
\begin{equation*}
d \Omega_{p}=\frac{d^{3} p}{(2 \pi)^{3} 2 p^{0}}, \quad p^{0}=\sqrt{\vec{p}^{2}+m^{2}} . \tag{1.32}
\end{equation*}
$$

We also recall a few facts about gamma matrices. They obey the anticommutations relations

$$
\begin{equation*}
\left\{\gamma^{\mu}, \gamma^{\nu}\right\}=\gamma^{\mu} \gamma^{\nu}+\gamma^{\nu} \gamma^{\mu}=2 \eta^{\mu \nu} \tag{1.33}
\end{equation*}
$$

[^7]with ${ }^{13} \eta_{\mu \nu}=\eta^{\mu \nu}=\operatorname{diag}(1,-1,-1,-1)$ the Minkowski (pseudo)metric tensor. The defining property Eq. 1.33) makes them the generators of a Clifford algebra. The set of matrices
\[

$$
\begin{equation*}
\mathbf{1}, \gamma^{\mu}, \sigma^{\mu \nu}, i \gamma^{5} \gamma^{\mu}, \gamma^{5} \tag{1.34}
\end{equation*}
$$

\]

where

$$
\begin{equation*}
\sigma^{\mu \nu}=\frac{1}{2 i}\left[\gamma^{\mu}, \gamma^{\nu}\right], \quad \gamma^{5}=-\frac{i}{4!} \varepsilon_{\mu \nu \rho \sigma} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}=-i \varepsilon_{0123} \gamma^{0} \gamma^{1} \gamma^{2} \gamma^{3}=i \gamma^{0} \gamma^{1} \gamma^{2} \gamma^{3} \tag{1.35}
\end{equation*}
$$

form a basis of the vector space of complex $4 \times 4$ matrices. Here $\varepsilon_{\mu \nu \rho \sigma}$ is the totally-antisymmetric tensor with $\varepsilon_{0123}=-1$. The sigma matrices are the generators of the relevant representation of the Lorentz group for the fermion fields: given an element of the (proper orthocronous) Lorentz group $\Lambda=e^{\frac{i}{2} \omega_{\mu \nu} J^{(\mu \nu)}}$, with $J^{(\mu \nu)}$ the group generators, the matrices $S\left(e^{\frac{i}{2} \omega_{\mu \nu} J^{(\mu \nu)}}\right) \equiv e^{\frac{i}{4} \omega_{\mu \nu} \sigma^{\mu \nu}}$ provide a finite-dimensional representation of the group. Moreover, if $U(\Lambda)$ are the unitary operators representing Lorentz transformations on single-particle states, then

$$
\begin{align*}
& U(\Lambda)^{\dagger} \psi(x) U(\Lambda)=S(\Lambda) \psi\left(\Lambda^{-1} x\right) \\
& U(\Lambda)^{\dagger} \bar{\psi}(x) U(\Lambda)=\bar{\psi}\left(\Lambda^{-1} x\right) S(\Lambda)^{-1} \tag{1.36}
\end{align*}
$$

Notice that $S(\Lambda)^{\dagger} \neq S(\Lambda)^{-1}$ (there are no finite-dimensional unitary representations of the Lorentz group), but $S(\Lambda)^{\dagger} \gamma^{0}=\gamma^{0} S(\Lambda)^{-1}$. Finally, the matrix $\gamma^{5}$ anticommute with all the $\gamma^{\mu}$,

$$
\begin{equation*}
\left\{\gamma^{5}, \gamma^{\mu}\right\}=0 \tag{1.37}
\end{equation*}
$$

and plays an important role, discussed below.
Chirality The weak interactions are said to be chiral, as they treat differently fields with different chirality. Given a generic Dirac bispinor $\psi$, it can be always written as $\psi=\psi_{+}+\psi_{-}$ with $\gamma^{5} \psi_{ \pm}= \pm \psi_{ \pm}$. The eigenvalue of $\gamma^{5}$ corresponding to $\psi_{ \pm}$is the chirality of $\psi_{ \pm}$. Clearly these eigenvalues can only be $\pm 1$, since $\left(\gamma^{5}\right)^{2}=1$. It is then possible to decompose $\psi$ making use of the chiral projectors $P_{ \pm}$,

$$
\begin{equation*}
P_{ \pm}=\frac{1 \pm \gamma^{5}}{2}, \quad P_{ \pm}=P_{ \pm}^{\dagger}=P_{ \pm}^{2}, \quad P_{+} P_{-}=P_{-} P_{+}=0, \quad P_{+}+P_{-}=\mathbf{1} \tag{1.38}
\end{equation*}
$$

Clearly $\gamma^{5} P_{ \pm}= \pm P_{ \pm}$, so $\psi_{ \pm}=P_{ \pm} \psi$. Notice that $\gamma^{\alpha} P_{ \pm}=P_{\mp} \gamma^{\alpha}$. Since $\mathcal{O}_{L}^{\alpha}$ contains only $P_{-}$, the charged current only involves the fermionic fields with negative chirality, $f_{-}=P_{-} f{ }^{14}$ while the neutral current has different couplings for the terms involving $f_{-}$and $f_{+}=P_{+} f$.

The concept of chirality is often conflated with that of helicity, which is the projection of the particle's spin in the direction of motion. As a matter of fact the two concepts coincide only for massless fermions. It has nonetheless become customary to denote with $L$ and $R$ the negative and positive chirality components of the fields, respectively, although these refer to the "handedness" of the particle (see below).

[^8]Neutrinos As mentioned above, chirality and helicity can be safely identified in the case of a massless fermion. If we insist on treating the neutrinos as massless, then we are forced to drop one of the two helicity components, as it does not appear anywhere in the interaction Lagrangian ${ }^{[5]}$ Eqs. (1.12) and 1.13 . In order to see this, let us work out in detail the solutions to the Dirac equation in the massless case.

Since $\left\{\not \partial, \gamma^{5}\right\}=0$, the solution of the massless Dirac equation can be chosen with definite chirality ${ }^{16}$

$$
\begin{equation*}
i \not \supset \psi_{ \pm}=0, \quad \gamma^{5} \psi_{ \pm}= \pm \psi_{ \pm} . \tag{1.39}
\end{equation*}
$$

Let us look first at positive-energy plane-wave solutions $\psi=u e^{-i p \cdot x}$, for which $\not p u=0$. These read for a massless particle

$$
\begin{equation*}
u(\vec{p})=\sqrt{|\vec{p}|}\binom{\xi}{\hat{p} \cdot \vec{\sigma} \xi}, \tag{1.40}
\end{equation*}
$$

where $\xi^{\dagger} \xi=1$, having chosen the normalisation $\bar{u} \gamma^{0} u=2 p^{0}=2|\vec{p}|$. If we take solutions with definite helicity, i.e., using $\hat{p} \cdot \vec{\sigma} \xi_{R, L}= \pm \xi_{R, L}$, then

$$
\begin{equation*}
u_{R, L}(\vec{p})=\sqrt{|\vec{p}|}\binom{\xi_{R, L}}{ \pm \xi_{R, L}}, \quad \gamma^{5} u_{R, L}(\vec{p})= \pm u_{R, L}(\vec{p}) . \tag{1.41}
\end{equation*}
$$

A positive-helicity particle is said to be right-handed, while a negative-helicity one is said to be left-handed. This means that in the massless case definite-helicity particle solutions are also definite-chirality solutions, with chirality equal to helicity. For negative-energy solutions $\psi=v e^{i p \cdot x}$, one again has $\not p v=0$, and

$$
\begin{equation*}
v(\vec{p})=\sqrt{|\vec{p}|}\binom{\hat{p} \cdot \overrightarrow{\tilde{\sigma}} \tilde{\xi}}{\tilde{\xi}} . \tag{1.42}
\end{equation*}
$$

Lorentz transformation properties of the fermion field tell us that if $u$ with a two-spinor $\xi$ represents a particle state with definite spin $s$ in a certain direction, then the antiparticle state described by $v$ will have the same definite spin $s$ in the same direction if we choose $\tilde{\xi}=-i \sigma_{2} \xi^{*}$. Then to have a state with positive or negative helicity we need $\tilde{\xi}_{R, L}=-i \sigma_{2} \xi_{R, L}^{*}$, so that

$$
\begin{equation*}
\hat{p} \cdot \vec{\sigma} \tilde{\xi}_{R, L}=\hat{p} \cdot \vec{\sigma}\left(-i \sigma_{2}\right) \xi_{R, L}^{*}=i \sigma_{2}\left(\hat{p} \cdot \vec{\sigma} \xi_{R, L}\right)^{*}=\mp\left(-i \sigma_{2}\right) \xi_{R, L}^{*}=\mp \tilde{\xi}_{R, L} \tag{1.43}
\end{equation*}
$$

It then follows

$$
\begin{equation*}
v_{R, L}(\vec{p})=\sqrt{|\vec{p}|}\binom{\mp \tilde{\xi}_{R, L}}{\tilde{\xi}_{R, L}}, \quad \gamma^{5} v_{R, L}(\vec{p})=\mp v_{R, L}(\vec{p}) . \tag{1.44}
\end{equation*}
$$

This means that definite-helicity antiparticle solutions are also definite-chirality solutions, with chirality equal to minus the helicity. It is worth mentioning that helicity is a Lorentz-invariant quantity only in the massless case: for massive particles one can always choose a fast enough reference frame to overtake the particle and flip its momentum, thus flipping its helicity. For massive particles, helicity becomes better and better a quantum number as the energy increases, for in that case the particle is closer and closer to behave as approximately massless.

[^9]The results of Eqs. (1.41) and 1.44) imply that since the neutrino field is coupled by the charged weak interaction with the projector $P_{-}$, what is relevant is the combination

$$
\begin{align*}
P_{-} \nu(x) & =\frac{1-\gamma^{5}}{2} \int d \Omega_{p} \sum_{h=R, L}\left\{b_{h}(\vec{p}) u_{h}(\vec{p}) e^{-i p \cdot x}+d_{h}(\vec{p})^{\dagger} v_{h}(\vec{p}) e^{i p \cdot x}\right\}  \tag{1.45}\\
& =\int d \Omega_{p}\left\{b_{L}(\vec{p}) u_{L}(\vec{p}) e^{-i p \cdot x}+d_{R}(\vec{p})^{\dagger} v_{R}(\vec{p}) e^{i p \cdot x}\right\} \equiv \nu_{L}(x)
\end{align*}
$$

The field $\nu_{L}(x)=P_{-} \nu(x)$ annihilates a left-handed neutrino and creates a right-handed antineutrino; the adjoint field $\bar{\nu}_{L}(x)=\left(\nu_{L}(x)\right)^{\dagger} \gamma^{0}=\bar{\nu} P_{+}$creates a left-handed neutrino and annihilates a right-handed antineutrino. Since the field $\nu_{R}(x)=P_{+} \nu(x)$ never appears in the weak Lagrangian [check also Eq. 1.22]], we conclude that massless neutrinos can only be left-handed, and antineutrinos can only be right-handed.
$P, C$, and $C P$ symmetries We conclude this overview discussing the properties of the weak Lagrangian under the discrete transformations of parity and charge conjugation. To this end we need the known transformation properties of the fermion bilinears $V^{\alpha} \equiv \bar{f} \gamma^{\alpha} f$ and $A^{\alpha} \equiv \bar{f} \gamma^{\alpha} \gamma^{5} f$,

$$
\begin{array}{ll}
V^{\alpha} \underset{P}{\rightarrow} \mathcal{P}^{\alpha}{ }_{\beta} V^{\beta}, & V^{\alpha} \underset{C}{\vec{~}}-V^{\alpha \dagger} \\
A^{\alpha} \underset{P}{\rightarrow}-\mathcal{P}^{\alpha}{ }_{\beta} A^{\beta}, & A^{\alpha} \underset{C}{\rightarrow} A^{\alpha \dagger} \tag{1.46}
\end{array}
$$

where $\mathcal{P}^{\alpha}{ }_{\beta}=\operatorname{diag}(1,-1,-1,-1)$. The $V-A$ structure of the charged-interaction Lagrangian reads schematically

$$
\begin{equation*}
\mathscr{L}=\left(V^{\alpha \dagger}-A^{\alpha \dagger}\right)\left(V_{\alpha}-A_{\alpha}\right)=V^{\alpha \dagger} V_{\alpha}+A^{\alpha \dagger} A_{\alpha}-V^{\alpha \dagger} A_{\alpha}-A^{\alpha \dagger} V_{\alpha} . \tag{1.47}
\end{equation*}
$$

It is easy to see that under $P$ and $C$

$$
\begin{align*}
& \mathscr{L} \underset{P}{\longrightarrow} \mathcal{P}^{\alpha}{ }_{\beta} \mathcal{P}_{\alpha}{ }^{\gamma}\left(V^{\beta \dagger} V_{\gamma}+A^{\beta \dagger} A_{\gamma}+V^{\beta \dagger} A_{\gamma}+A^{\beta \dagger} V_{\gamma}\right)=\left(V^{\alpha \dagger}+A^{\alpha \dagger}\right)\left(V_{\alpha}+A_{\alpha}\right) \\
& \mathscr{L} \underset{C}{\rightarrow} V^{\alpha} V_{\alpha}^{\dagger}+A^{\alpha} A_{\alpha}^{\dagger}+V^{\alpha} A_{\alpha}^{\dagger}+A^{\alpha} V_{\alpha}^{\dagger}=\left(V^{\alpha \dagger}+A^{\alpha \dagger}\right)\left(V_{\alpha}+A_{\alpha}\right) . \tag{1.48}
\end{align*}
$$

i.e., both $P$ and $C$ are broken ${ }^{17}$ The clearest example is provided by neutrinos: a left-handed neutrino is transformed by $P$ into a right-handed neutrino, and by $C$ into a left-handed antineutrino, which do not appear in the weak Lagrangian ${ }^{18}$

The schematic structure Eq. (1.47) is good enough for the leptonic part of the current, and (barring neutrino mixing for the time being) shows that the combined transformation $C P$ is a good symmetry in the leptonic sector. However, Eq. (1.47) is in general not good enough for the hadronic part, which reads

$$
\begin{align*}
J_{h}^{\alpha} & =\bar{d}^{\prime} \mathcal{O}_{L}^{\alpha} u+\bar{s}^{\prime} \mathcal{O}_{L}^{\alpha} c+\bar{b}^{\prime} \mathcal{O}_{L}^{\alpha} t=\sum_{q_{1}=u, c, t, t} \sum_{q_{2}=d, s, b}\left(V_{\mathrm{CKM}}\right)_{q_{1} q_{2}} \bar{q}_{2} \mathcal{O}_{L}^{\alpha} q_{1} \\
& =\sum_{q_{1}=u, c, t, t} \sum_{q_{2}=d, s, b}\left(V_{\mathrm{CKM}}\right)_{q_{1} q_{2}}\left(V_{q_{2} q_{1}}^{\alpha}-A_{q_{2} q_{1}}^{\alpha}\right) \equiv J_{h}^{\alpha}\left(V_{\mathrm{CKM}}\right) \tag{1.49}
\end{align*}
$$

[^10]Under $C P$ one has

$$
\begin{equation*}
J_{h}^{\alpha} \underset{C P}{\longrightarrow}-\mathcal{P}^{\alpha}{ }_{\beta} \sum_{q_{1}=u, c, t} \sum_{q_{2}=d, s, b}\left(V_{\mathrm{CKM}}\right)_{q_{1} q_{2}}\left(V_{q_{2} q_{1}}^{\beta}-A_{q_{2} q_{1}}^{\beta}\right)^{\dagger}=-\mathcal{P}^{\alpha} J_{h}^{\beta}\left(V_{\mathrm{CKM}}^{*}\right)^{\dagger} . \tag{1.50}
\end{equation*}
$$

If $V_{\text {CKM }}$ is real, then $C P$ is a symmetry. With only two families of quarks one can redefine the fermion fields to make $V_{\text {CKM }}$ real [see Eq. (1.19]], so there can be no $C P$ violation. On the other hand, with three families there is one phase factor that cannot be transformed to 1 , and so there is the possibility of $C P$ violation (and thus of $T$ violation). In general, $V_{\text {CKM }}$ is expressed in terms of sines and cosines of three angles and one phase factor.

A similar $C P$-violating phase can appear in the lepton sector, assuming that a nontrivial mixing matrix exists there. This is not possible if the neutrinos are massless (or more generally mass-degenerate): any nontrivial mixing matrix $U_{\ell^{\prime} \ell}$ could be transformed away by redefining the neutrino fields $\nu_{\ell}^{\prime}=U_{\ell \ell^{\prime}} \nu_{\ell^{\prime}}$, which does not affect the free part of the Lagrangian. As a matter of fact, neutrinos are massive, and a nontrivial, physically relevant mixing matrix $U_{\text {PMNS }}$ (Pontecorvo-Maki-Nakagawa-Sakata matrix) appears.

As long as weak interactions are described by a Poincaré-invariant quantum field theory, it is guaranteed that $\Theta=C P T$ is a good (antiunitary) symmetry. This is enough to show that particles and antiparticles have the same mass, and the same decay width/lifetime if they are unstable. Indeed, since for a particle with quantum numbers $\alpha$, momentum $\vec{p}$ and spin component $s$ one finds, $\Theta|\alpha ; \vec{p}, s\rangle=|\bar{\alpha} ; \vec{p},-s\rangle$ (with the appropriate choice of phases), where $\bar{\alpha}$ denotes the quantum numbers of the corresponding antiparticle, one has

$$
\begin{equation*}
\left\langle\bar{\alpha} ; \vec{p}^{\prime},-s^{\prime}\right| P^{2}|\bar{\alpha} ; \vec{p},-s\rangle=\left\langle\alpha ; \vec{p}^{\prime}, s^{\prime}\right| \Theta^{\dagger} P^{2} \Theta|\alpha ; \vec{p}, s\rangle=\left\langle\alpha ; \vec{p}^{\prime}, s^{\prime}\right| P^{2}|\alpha ; \vec{p}, s\rangle \tag{1.51}
\end{equation*}
$$

from which $m_{\bar{\alpha}}=m_{\alpha}$ follows. Equality of decay widths will be discussed in the next subsection.
Baryon, lepton and lepton family number While the very nature of the charged currents makes flavour not a good quantum number, both for quarks and leptons, the mixing of quarks prevents also "quark family" from being a good quantum number. On the other hand, quark number, or equivalently baryon number, is conserved. In the absence of a leptonic mixing matrix, lepton family number is a conserved quantity, and so a fortiori is lepton number. The usual assignment of lepton family numbers is $l_{\ell}=1$ for $\ell$ and $\nu_{\ell}, l_{\ell}=-1$ for $\bar{\ell}$ and $\bar{\nu}_{\ell}$, and $l_{\ell}=0$ for anything else. Of course, given a system of particles one has for $l_{\ell}$ of the system the sum of the individual lepton family numbers. Lepton number is just $L=\sum_{\ell} l_{\ell}$. It is now known that a nontrivial lepton mixing matrix is present, and so (small) violations of lepton family number conservation are expected. Ignoring these, the approximate lepton family number conservation forbids a number of processes that would otherwise be allowed using only phasespace considerations and electric charge conservation, for example

$$
\begin{equation*}
\mu^{-} \rightarrow e^{-} \gamma, \quad \mu^{-} \rightarrow e^{-} e^{+} e^{-}, \tag{1.52}
\end{equation*}
$$

and neutrino-nucleus scattering processes like

$$
\begin{equation*}
{ }_{A}^{Z} \mathrm{~N}+\nu_{\mu} \rightarrow{ }_{A}^{Z+1} \mathrm{~N}+e^{-}, \tag{1.53}
\end{equation*}
$$

where ${ }_{A}^{Z} \mathrm{~N}$ denotes a nucleus with atomic number $Z$ and mass number $A$. On the other hand, the following processes are allowed,

$$
\begin{equation*}
{ }_{A}^{Z} \mathrm{~N}+\nu_{e} \rightarrow{ }_{A}^{Z+1} \mathrm{~N}+e^{-}, \quad{ }_{A}^{Z} \mathrm{~N}+\bar{\nu}_{e} \rightarrow{ }_{A}^{Z-1} \mathrm{~N}+e^{+} . \tag{1.54}
\end{equation*}
$$

Lepton family number conservation also forbids the neutrinoless double-beta decay ${ }^{19}$

$$
\begin{equation*}
{ }_{A}^{Z} \mathrm{~N} \rightarrow{ }_{A}^{Z+2} \mathrm{~N}+2 e^{-} . \tag{1.55}
\end{equation*}
$$

### 1.3 Decay of unstable particles

In the following we will deal with a wide variety of decays of unstable particles. Here we briefly summarise the main technical points required to deal with the theoretical description of these phenomena.

The decay rate of an unstable particle is the probability per unit time that it decays in any of the allowed final states. It is also called the (total) decay width, and is usually denoted with the symbol $\Gamma$. The probability per unit time to decay into a final state with a specified set of products, i.e., into a specific channel, is called partial width. The ratio of a partial width over the total width is the branching ratio (or fraction) of the given channel, and tells us how likely a certain decay mode is among all those allowed for the unstable particle under consideration. For a large sample of (independent) unstable particles, one expects an exponential decay of the population with lifetime $\tau=1 / \Gamma$.

The probability per unit time for the unstable particle to decay into a specific channel with a specified final state (i.e., with definite momenta and/or spins) is called differential decay rate/width. For an unstable particle with four-momentum $p$ decaying into an $n$-particle final state, the differential decay rate $d \Gamma^{(n)}$ is given by

$$
\begin{equation*}
d \Gamma^{(n)}=\frac{\left|\mathcal{M}_{\mathrm{f}}\right|^{2}}{2 p^{0}} d \Phi^{(n)}, \tag{1.56}
\end{equation*}
$$

where $d \Phi^{(n)}$ is the infinitesimal element of invariant $n$-particle phase space,

$$
\begin{equation*}
d \Phi^{(n)}=(2 \pi)^{4} \delta^{(4)}\left(p-\sum_{i=1}^{n} p_{i}\right) \prod_{i=1}^{n} \frac{d^{3} p_{i}}{(2 \pi)^{3} 2 p_{i}^{0}}, \tag{1.57}
\end{equation*}
$$

and $\mathcal{M}_{\mathrm{fi}}$ is the matrix element of the decay operator between the initial and final states. We will not need here to fully develop the formal theory of decay since, given the weakness of weak interactions, the first-order perturbative approximation will almost always suffice. In such an approximation the relevant matrix element is

$$
\begin{equation*}
(2 \pi)^{4} \delta^{(4)}\left(P_{f}-P_{i}\right) \mathcal{M}_{\mathrm{fi}}=-\int d x^{0}\langle f| H_{W}^{\mathrm{int}}\left(x^{0}\right)|i\rangle \tag{1.58}
\end{equation*}
$$

where $|i\rangle$ and $|f\rangle$ are the initial and final free-particle states, with relativistic normalisation, and $H_{W}^{\mathrm{int}}\left(x^{0}\right)$ is the weak interaction Hamiltonian in the interaction picture, which since there is no derivative coupling reads

$$
\begin{equation*}
H_{W}^{\mathrm{int}}\left(x^{0}\right)=\int d^{3} x \mathscr{H}_{W}^{\mathrm{int}}\left(f_{j}(x), \bar{f}_{j}(x)\right)=-\int d^{3} x \mathscr{L}_{W}^{\mathrm{int}}\left(f_{j}(x), \bar{f}_{j}(x)\right), \tag{1.59}
\end{equation*}
$$

[^11]where we made explicit the dependence on the fermion fields $\left\{f_{j}, \bar{f}_{j}\right\}$ (which are here fields in the interaction representation). Substituting Eq. (1.59) in Eq. (1.58) we find
\[

$$
\begin{equation*}
\mathcal{M}_{\mathrm{fi}}=\langle f| \mathscr{L}_{W}^{\operatorname{int}}\left(f_{j}(0), \bar{f}_{j}(0)\right)|i\rangle \tag{1.60}
\end{equation*}
$$

\]

having used translation invariance to integrate over spacetime, and having dropped the momen-tum-conserving delta function. These matrix elements are efficiently evaluated making use of Feynman diagrams, and of the Feynman rules in momentum space. These are easily derived based on the following considerations. The interaction vertex couples four fermionic fields, or more precisely two fermionic currents $j_{1}^{\alpha}$ and $j_{2}^{\alpha}$, that can create or annihilate initial or final (anti)particles. The vertex couples them in the form $-\frac{G}{\sqrt{2}} j_{1}^{\alpha} j_{2 \alpha}$, and therefore

1. for each vertex, draw a dot and include a factor $-\frac{G}{\sqrt{2}}$.

The currents are of the general form $g_{a b} \bar{f}_{a} \mathcal{O}^{\alpha} f_{b}$ with $g_{a b}$ some coupling (e.g., $V_{\text {CKM }}$ matrix elements) and $\mathcal{O}^{\alpha}$ a combination of gamma matrices. Depending on the process, we will then include Dirac bispinors $\bar{w}_{a}$ and $w_{b}$ corresponding to the fields $\bar{f}_{a}$ and $f_{b}$ creating or destroying particles in the initial and final states, as follows:
2. a bispinor $u_{s}(\vec{p})$ for each particle in the initial state;
3. a bispinor $\bar{u}_{s}(\vec{p})$ for each particle in the final state;
4. a bispinor $\bar{v}_{s}(\vec{p})$ for each antiparticle in the initial state;
5. a bispinor $v_{s}(\vec{p})$ for each antiparticle in the final state.

These are represented as oriented external lines attached to the vertex containing the field responsible for the creation/annihilation of the corresponding particle, either flowing in the vertex (initial particle/final antiparticle) or out of the vertex (initial antiparticle/final particle). All remaining fermion fields must be contracted with each other, yielding fermion propagators that connect different vertices. These are represented as oriented internal lines, running from the vertex containing the field $\bar{f}$ to that containing the field $f$ of the relevant contraction. At this point the Lorentz indices of the bispinors and of the propagators must be contracted according to the structure of the currents, and all missing factors should be included:
6. connect bispinors and propagators along each uninterrupted fermion line, starting from the end and moving backwards, and including the appropriate vertex factors along the way;
7. contract the Lorentz indices of currents coupled at a vertex.

The first of these rules simplifies a lot in the case of a single vertex:
$6^{\prime}$ (for a single vertex) connect the pairs belonging to the same current with the appropriate factor $\mathcal{O}^{\alpha}$ (e.g., $\mathcal{O}_{L}^{\alpha}$ for charged currents), forming bilinears of the type $\bar{w}_{a} \mathcal{O}^{\alpha} w_{b}$, including the appropriate $g_{a b}$ factor.

At this point all that is left is standard practice:
8. impose conservation of momentum at each vertex;
9. integrate over internal momenta (i.e., momenta flowing through the propagators) with measure $\frac{d^{4} q}{(2 \pi)^{4}}$;
10. include minus signs for each fermionic loop, and each fermionic line crossing the diagram from top to bottom (i.e., an antifermionic line across the whole diagram);
11. include the appropriate numerical factors counting the number of ways a certain diagram can be obtained.

We conclude this subsection showing that CPT-invariance implies equality of the lifetimes of an unstable particle and the corresponding antiparticle. In fact, since (working in the rest frame of the decaying particle)

$$
\begin{equation*}
\left.\Gamma=\frac{1}{2 m} \sum_{n} \int d \Phi^{(n)}\left|\mathcal{M}_{\mathrm{i} \rightarrow \mathrm{n}}\right|^{2}=\frac{1}{2 m} \sum_{n} \int d \Phi^{(n)}\left|\langle n| \mathscr{H}^{\mathrm{int}}(0)\right| i, s\right\rangle\left.\right|^{2}, \tag{1.61}
\end{equation*}
$$

for a decay governed by an interaction Hamiltonian density $\mathscr{H}^{\text {int }}$, using completeness of the set of states $|n\rangle$ one finds

$$
\begin{equation*}
\Gamma=\frac{1}{2 m}\langle i, s| \mathscr{H}^{\text {int }}(0) \mathscr{H}^{\text {int }}(0)|i, s\rangle . \tag{1.62}
\end{equation*}
$$

Using $C P T$ invariance, one has then for the decay width $\bar{\Gamma}$ of the antiparticle $\bar{\imath}$ (notice that the two particles must have the same mass)

$$
\begin{align*}
\bar{\Gamma} & =\frac{1}{2 \bar{m}}\langle\bar{\imath}, s| \mathscr{H}^{\mathrm{int}}(0) \mathscr{H}^{\mathrm{int}}(0)|\bar{\imath}, s\rangle=\frac{1}{2 m}\langle i,-s| \mathscr{H}^{\mathrm{int}}(0) \mathscr{H}^{\mathrm{int}}(0)|i,-s\rangle  \tag{1.63}\\
& =\frac{1}{2 m}\langle i, s| \mathscr{H}^{\mathrm{int}}(0) \mathscr{H}^{\mathrm{int}}(0)|i, s\rangle=\Gamma,
\end{align*}
$$

where in the last passage we have used rotation invariance (which by the way implies that the total decay width is independent of the polarisation of the unstable particle).


Figure 3: Muon decay.

## 2 Muon decay

We now begin to discuss applications to phenomenology, starting with the simplest example, namely, the main decay mode of the muon (see Fig. 33),

$$
\begin{equation*}
\mu^{-} \rightarrow e^{-} \bar{\nu}_{e} \nu_{\mu} \tag{2.1}
\end{equation*}
$$

Following Eq. 1.60, the relevant quantity to compute is

$$
\begin{equation*}
\mathcal{M}_{\mathrm{fi}}=-\frac{G}{\sqrt{2}}\left\langle e^{-}\left(k, s_{e}\right) \bar{\nu}_{e}\left(q_{1}\right) \nu_{\mu}\left(q_{2}\right)\right|\left(\bar{\nu}_{\mu}(0) \mathcal{O}_{L}^{\alpha} \mu(0)\right)\left(\bar{e}(0) \mathcal{O}_{L \alpha} \nu_{e}(0)\right)\left|\mu^{-}\left(p, s_{\mu}\right)\right\rangle \tag{2.2}
\end{equation*}
$$

where we have already selected the only combination of currents that contributes. Here $p_{\mu, e}$ and $s_{\mu, e}$ are the four-momentum and the spin of muon and electron, and $q_{1,2}$ are the momenta of the neutrinos. We assume neutrinos to be massless, so there is no need to specify their helicity since it is fixed. This matrix element is easily evaluated going over to momentum space, or directly using the Feynman rules listed above in section 1.3, and equals

$$
\begin{equation*}
\mathcal{M}_{\mathrm{fi}}=-\frac{G}{\sqrt{2}}\left(\bar{u}^{\left(\nu_{\mu}\right)}\left(q_{2}\right) \mathcal{O}_{L}^{\alpha} u^{(\mu)}\left(p, s_{\mu}\right)\right)\left(\bar{u}^{(e)}\left(k, s_{e}\right) \mathcal{O}_{L \alpha} v^{\left(\nu_{e}\right)}\left(q_{1}\right)\right) . \tag{2.3}
\end{equation*}
$$

Here we have conveniently changed the notation for the Dirac bispinors in an obvious way. The decay width $d \Gamma$ involves the absolute value square of this matrix element,

$$
\begin{align*}
&\left|\mathcal{M}_{\mathrm{f}}\right|^{2}=\frac{G^{2}}{2}\left(\bar{u}^{\left(\nu_{\mu}\right)}\left(q_{2}\right) \mathcal{O}_{L}^{\alpha} u^{(\mu)}\left(p, s_{\mu}\right)\right)\left(\bar{u}^{(\mu)}\left(p, s_{\mu}\right) \mathcal{O}_{L}^{\beta} u^{\left(\nu_{\mu}\right)}\left(q_{2}\right)\right) \\
& \times\left(\bar{u}^{(e)}\left(k, s_{e}\right) \mathcal{O}_{L \alpha} v^{\left(\nu_{e}\right)}\left(q_{1}\right)\right)\left(\bar{v}^{\left(\nu_{e}\right)}\left(q_{1}\right) \mathcal{O}_{L \beta} u^{(e)}\left(k, s_{e}\right)\right)  \tag{2.4}\\
&=\frac{G^{2}}{2} \operatorname{tr}\left(u^{\left(\nu_{\mu}\right)}\left(q_{2}\right) \bar{u}^{\left(\nu_{\mu}\right)}\left(q_{2}\right) \mathcal{O}_{L}^{\alpha} u^{(\mu)}\left(p, s_{\mu}\right) \bar{u}^{(\mu)}\left(p, s_{\mu}\right) \mathcal{O}_{L}^{\beta}\right) \\
& \times \operatorname{tr}\left(v^{\left(\nu_{e}\right)}\left(q_{1}\right) \bar{v}^{\left(\nu_{e}\right)}\left(q_{1}\right) \mathcal{O}_{L \beta} u^{(e)}\left(k, s_{e}\right) \bar{u}^{(e)}\left(k, s_{e}\right) \mathcal{O}_{L \alpha}\right) .
\end{align*}
$$

We now study the amplitude with an increasing degree of detail.

An alternative but equivalent form of $\mathcal{M}_{\mathrm{fi}}$ is obtained making use of the following Fierz identity,

$$
\begin{equation*}
\left(\bar{a} O_{L}^{\alpha} b\right)\left(\bar{c} O_{L \alpha} d\right)=-\left(\bar{a} O_{L}^{\alpha} d\right)\left(\bar{c} O_{L \alpha} b\right), \tag{2.5}
\end{equation*}
$$

derived in Section 2.4 below. This results in the following alternative form of Eq. 2.3,

$$
\begin{equation*}
\mathcal{M}_{\mathrm{fi}}=\frac{G}{\sqrt{2}}\left(\bar{u}^{\left(\nu_{\mu}\right)}\left(q_{2}\right) \mathcal{O}_{L}^{\alpha} v^{\left(\nu_{e}\right)}\left(q_{1}\right)\right)\left(\bar{u}^{(e)}\left(k, s_{e}\right) \mathcal{O}_{L \alpha} u^{(\mu)}\left(p, s_{\mu}\right)\right) \tag{2.6}
\end{equation*}
$$

The complex conjugate of this can be combined with Eq. 2.3 to yield the following equivalent form of Eq. 2.4.,

$$
\begin{align*}
\left|\mathcal{M}_{\mathrm{f}}\right|^{2}=- & \frac{G^{2}}{2}\left(\bar{v}^{\left(\nu_{e}\right)}\left(q_{1}\right) \mathcal{O}_{L}^{\alpha} u^{\left(\nu_{\mu}\right)}\left(q_{2}\right)\right)\left(\bar{u}^{(\mu)}\left(p, s_{\mu}\right) \mathcal{O}_{L \alpha} u^{(e)}\left(k, s_{e}\right)\right) \\
& \times\left(\bar{u}^{\left(\nu_{\mu}\right)}\left(q_{2}\right) \mathcal{O}_{L}^{\beta} u^{(\mu)}\left(p, s_{\mu}\right)\right)\left(\bar{u}^{(e)}\left(k, s_{e}\right) \mathcal{O}_{L \beta} v^{\left(\nu_{e}\right)}\left(q_{1}\right)\right)  \tag{2.7}\\
=- & \frac{G^{2}}{2}\left(\bar{v}^{\left(\nu_{e}\right)}\left(q_{1}\right) \mathcal{O}_{L}^{\alpha} u^{\left(\nu_{\mu}\right)}\left(q_{2}\right)\right)\left(\bar{u}^{\left(\nu_{\mu}\right)}\left(q_{2}\right) \mathcal{O}_{L}^{\beta} u^{(\mu)}\left(p, s_{\mu}\right)\right) \\
& \times\left(\bar{u}^{(\mu)}\left(p, s_{\mu}\right) \mathcal{O}_{L \alpha} u^{(e)}\left(k, s_{e}\right)\right)\left(\bar{u}^{(e)}\left(k, s_{e}\right) \mathcal{O}_{L \beta} v^{\left(\nu_{e}\right)}\left(q_{1}\right)\right) .
\end{align*}
$$

We will use the form Eq. 2.4, commenting on how the calculation develops when using instead the alternative expression Eq. 2.7.

### 2.1 Unpolarised muons, electron spin not measured

In this case we have to sum the decay width over the spin of the electron, and average over the spin of the muon. To this end, for $m \neq 0$ one makes use of the following completeness relations,

$$
\begin{equation*}
\sum_{s} u_{s}(\vec{p}) \bar{u}_{s}(\vec{p})=\not p+m, \quad \sum_{s} v_{s}(\vec{p}) \bar{v}_{s}(\vec{p})=\not p-m \tag{2.8}
\end{equation*}
$$

while for massless fermions of definite helicity $h= \pm 1$ the bispinors satisfy

$$
\begin{equation*}
u_{h}(\vec{p}) \bar{u}_{h}(\vec{p})=\not p \frac{1-h \gamma^{5}}{2}, \quad v_{h}(\vec{p}) \bar{v}_{h}(\vec{p})=\not p \frac{1+h \gamma^{5}}{2} \tag{2.9}
\end{equation*}
$$

Summing over spins in Eq. (2.4), and taking into account that $h=-1$ (resp. $h=+1$ ) for neutrinos (resp. antineutrinos) one then obtains

$$
\begin{equation*}
\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{fi}}\right|^{2}\right\rangle\right\rangle \equiv \sum_{s_{\mu}, s_{e}}\left|\mathcal{M}_{\mathrm{fi}}\right|^{2}=\frac{G^{2}}{2} \operatorname{tr}\left(\not q_{2} \mathcal{O}_{L}^{\alpha}\left(\not p+m_{\mu}\right) \mathcal{O}_{L}^{\beta}\right) \operatorname{tr}\left(\not q_{1} \mathcal{O}_{L \beta}\left(\not \nless+m_{e}\right) \mathcal{O}_{L \alpha}\right) \tag{2.10}
\end{equation*}
$$

having used $\frac{1+\gamma^{5}}{2} \mathcal{O}_{L}^{\alpha}=\frac{1+\gamma^{5}}{2} \gamma^{\alpha}\left(1-\gamma^{5}\right)=\gamma^{\alpha} \frac{1-\gamma^{5}}{2}\left(1-\gamma^{5}\right)=\mathcal{O}_{L}^{\alpha}$. The terms proportional to the fermion masses drop, since they involve traces of an odd number of gamma matrices $\left(\gamma^{5}\right.$ counts as four gamma matrices) which automatically vanish. We are then left with

$$
\begin{align*}
\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle & =\frac{G^{2}}{2} \operatorname{tr}\left(\not q_{2} \mathcal{O}_{L}^{\alpha} \not p \mathcal{O}_{L}^{\beta}\right) \operatorname{tr}\left(\not q_{1} \mathcal{O}_{L \beta} \not \mathcal{K}_{L \alpha}\right) \\
& =\frac{G^{2}}{2} \operatorname{tr}\left(\not q_{2} \gamma^{\alpha}\left(1-\gamma^{5}\right) \not p \gamma^{\beta}\left(1-\gamma^{5}\right)\right) \operatorname{tr}\left(\not q_{1} \gamma_{\beta}\left(1-\gamma^{5}\right) \not k \gamma_{\alpha}\left(1-\gamma^{5}\right)\right)  \tag{2.11}\\
& =\frac{G^{2}}{2} 4 \operatorname{tr}\left(\not q_{2} \gamma^{\alpha} \not p \gamma^{\beta}\left(1-\gamma^{5}\right)\right) \operatorname{tr}\left(\not q_{1} \gamma_{\beta} \not k \gamma_{\alpha}\left(1-\gamma^{5}\right)\right)
\end{align*}
$$

having used $\left(1-\gamma^{5}\right)^{2}=2\left(1-\gamma^{5}\right)$.

Starting from the alternative expression Eq. 2.7) one obtains the following single-trace formula instead of Eq. 2.10,

$$
\begin{align*}
\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle & =-\frac{G^{2}}{2} \operatorname{tr} \not q_{1} \mathcal{O}_{L}^{\alpha} q_{2} \mathcal{O}_{L}^{\beta}\left(\not p+m_{\mu}\right) \mathcal{O}_{L \alpha}\left(\not \nless+m_{e}\right) \mathcal{O}_{L \beta}  \tag{2.12}\\
& =-\frac{G^{2}}{2} \operatorname{tr} \not q_{1} \mathcal{O}_{L}^{\alpha} q_{2} \mathcal{O}_{L}^{\beta} \not p \mathcal{O}_{L \alpha} \not k \mathcal{O}_{L \beta}
\end{align*}
$$

The terms linear in $m_{\mu}$ and $m_{e}$ vanish since they contain the trace of an odd number of gamma matrices, while the term proportional to $m_{\mu} m_{e}$ vanishes since $\mathcal{O}_{L \alpha} \mathcal{O}_{L \beta}=\gamma_{\alpha}\left(1-\gamma^{5}\right) \gamma_{\beta}\left(1-\gamma^{5}\right)=$ $\gamma_{\alpha}\left(1-\gamma^{5}\right)\left(1+\gamma^{5}\right) \gamma_{\beta}=0$.

We now need the following identities for traces of products of gamma matrices:

$$
\begin{align*}
\operatorname{tr} \gamma^{\mu} \gamma^{\alpha} \gamma^{\nu} \gamma^{\beta} & =4\left(\eta^{\mu \alpha} \eta^{\nu \beta}-\eta^{\mu \nu} \eta^{\alpha \beta}+\eta^{\mu \beta} \eta^{\nu \alpha}\right) \equiv 4 S^{\mu \alpha \nu \beta}  \tag{2.13}\\
\operatorname{tr} \gamma^{5} \gamma^{\mu} \gamma^{\alpha} \gamma^{\nu} \gamma^{\beta} & =-4 i \varepsilon^{\mu \alpha \nu \beta}
\end{align*}
$$

with $\varepsilon^{\mu \alpha \nu \beta}$ the totally antisymmetric tensor with $\varepsilon^{0123}=1$. Using the symmetries of the two tensors we can drop crossed terms in Eq. 2.11 and write

$$
\begin{align*}
\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle & =\frac{G^{2}}{2} 4^{3}\left(S^{\mu \alpha \nu \beta}+i \varepsilon^{\mu \alpha \nu \beta}\right)\left(S_{\rho \alpha \sigma \beta}+i \varepsilon_{\rho \beta \sigma \alpha}\right) q_{2 \mu} p_{\nu} q_{1}^{\rho} k^{\sigma}  \tag{2.14}\\
& =32 G^{2}\left(S^{\mu \alpha \nu \beta} S_{\rho \alpha \sigma \beta}+\varepsilon^{\mu \alpha \nu \beta} \varepsilon_{\rho \alpha \sigma \beta}\right) q_{2 \mu} p_{\nu} q_{1}^{\rho} k^{\sigma}
\end{align*}
$$

A straightforward calculation shows that

$$
\begin{equation*}
S^{\mu \alpha \nu \beta} S_{\rho \alpha \sigma \beta}=2\left(\delta_{\sigma}^{\mu} \delta_{\rho}^{\nu}+\delta_{\rho}^{\mu} \delta_{\sigma}^{\nu}\right) \tag{2.15}
\end{equation*}
$$

The other contraction reads instead

$$
\begin{equation*}
\varepsilon^{\mu \alpha \nu \beta} \varepsilon_{\rho \alpha \sigma \beta}=2\left(\delta_{\sigma}^{\mu} \delta_{\rho}^{\nu}-\delta_{\rho}^{\mu} \delta_{\sigma}^{\nu}\right) \tag{2.16}
\end{equation*}
$$

Proof: The left hand-side in Eq. (2.16) is a Lorentz-invariant tensor $T_{\rho \sigma}^{\mu \nu}=T^{\mu \nu \alpha \beta} \eta_{\alpha \rho} \eta_{\beta \sigma}$, with $T^{\mu \nu \alpha \beta}$ invariant, and antisymmetric in both the first and second pair of indices, and symmetric under exchange of the two pairs. It is furthermore invariant under parity. The only such tensor is $T^{\mu \nu \alpha \beta}=A\left(\eta^{\mu \alpha} \eta^{\nu \beta}-\eta^{\mu \beta} \eta^{\nu \alpha}\right)$ for some constant $A$, so $T_{\rho \sigma}^{\mu \nu}=A\left(\delta^{\mu}{ }_{\rho} \delta^{\nu}{ }_{\sigma}-\delta^{\mu}{ }_{\sigma} \delta^{\nu}{ }_{\rho}\right)$. Contracting $\mu$ with $\rho$ and $\nu$ with $\sigma, T_{\mu \nu}^{\mu \nu}=12 A=\varepsilon^{\mu \alpha \nu \beta} \varepsilon_{\mu \alpha \nu \beta}=-24$, where the last passage follows from counting the nonzero entries of the Levi-Civita tensor (which are $4!=24$ ), recalling that they are equal to $\pm 1$, and that $\varepsilon_{0123}=-\varepsilon^{0123}=-1$.

Plugging Eqs. (2.15) and (2.16) into Eq. (2.14) we find

$$
\begin{equation*}
\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle=128 G^{2} \delta^{\mu}{ }_{\sigma} \delta^{\nu}{ }_{\rho} q_{2 \mu} p_{\nu} q_{1}^{\rho} k^{\sigma}=128 G^{2}\left(p \cdot q_{1}\right)\left(k \cdot q_{2}\right) \tag{2.17}
\end{equation*}
$$

for the square amplitude summed over spins.
If one wants to use instead Eq. 2.12 to find the decay width, then one needs the following identities:

$$
\begin{equation*}
\gamma^{\alpha} A \mathscr{A} \gamma_{\alpha}=4 A \cdot B, \quad \gamma^{\alpha} A \mathscr{A} \not \subset \gamma_{\alpha}=-2 \not \subset \not B A A \tag{2.18}
\end{equation*}
$$

which can be proved straightforwardly making only use of the anticommutation relations of gamma matrices. Using these, one shows that

$$
\begin{align*}
& \operatorname{tr} \not q_{1} \mathcal{O}_{L}^{\alpha} q_{2} \mathcal{O}_{L}^{\beta} \not p \mathcal{O}_{L \alpha} \not k \mathcal{O}_{L \beta}=\operatorname{tr} q_{1} \gamma^{\alpha}\left(1-\gamma^{5}\right) q_{2} \gamma^{\beta}\left(1-\gamma^{5}\right) \not p \gamma_{\alpha}\left(1-\gamma^{5}\right) \not k \gamma_{\beta}\left(1-\gamma^{5}\right) \\
& =\operatorname{tr} \not q_{1} \gamma^{\alpha} \not q_{2} \gamma^{\beta} \not p \gamma_{\alpha} \not k \gamma_{\beta}\left(1-\gamma^{5}\right)^{4}=-16 \operatorname{tr} \not q_{1} \not p \gamma^{\beta} \not q_{2} \not k \gamma_{\beta}\left(1-\gamma^{5}\right)=-64 \operatorname{tr} \not q_{1} \not p\left(k \cdot q_{2}\right)\left(1-\gamma^{5}\right)  \tag{2.19}\\
& -64 \operatorname{tr} \not q_{1} \not p\left(k \cdot q_{2}\right)=-256\left(p \cdot q_{1}\right)\left(k \cdot q_{2}\right)
\end{align*}
$$

from which Eq. 2.17 follows.
Taking into account a factor $1 /(2 s+1)=1 / 2$ due to averaging (not summing) over the muon spin, according to Eq. 1.56 the differential decay width reads in the muon rest frame

$$
\begin{equation*}
d \Gamma=\frac{1}{2 m_{\mu}} \frac{\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle}{2} d \Phi^{(3)}=\frac{32 G^{2}}{m_{\mu}}\left(p \cdot q_{1}\right)\left(k \cdot q_{2}\right) d \Phi^{(3)} \tag{2.20}
\end{equation*}
$$

The phase-space element reads

$$
\begin{equation*}
d \Phi^{(3)}=(2 \pi)^{4} \delta^{(4)}\left(p-k-q_{1}-q_{2}\right) \frac{d^{3} k}{(2 \pi)^{3} 2 E} \frac{d^{3} q_{1}}{(2 \pi)^{3} 2 \omega_{1}} \frac{d^{3} q_{2}}{(2 \pi)^{3} 2 \omega_{2}} \tag{2.21}
\end{equation*}
$$

where $E=k^{0}=\sqrt{\vec{k}^{2}+m_{e}^{2}}$ and $\omega_{i}=q_{i}^{0}=\left|\vec{q}_{i}\right|$. Four-momentum conservation imposes that $p-k=q_{1}+q_{2}$ be a timelike or lightlike vector, as it satisfies $(p-q)^{2}=\left(q_{1}+q_{2}\right)^{2}=2 q_{1} \cdot q_{2}=$ $2 \omega_{1} \omega_{2}\left(1-\cos \theta_{12}\right) \geq 0$, where $\theta_{12}$ is the relative angle between the trajectories of the neutrinos. As a consequence, the electron energy in the muon rest frame is bounded by $m_{\mu}^{2}+m_{e}^{2}-2 m_{\mu} E \geq 0$, i.e.,

$$
\begin{equation*}
E \leq \frac{m_{\mu}^{2}+m_{e}^{2}}{2 m_{\mu}}=\frac{m_{\mu}}{2}\left(1+\mathcal{O}\left(\frac{m_{e}^{2}}{m_{\mu}^{2}}\right)\right) \tag{2.22}
\end{equation*}
$$

Typically neutrinos are not detected, and measurements are made only on the electron. We then integrate over the neutrino momenta, and get

$$
\begin{align*}
d \Gamma & =\frac{32 G^{2}}{m_{\mu}} \frac{d^{3} k}{(2 \pi)^{5} 8 E} \int \frac{d^{3} q_{1}}{\omega_{1}} \int \frac{d^{3} q_{2}}{\omega_{2}} \delta^{(4)}\left(p-k-q_{1}-q_{2}\right)\left(p \cdot q_{1}\right)\left(k \cdot q_{2}\right) \\
& =\frac{G^{2}}{8 m_{\mu} \pi^{5}} \frac{d^{3} k}{E} p^{\alpha} k^{\beta} I_{\alpha \beta}(p-k),  \tag{2.23}\\
I_{\alpha \beta}(q) & \equiv \int \frac{d^{3} q_{1}}{\omega_{1}} \int \frac{d^{3} q_{2}}{\omega_{2}} \delta^{(4)}\left(q-q_{1}-q_{2}\right) q_{1 \alpha} q_{2 \beta}
\end{align*}
$$

Evaluation of $I_{\alpha \beta}(q)$ is made easier by exploiting its properties under Lorentz transformation: since it must be a symmetric tensor of mass dimension 2 built out of $q$, one must have

$$
\begin{equation*}
I_{\alpha \beta}(q)=A q^{2} \eta_{\alpha \beta}+B q_{\alpha} q_{\beta} \tag{2.24}
\end{equation*}
$$

with $A$ and $B$ dimensionless functions of $q^{2}$. Since there is no other dimensionful invariant available besides $q^{2}$, they must simply be numerical constants. Next, notice that thanks to the delta function one can replace $q^{2} \rightarrow 2 q_{1} \cdot q_{2}$ in the integrand, and so

$$
\begin{align*}
\eta^{\alpha \beta} I_{\alpha \beta}(q) & =q^{2}(4 A+B)=\int \frac{d^{3} q_{1}}{\omega_{1}} \int \frac{d^{3} q_{2}}{\omega_{2}} \delta^{(4)}\left(q-q_{1}-q_{2}\right) q_{1} \cdot q_{2}=\frac{q^{2}}{2} C \\
q^{\alpha} q^{\beta} I_{\alpha \beta}(q) & =\left(q^{2}\right)^{2}(A+B)=\int \frac{d^{3} q_{1}}{\omega_{1}} \int \frac{d^{3} q_{2}}{\omega_{2}} \delta^{(4)}\left(q-q_{1}-q_{2}\right)\left(q_{1} \cdot q_{2}\right)^{2}=\frac{\left(q^{2}\right)^{2}}{4} C  \tag{2.25}\\
C & \equiv \int \frac{d^{3} q_{1}}{\omega_{1}} \int \frac{d^{3} q_{2}}{\omega_{2}} \delta^{(4)}\left(q-q_{1}-q_{2}\right)
\end{align*}
$$

The resulting system of equations for $A$ and $B$ is easily solved to give $A=C / 12, B=C / 6$. Since $C$ is a Lorentz invariant it can be equivalently evaluated in any frame. Since $q_{1,2}$ are lightlike vectors, due to the delta function imposing $q=q_{1}+q_{2}$ one has that $I_{\alpha \beta}(q)$ can be nonzero only if $q$ is a timelike vector ${ }^{20}$ so we can choose the frame where $\vec{q}=0$, which is the centre-of-mass frame of the neutrino-antineutrino system. Upon integration we then find

$$
\begin{align*}
C & =\int \frac{d^{3} q_{1}}{\omega_{1}} \int \frac{d^{3} q_{2}}{\omega_{2}} \delta\left(q^{0}-\omega_{1}-\omega_{2}\right) \delta^{(3)}\left(\vec{q}_{1}+\vec{q}_{2}\right)=\int \frac{d^{3} q_{1}}{\omega_{1}^{2}} \delta\left(q^{0}-2 \omega_{1}\right) \\
& =\frac{1}{2} \int d \Omega \int \frac{d \omega_{1} \omega_{1}^{2}}{\omega_{1}^{2}} \delta\left(\frac{1}{2} q^{0}-\omega_{1}\right)=2 \pi \tag{2.26}
\end{align*}
$$

and so

$$
\begin{equation*}
I_{\alpha \beta}(q)=\frac{\pi}{6}\left(q^{2} \eta_{\alpha \beta}+2 q_{\alpha} q_{\beta}\right) . \tag{2.27}
\end{equation*}
$$

The case where neutrinos are replaced by particles of arbitrary mass is discussed in Section 2.5. Plugging this result back into Eq. (2.23) we find

$$
\begin{align*}
d \Gamma & =\frac{G^{2}}{8 m_{\mu} \pi^{5}} \frac{d^{3} k}{E} p^{\alpha} k^{\beta} \frac{\pi}{6}\left((p-k)^{2} \eta_{\alpha \beta}+2(p-k)_{\alpha}(p-k)_{\beta}\right) \\
& =\frac{G^{2}}{48 m_{\mu} \pi^{4}} \frac{d^{3} k}{E}\left((p-k)^{2} p \cdot k+2 p \cdot(p-k) k \cdot(p-k)\right) \\
& =\frac{G^{2}}{48 m_{\mu} \pi^{4}} \frac{d^{3} k}{E}\left(\left(p^{2}+k^{2}-2 p \cdot k\right) p \cdot k-2\left(p^{2}-p \cdot k\right)\left(k^{2}-p \cdot k\right)\right)  \tag{2.28}\\
& =\frac{G^{2}}{48 m_{\mu} \pi^{4}} \frac{d^{3} k}{E}\left(3\left(p^{2}+k^{2}\right) p \cdot k-4(p \cdot k)^{2}-2 p^{2} k^{2}\right) \\
& =\frac{G^{2}}{48 m_{\mu} \pi^{4}} \frac{d^{3} k}{E}\left(3\left(m_{\mu}^{2}+m_{e}^{2}\right) m_{\mu} E-4\left(m_{\mu} E\right)^{2}-2 m_{\mu}^{2} m_{e}^{2}\right) .
\end{align*}
$$

We now make approximations based on the fact that the electron mass is much smaller than the muon mass, so that the last term is much smaller than the first (in which we can ignore the electron mass), since $m_{e}^{2} /\left(m_{\mu} E\right)<m_{e} / m_{\mu} \ll 1$, and that the electron is typically ultrarelativistic (we will check this assumption self-consistently at the end of the calculation), $m_{e} / E \ll 1$, so that the last term is much smaller than the second. We can therefore neglect $m_{e}^{2}$ in the first term, and the last term altogether, and write

$$
\begin{equation*}
d \Gamma=\frac{G^{2}}{48 m_{\mu} \pi^{4}} \frac{d^{3} k}{E}(p \cdot k)\left(3 p^{2}-4(p \cdot k)\right)=\frac{G^{2}}{48 m_{\mu} \pi^{4}} \frac{d^{3} k}{E} m_{\mu}^{2} E\left(3 m_{\mu}-4 E\right) \tag{2.29}
\end{equation*}
$$

up to terms of order $\left(m_{e} / m_{\mu}\right)^{2},\left(m_{e} / m_{\mu}\right)\left(m_{e} / E\right)$, and $\left(m_{e} / E\right)^{2}$. Integrating over the direction of the electron, and using $k d k=E d E$, we get

$$
\begin{align*}
d \Gamma & =\frac{G^{2}}{48 m_{\mu} \pi^{4}} 4 \pi d E E \sqrt{E^{2}-m_{e}^{2}} m_{\mu}^{2}\left(3 m_{\mu}-4 E\right)  \tag{2.30}\\
& =\frac{G^{2} m_{\mu}}{12 \pi^{3}}\left(3 m_{\mu}-4 E\right) E \sqrt{E^{2}-m_{e}^{2}} d E .
\end{align*}
$$

[^12]

Figure 4: Probability distribution function for the energy of the electron produced in the muon decay $\mu^{-} \rightarrow e^{-} \bar{\nu}_{e} \nu_{\mu}$.

The maximal energy that the electron can reach is $E_{\max }=\frac{m_{\mu}}{2}$, up to corrections of relative order $\mathcal{O}\left(m_{e}^{2} / m_{\mu}^{2}\right)$, and expressing $d \Gamma$ in terms of the variable $\varepsilon \equiv E / E_{\max }=2 m_{\mu} / E$ we find

$$
\begin{align*}
d \Gamma & =\frac{G^{2} m_{\mu}^{2}}{12 \pi^{3}}\left(3-\frac{2 E}{m_{\mu} / 2}\right)\left(\frac{m_{\mu}}{2}\right)^{3} \frac{E}{m_{\mu} / 2} \sqrt{\left(\frac{E}{m_{\mu} / 2}\right)^{2}-\left(\frac{m_{e}}{m_{\mu} / 2}\right)^{2}} \frac{d E}{m_{\mu} / 2}  \tag{2.31}\\
& =\frac{G^{2} m_{\mu}^{5}}{96 \pi^{3}}(3-2 \varepsilon) \varepsilon^{2} d \varepsilon
\end{align*}
$$

having consistently neglected the term of order $\left(m_{e} / m_{\mu}\right)^{2}$ in the square root, since it gives a correction of relative order $\mathcal{O}\left(m_{e}^{2} / E^{2}\right)$ to the width. Similarly, the lower limit of integration is $m_{e} / E_{\max }=\mathcal{O}\left(m_{e} / m_{\mu}\right)$, which is negligible. In our approximation, the variable $\varepsilon$ thus runs from 0 to 1 . The total width is obtained integrating Eq. (2.31) over $\varepsilon$, and equals

$$
\begin{equation*}
\Gamma=\frac{G^{2} m_{\mu}^{5}}{96 \pi^{3}} \int_{0}^{1} d \varepsilon(3-2 \varepsilon) \varepsilon^{2}=\frac{G^{2} m_{\mu}^{5}}{192 \pi^{3}} . \tag{2.32}
\end{equation*}
$$

This can be compared to experiments to extract the Fermi constant $G$ (after the appropriate electromagnetic radiative corrections are included). Using Eq. (2.32) we can recast Eq. (2.31) as

$$
\begin{equation*}
\frac{1}{\Gamma} \frac{d \Gamma}{d \varepsilon}=(6-4 \varepsilon) \varepsilon^{2} \tag{2.33}
\end{equation*}
$$

which provides the probability distribution function for the electron energy (see Fig. (4). Integrating this quantity one finds out that in more than $98 \%$ of the cases one has $\varepsilon>0.2$, which since $\varepsilon=\left(E / m_{e}\right)\left(2 m_{e} / m_{\mu}\right) \simeq 0.01 E / m_{e}$ corresponds to a gamma factor $\gamma=E / m_{e} \gtrsim 20$, i.e., an ultrarelativistic electron.

### 2.2 Polarised muons

The generalisation to the case when muons are polarised is rather easy, if we recall that for a fermion of mass $m$ with definite positive spin in direction $\vec{\eta}$ in its rest frame one has

$$
\begin{equation*}
u(p, s) \bar{u}(p, s)=(\not p+m) \frac{1+\gamma^{5} \nless}{2} . \tag{2.34}
\end{equation*}
$$

Here $s$ is a spacelike vector that in the fermion rest frame reads $s=(0, \vec{\eta})$, and in a generic reference frame is transformed $t{ }^{21}$

$$
\begin{equation*}
s=\left(\frac{\vec{\eta} \cdot \vec{p}}{m}, \vec{\eta}+\frac{\vec{p}(\vec{\eta} \cdot \vec{p})}{m\left(p^{0}+m\right)}\right) \tag{2.35}
\end{equation*}
$$

Clearly, $s^{2}=-\vec{\eta}^{2}=-1$ and $s \cdot p=0$. In the most general case, Eq. (2.4) is then equal to

$$
\begin{align*}
\left|\mathcal{M}_{\mathrm{f}}\right|^{2} & =\frac{G^{2}}{2} \frac{1}{2^{2}} \operatorname{tr}\left(\phi_{2} \mathcal{O}_{L}^{\alpha}\left(\not p+m_{\mu}\right)\left(1+\gamma^{5} \phi_{\mu}\right) \mathcal{O}_{L}^{\beta}\right) \operatorname{tr}\left(q_{1} \mathcal{O}_{L \beta}\left(\not k+m_{e}\right)\left(1+\gamma^{5} \phi_{e}\right) \mathcal{O}_{L \alpha}\right) \\
& =\frac{G^{2}}{2} \operatorname{tr}\left(q_{2} \gamma^{\alpha}\left(\not p+m_{\mu}\right)\left(1+\gamma^{5} \phi_{\mu}\right) \gamma^{\beta}\left(1-\gamma^{5}\right)\right) \operatorname{tr}\left(q_{1} \gamma_{\beta}\left(\not k+m_{e}\right)\left(1+\gamma^{5} \phi_{e}\right) \gamma_{\alpha}\left(1-\gamma^{5}\right)\right) . \tag{2.36}
\end{align*}
$$

The polarisation vectors of the muon and of the electron will be denoted simply with $\vec{\eta}_{\mu}=\vec{\eta}$ and $\vec{\eta}_{e}=\vec{\zeta}$. From Eq. (2.36) we recover $\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle$ by replacing $s_{\mu, e} \rightarrow 0$ and multiplying by a factor 4 , which exactly corresponds to summing over the two spin values. If we want to discuss partially polarised muons, we have to average this expression over $\vec{\eta}$ with some probability distribution, and since $s$ is linear in $\vec{\eta}$, we simply have to replace $\vec{\eta} \rightarrow\langle\vec{\eta}\rangle$ in the expression for $s$, i.e., $s(\vec{\eta}) \rightarrow s(\langle\vec{\eta}\rangle)=\bar{s}$. Notice that while one still has $\bar{s} \cdot p=0$, in general $-1 \leq \bar{s}^{2} \leq 0$.

Since the two factors in Eq. 2.36) have the same structure, it suffices to study only the first one to make progress. The only terms coming from $\left(\not p+m_{\mu}\right)\left(1+\gamma^{5} \phi_{\mu}\right)$ that contribute to the trace are those containing an odd number of gamma matrices, and so we can replace $\left(\not p+m_{\mu}\right)\left(1+\gamma^{5} \phi_{\mu}\right) \rightarrow\left(\not p+\gamma^{5} m_{\mu} \phi_{\mu}\right)$. Furthermore,

$$
\begin{equation*}
\operatorname{tr}\left(\mathscr{q}_{2} \gamma^{\alpha} \gamma^{5} \$_{\mu} \gamma^{\beta}\left(1-\gamma^{5}\right)\right)=\operatorname{tr}\left(\mathscr{q}_{2} \gamma^{\alpha} \phi_{\mu} \gamma^{\beta} \gamma^{5}\left(1-\gamma^{5}\right)\right)=-\operatorname{tr}\left(q_{2} \gamma^{\alpha} \phi_{\mu} \gamma^{\beta}\left(1-\gamma^{5}\right)\right) . \tag{2.37}
\end{equation*}
$$

This means that in practice all that we have to do is replace $p \rightarrow\left(p-m_{\mu} s_{\mu}\right) / 2$ and $k \rightarrow$ ( $\left.k-m_{e} s_{e}\right) / 2$ in Eq. 2.17), and obtain in the most general case

$$
\begin{equation*}
\left|\mathcal{M}_{\mathrm{f}}\right|^{2}=32 G^{2}\left[\left(p-m_{\mu} s_{\mu}\right) \cdot q_{1}\right]\left[\left(k-m_{e} s_{e}\right) \cdot q_{2}\right], \tag{2.38}
\end{equation*}
$$

and thus

$$
\begin{equation*}
d \Gamma=\frac{1}{2 m_{\mu}}\left|\mathcal{M}_{\mathrm{fi}}\right|^{2} d \Phi^{(3)}=\frac{16 G^{2}}{m_{\mu}}\left[\left(p-m_{\mu} s_{\mu}\right) \cdot q_{1}\right]\left[\left(k-m_{e} s_{e}\right) \cdot q_{2}\right] d \Phi^{(3)} . \tag{2.39}
\end{equation*}
$$

Integrating over the momenta of neutrinos we find

$$
\begin{align*}
d \Gamma= & \frac{G^{2}}{16 m_{\mu} \pi^{5}} \frac{d^{3} k}{E}\left(p-m_{\mu} s_{\mu}\right)^{\alpha}\left(k-m_{e} s_{e}\right)^{\beta} I_{\alpha \beta}(p-k) \\
= & \frac{G^{2}}{96 m_{\mu} \pi^{4}} \frac{d^{3} k}{E}\left[(p-k)^{2}\left(p-m_{\mu} s_{\mu}\right) \cdot\left(k-m_{e} s_{e}\right)\right.  \tag{2.40}\\
& \left.+2(p-k) \cdot\left(p-m_{\mu} s_{\mu}\right)(p-k) \cdot\left(k-m_{e} s_{e}\right)\right] .
\end{align*}
$$

[^13]Setting $\tilde{p}=p-m_{\mu} s_{\mu}, \tilde{k}=k-m_{e} s_{e}$, we have for the term in square brackets

$$
\begin{align*}
{[\cdots]=} & (p-k)^{2} \tilde{p} \cdot \tilde{k}+2(p-k) \cdot \tilde{p}(p-k) \cdot \tilde{k} \\
= & (p-k)^{2}\left(p \cdot \tilde{k}-m_{\mu} s_{\mu} \cdot \tilde{k}\right)+2\left[(p-k) \cdot p+m_{\mu} s_{\mu} \cdot k\right]\left(p \cdot \tilde{k}-k^{2}\right) \\
= & (p \cdot \tilde{k})\left[(p-k)^{2}+2(p-k) \cdot p\right]  \tag{2.41}\\
& -m_{\mu}\left[(p-k)^{2} s_{\mu} \cdot \tilde{k}-2\left(p \cdot \tilde{k}-k^{2}\right) s_{\mu} \cdot k\right]-2 k^{2}(p-k) \cdot p \\
= & (p \cdot \tilde{k})\left(3 p^{2}-4 p \cdot k+k^{2}\right) \\
& -m_{\mu}\left[\left(p^{2}-2 p \cdot k+k^{2}\right) s_{\mu} \cdot \tilde{k}-2\left(p \cdot \tilde{k}-k^{2}\right) s_{\mu} \cdot k\right]-2 k^{2}\left(p^{2}-p \cdot k\right) .
\end{align*}
$$

Since $m_{\mu} \gg m_{e}$, we can neglect $k^{2}=m_{e}^{2}$ against $p^{2}=m_{\mu}^{2}$ and $p \cdot k=E m_{\mu}$, and against $p \cdot \tilde{k}=m_{\mu}(E-\vec{\zeta} \cdot \vec{k})$. In this approximation,

$$
\begin{equation*}
[\cdots]=(p \cdot \tilde{k})\left(3 p^{2}-4 p \cdot k\right)-m_{\mu}\left[\left(p^{2}-2 p \cdot k\right) s_{\mu} \cdot \tilde{k}-2 p \cdot \tilde{k} s_{\mu} \cdot k\right] . \tag{2.42}
\end{equation*}
$$

If we sum over the electron spin states, this becomes

$$
\begin{equation*}
[\ldots] \xrightarrow[\text { sum over } s_{e}]{ } 2\left[(p \cdot k)\left(3 p^{2}-4 p \cdot k\right)-m_{\mu} s_{\mu} \cdot k\left(p^{2}-4 p \cdot k\right)\right] . \tag{2.43}
\end{equation*}
$$

and so

$$
\begin{align*}
d \Gamma & =\frac{G^{2}}{48 m_{\mu} \pi^{4}} \frac{d^{3} k}{E}\left[(p \cdot k)\left(3 p^{2}-4 p \cdot k\right)-m_{\mu} s_{\mu} \cdot k\left(p^{2}-4 p \cdot k\right)\right] \\
& =\frac{G^{2}}{48 m_{\mu} \pi^{4}} \frac{d^{3} k}{E}\left[m_{\mu} E\left(3 m_{\mu}^{2}-4 m_{\mu} E\right)+m_{\mu} \vec{\eta} \cdot \vec{n}|\vec{k}|\left(m_{\mu}^{2}-4 m_{\mu} E\right)\right]  \tag{2.44}\\
& =\frac{G^{2}}{48 \pi^{4}} d^{3} k\left[\left(3 m_{\mu}^{2}-4 m_{\mu} E\right)+\vec{\eta} \cdot \vec{n} \frac{|\vec{k}|}{E}\left(m_{\mu}^{2}-4 m_{\mu} E\right)\right],
\end{align*}
$$

where $\vec{n}$ is the direction of the electron momentum. Going over to $\varepsilon$ and neglecting powers of $m_{e} / m_{\mu}$ we find

$$
\begin{align*}
d \Gamma & =\frac{G^{2} m_{\mu}^{5}}{48 \pi^{4}} \frac{d \Omega d \varepsilon \varepsilon^{2}}{8}(3-2 \varepsilon+\vec{\eta} \cdot \vec{n}(1-2 \varepsilon)) \\
& =\frac{G^{2} m_{\mu}^{5}}{96 \pi^{3}}(3-2 \varepsilon+\vec{\eta} \cdot \vec{n}(1-2 \varepsilon)) d \varepsilon \varepsilon^{2} \frac{d \Omega}{4 \pi}  \tag{2.45}\\
& =\frac{G^{2} m_{\mu}^{5}}{192 \pi^{3}}(3-2 \varepsilon+\cos \theta(1-2 \varepsilon)) d \varepsilon \varepsilon^{2} d \cos \theta \\
& =\Gamma(3-2 \varepsilon+(1-2 \varepsilon) \cos \theta) \varepsilon^{2} d \varepsilon d \cos \theta,
\end{align*}
$$

where $\theta$ is the angle between the electron momentum and the muon polarisation. Integrating over energy we find the angular distribution of the electron

$$
\begin{equation*}
\frac{1}{\Gamma} \frac{d \Gamma}{d \cos \theta}=\int_{0}^{1} d \varepsilon \varepsilon^{2}(3-2 \varepsilon+(1-2 \varepsilon) \cos \theta)=\frac{1}{2}\left(1-\frac{1}{3} \cos \theta\right) \tag{2.46}
\end{equation*}
$$

If we do not sum over but instead observe the electron spin, considering the (typical) high-energy case $E \gg m_{e}$, we have that

$$
\begin{equation*}
m_{e} s_{e}=\left(\vec{\zeta} \cdot \vec{k}, m_{e} \vec{\zeta}+\vec{k} \frac{\vec{\zeta} \cdot \vec{k}}{E+m_{e}}\right) \simeq E(\vec{\zeta} \cdot \vec{n})(1, \vec{n}) \simeq(\vec{\zeta} \cdot \vec{n}) k \tag{2.47}
\end{equation*}
$$

i.e., $\tilde{k} \simeq(1-\vec{\zeta} \cdot \vec{n}) k$, and so

$$
\begin{equation*}
[\ldots] \underset{E \gg m_{e}}{ }(1-\vec{\zeta} \cdot \vec{n})\left\{(p \cdot k)\left(3 p^{2}-4 p \cdot k\right)-m_{\mu} s_{\mu} \cdot k\left(p^{2}-4 p \cdot k\right)\right\} \tag{2.48}
\end{equation*}
$$

To obtain the differential decay width we then have to multiply Eq. (2.44) by $\frac{1}{2}(1-\vec{\zeta} \cdot \vec{n})$. Going over to $\varepsilon$ we then get

$$
\begin{align*}
d \Gamma & =\frac{G^{2} m_{\mu}^{5}}{192 \pi^{3}}(1-\vec{\zeta} \cdot \vec{n})(3-2 \varepsilon+\vec{\eta} \cdot \vec{n}(1-2 \varepsilon)) d \varepsilon \varepsilon^{2} \frac{d \Omega}{4 \pi}  \tag{2.49}\\
& =\Gamma(1-\vec{\zeta} \cdot \vec{n})(3-2 \varepsilon+\vec{\eta} \cdot \vec{n}(1-2 \varepsilon)) d \varepsilon \varepsilon^{2} \frac{d \Omega}{4 \pi}
\end{align*}
$$

If we choose to measure the azimuthal angle of $\vec{n}$ from $\vec{\zeta}$, then

$$
\begin{equation*}
\cos \phi=\frac{[\vec{\zeta}-\vec{\eta}(\vec{\eta} \cdot \vec{\zeta})] \cdot[\vec{n}-\vec{\eta}(\vec{\eta} \cdot \vec{n})]}{|\vec{\zeta}-\vec{\eta}(\vec{\eta} \cdot \vec{\zeta})||\vec{n}-\vec{\eta}(\vec{\eta} \cdot \vec{n})|}=\frac{\vec{\zeta} \cdot \vec{n}-(\vec{\eta} \cdot \vec{\zeta})(\vec{\eta} \cdot \vec{n})}{\sqrt{1-(\vec{\eta} \cdot \vec{\zeta})^{2}} \sqrt{1-(\vec{\eta} \cdot \vec{n})^{2}}}=\frac{\vec{\zeta} \cdot \vec{n}-\cos \theta \cos \delta}{\sin \theta \sin \delta} \tag{2.50}
\end{equation*}
$$

where $\cos \delta=\vec{\eta} \cdot \vec{\zeta}$. We have then $\vec{\zeta} \cdot \vec{n}=\cos \theta \cos \delta+\sin \theta \sin \delta \cos \phi$, and

$$
\begin{equation*}
\frac{d \Gamma}{\Gamma}=(1-\cos \theta \cos \delta-\sin \theta \sin \delta \cos \phi)(3-2 \varepsilon+\cos \theta(1-2 \varepsilon)) d \varepsilon \varepsilon^{2} \frac{d \Omega}{4 \pi} \tag{2.51}
\end{equation*}
$$

Notice that if we average over the muon polarisation in Eq. 2.49 , we get

$$
\begin{equation*}
\frac{d \Gamma}{\Gamma}=(1-\vec{\zeta} \cdot \vec{n})(3-2 \varepsilon) d \varepsilon \varepsilon^{2} \frac{d \Omega}{4 \pi}=\frac{1-\cos \tilde{\theta}}{2}(3-2 \varepsilon) d \varepsilon \varepsilon^{2} d \cos \tilde{\theta} \tag{2.52}
\end{equation*}
$$

with $\tilde{\theta}$ the angle between the electron momentum and polarisation.
Before discussing the implications of these results, let us derive the corresponding formulas in the case of antimuon decay,

$$
\begin{equation*}
\mu^{+} \rightarrow e^{+} \nu_{e} \bar{\nu}_{\mu} \tag{2.53}
\end{equation*}
$$

The relevant amplitude is easily seen to be

$$
\begin{equation*}
\mathcal{M}_{\mathrm{fi}}=\frac{G}{\sqrt{2}}\left(\bar{v}^{(\mu)}\left(p, s_{\mu}\right) \mathcal{O}_{L}^{\alpha} v^{\left(\nu_{\mu}\right)}\left(q_{2}\right)\right)\left(\bar{u}^{\left(\nu_{e}\right)}\left(q_{1}\right) \mathcal{O}_{L \alpha} v^{(e)}\left(k, s_{e}\right)\right) \tag{2.54}
\end{equation*}
$$

and all that is required to obtain the desired decay widths is to make the replacements

$$
\begin{equation*}
\left(\not p+m_{\mu}\right) \frac{1+\gamma^{5} \phi_{\mu}}{2} \rightarrow\left(\not p-m_{\mu}\right) \frac{1+\gamma^{5} \phi_{\mu}}{2}, \quad\left(\not k+m_{e}\right) \frac{1+\gamma^{5} \phi_{e}}{2} \rightarrow\left(\not k-m_{e}\right) \frac{1+\gamma^{5} \phi_{e}}{2} \tag{2.55}
\end{equation*}
$$

which after evaluating the traces boils down to the replacements

$$
\begin{equation*}
p-m_{\mu} s_{\mu} \rightarrow p+m_{\mu} s_{\mu}, \quad k-m_{e} s_{e} \rightarrow k+m_{e} s_{e} \tag{2.56}
\end{equation*}
$$

in the matrix element, which in turn corresponds simply to

$$
\begin{equation*}
\vec{\eta} \rightarrow-\vec{\eta}, \quad \vec{\zeta} \rightarrow-\vec{\zeta} \tag{2.57}
\end{equation*}
$$

The phase-space element is left unchanged, and so we find in the most general case of polarised muons and measured electron spin

$$
\begin{align*}
& d \Gamma_{\mu^{-}}=\Gamma(1-\vec{\zeta} \cdot \vec{n})(3-2 \varepsilon+\vec{\eta} \cdot \vec{n}(1-2 \varepsilon)) d \varepsilon \varepsilon^{2} \frac{d \Omega}{4 \pi} \\
& d \Gamma_{\mu^{+}}=\Gamma(1+\vec{\zeta} \cdot \vec{n})(3-2 \varepsilon-\vec{\eta} \cdot \vec{n}(1-2 \varepsilon)) d \varepsilon \varepsilon^{2} \frac{d \Omega}{4 \pi} \tag{2.58}
\end{align*}
$$

where we have also reported the $\mu^{-}$decay width for comparison.

### 2.3 Qualitative discussion

Let us highlight the most interesting features of the results Eqs. (2.32) and (2.58).

- The muon mass dependence $\Gamma \propto m_{\mu}^{5}$ follows for dimensional reasons from

$$
\begin{equation*}
\Gamma=m_{\mu} F\left(G m_{\mu}^{2}, m_{e} / m_{\mu}\right), \tag{2.59}
\end{equation*}
$$

for some dimensionless function $F$, once we take into account that $m_{e} / m_{\mu} \ll 1$, so that it can be neglected, and that to the given perturbative order $F(x, 0)=F_{0} x^{2}$, i.e.,

$$
\begin{equation*}
\Gamma=m_{\mu} F\left(G m_{\mu}^{2}, 0\right)=m_{\mu}\left(G m_{\mu}^{2}\right)^{2} F_{0} \tag{2.60}
\end{equation*}
$$

- The decay widths in Eq. (2.58) break both parity and charge conjugation symmetry. Indeed, under these transformations

$$
\begin{align*}
& d \Gamma_{\mu^{\mp}}(\vec{n}, \vec{\eta}, \vec{\zeta}) \underset{C}{\rightarrow} d \Gamma_{\mu^{ \pm}}(\vec{n}, \vec{\eta}, \vec{\zeta}) \neq d \Gamma_{\mu^{\mp}}(\vec{n}, \vec{\eta}, \vec{\zeta}), \\
& d \Gamma_{\mu^{\mp}}(\vec{n}, \vec{\eta}, \vec{\zeta}) \underset{P}{\rightarrow} d \Gamma_{\mu^{\mp}}(-\vec{n}, \vec{\eta}, \vec{\zeta})=d \Gamma_{\mu^{ \pm}}(\vec{n}, \vec{\eta}, \vec{\zeta}), \tag{2.61}
\end{align*}
$$

which also shows that the combined transformation $C P$ is instead a symmetry,

$$
\begin{equation*}
d \Gamma_{\mu^{\mp}}(\vec{n}, \vec{\eta}, \vec{\zeta}) \underset{P}{\rightarrow} d \Gamma_{\mu^{\mp}}(-\vec{n}, \vec{\eta}, \vec{\zeta})=d \Gamma_{\mu^{ \pm}}(\vec{n}, \vec{\eta}, \vec{\zeta}) \underset{C}{\rightarrow} d \Gamma_{\mu^{\mp}}(\vec{n}, \vec{\eta}, \vec{\zeta}) . \tag{2.62}
\end{equation*}
$$

This is reflected, for example, in the different angular distributions of the electron and the positron (summed over final spins),

$$
\begin{equation*}
\left.\frac{1}{\Gamma} \frac{d \Gamma}{d \cos \theta}\right|_{e^{-}}=\frac{1}{2}\left(1-\frac{1}{3} \cos \theta\right),\left.\quad \frac{1}{\Gamma} \frac{d \Gamma}{d \cos \theta}\right|_{e^{+}}=\frac{1}{2}\left(1+\frac{1}{3} \cos \theta\right) . \tag{2.63}
\end{equation*}
$$

- The breaking of parity can be inferred also from the fact that the decay of unpolarised muons produces polarised electrons (with polarisation that can only be along the same direction of their momentum due to rotation invariance).
- The factor $(1-\vec{\zeta} \cdot \vec{n})$ suppresses high-energy electrons with polarisation parallel to their momentum. This is a consequence of the chiral coupling of the charged currents, which suppresses massless particles (resp. antiparticles) with positive (resp. negative) helicity, and of the obvious fact that a high-energy particle effectively resembles a massless one.
- The angular asymmetry in the emission of electrons is a consequence of angular momentum conservation and of the fixed helicity of neutrinos and antineutrinos (see Fig. 5). For $\varepsilon \simeq 1$, one has $(p-k)^{2} \simeq m_{\mu}^{2}-2 m_{\mu} E \simeq 0$, so $q_{1} \cdot q_{2}=\omega_{1} \omega_{2}\left(1-\cos \theta_{\nu}\right) \simeq 0$, and since low values of $\omega_{1,2}$ are suppressed by the phase-space element as $d q_{i} / \omega_{i}=d \Omega_{i} d \omega_{i} \omega_{i}$, one has $\cos \theta_{\nu} \simeq 1$, i.e., the neutrino and the antineutrino momenta are parallel. This means that their spins add up to zero in the direction of their motion, leading to a state of zero angular momentum ${ }^{22}$ Furthermore, in the high-energy limit the electron has essentially negative helicity, and since its spin must be in the same direction as that of the muon, it

[^14]

Figure 5: Alignment of momenta and spin in the decay of the muon at high (top) and low (bottom) energy. Thin lines correspond to momenta, white arrows to spins.
must be emitted in the direction opposite to the muon spin. In the low energy limit $\vec{k} \simeq 0$ one has instead that the neutrino and antineutrino travel in opposite directions, so that their spins add up to 1 in the direction of the antineutrino. This means that the electron spin must be opposite to the muon spin, and parallel to the direction of motion of the neutrino. Since negative helicity is favoured by the chiral coupling, the electron is then emitted preferentially in the direction of the muon polarisation (see Fig. 5).

### 2.4 Appendix: Fierz identities

Equation (2.5) is a particular case of the so-called Fierz identities. The proof of these identities revolves around the basic fact that the set of sixteen matrices $\left\{\Gamma^{A}\right\}$,

$$
\begin{equation*}
\left\{\Gamma^{A}\right\}=\left\{\mathbf{1} ; \quad \gamma^{\mu}, 0 \leq \mu \leq 3 ; \quad \sigma^{\mu \nu}, 0 \leq \mu<\nu \leq 3 ; \quad i \gamma^{5} \gamma^{\mu}, 0 \leq \mu \leq 3 ; \quad \gamma^{5}\right\} \tag{2.64}
\end{equation*}
$$

provides a basis of the linear space of complex $4 \times 4$ matrices. This is in fact a 16 -dimensional space, and the matrices $\Gamma^{A}$ are necessarily linearly independent since they transform differently under proper orthocronous Lorentz and parity transformations. In particular, this implies that $\operatorname{tr} \Gamma^{A} \Gamma^{B}=0$ unless $A=B$ : the object $\operatorname{tr} \Gamma^{A} \Gamma^{B}$ must in fact be a symmetric, Lorentz- and parity-invariant object. No invariant can be obtained from matrices transforming differently under Lorentz and parity, and so this object must vanish unless $\Gamma^{A}$ and $\Gamma^{B}$ are of the same type, i.e., they belong to the same Lorentz multiplet. The only symmetric invariants can be built using $\eta^{\mu \nu}$, and one can then show that $\operatorname{tr} \Gamma^{A} \Gamma^{B}=\alpha_{A} \delta^{A B}$. The values of $\alpha_{A}$ are obtained noticing that

$$
\begin{align*}
\operatorname{tr} \mathbf{1 1} & =4, \\
\operatorname{tr} \gamma^{\alpha} \gamma^{\beta} & =\frac{1}{2} \operatorname{tr}\left\{\gamma^{\alpha}, \gamma^{\beta}\right\}=4 \eta^{\alpha \beta}, \\
\operatorname{tr}\left[\sigma^{\mu \nu} \sigma^{\alpha \beta}\right]_{\alpha<\beta}^{\mu<\nu} & =-\operatorname{tr}\left[\gamma^{\mu} \gamma^{\nu} \gamma^{\alpha} \gamma^{\beta}\right]_{\alpha<\beta}^{\mu<\nu}=-4\left[\eta^{\mu \nu} \eta^{\alpha \beta}-\eta^{\mu \alpha} \eta^{\nu \beta}+\eta^{\mu \beta} \eta^{\nu \alpha}\right]_{\alpha<\beta}^{\mu<\nu}=4 \eta^{\mu \alpha} \eta^{\nu \beta},  \tag{2.65}\\
\operatorname{tr} i \gamma^{5} \gamma^{\alpha} i \gamma^{5} \gamma^{\beta} & =\operatorname{tr} \gamma^{\alpha} \gamma^{\beta}=4 \eta^{\alpha \beta}, \\
\operatorname{tr} \gamma^{5} \gamma^{5} & =4 .
\end{align*}
$$

If we denote with $\Gamma_{A}$ the matrix obtained lowering the Lorentz indices with the metric tensor, then it follows that

$$
\begin{equation*}
\frac{1}{4} \operatorname{tr} \Gamma^{A} \Gamma_{B}=\delta_{B}^{A} . \tag{2.66}
\end{equation*}
$$

Using this identity one proves straightforwardly the linear independence of the $\Gamma^{A}$ (write any linear combination, and extract the single coefficients by taking the appropriate trace). The $\left\{\Gamma^{A}\right\}$ thus form a complete set, and one can write a generic $4 \times 4$ complex matrix as $M=\sum_{A} C_{A} \Gamma^{A}$, with

$$
\begin{equation*}
\frac{1}{4} \operatorname{tr} M \Gamma_{A}=\frac{1}{4} \sum_{B} C_{B} \operatorname{tr} \Gamma^{B} \Gamma_{A}=C_{A} . \tag{2.67}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
M=\sum_{A} \frac{1}{4} \operatorname{tr}\left[M \Gamma_{A}\right] \Gamma^{A}, \quad M_{j}^{i}=\sum_{A} \frac{1}{4} \operatorname{tr}\left[M \Gamma_{A}\right]\left(\Gamma^{A}\right)_{j}^{i}=\sum_{A} \frac{1}{4} M_{l}^{k}\left(\Gamma_{A}\right)_{k}^{l}\left(\Gamma^{A}\right)_{j}^{i}, \tag{2.68}
\end{equation*}
$$

and since this must hold for any matrix $M$, the following completeness relation follows,

$$
\begin{equation*}
\delta_{k}^{i} \delta_{j}^{l}{ }_{j}=\frac{1}{4} \sum_{A}\left(\Gamma^{A}\right)_{j}^{i}{ }_{j}\left(\Gamma_{A}\right)^{l}{ }_{k} . \tag{2.69}
\end{equation*}
$$

This can be contracted with any pair of matrices $F$ and $G$ to yield

$$
\begin{equation*}
F_{j}^{l} G_{k}^{i}=\frac{1}{4} \sum_{A}\left(\Gamma^{A}\right)_{j}^{i}\left(F \Gamma_{A} G\right)_{k}^{l}, \tag{2.70}
\end{equation*}
$$

and further with bispinors $\bar{a}, b, \bar{c}$ and $d$ to get

$$
\begin{equation*}
(\bar{a} F b)(\bar{c} G d)=\frac{1}{4} \sum_{A}\left(\bar{c} \Gamma^{A} b\right)\left(\bar{a} F \Gamma_{A} G d\right) . \tag{2.71}
\end{equation*}
$$

This is the starting point to derive a number of useful relations by taking such $F$ and $G$ that lead to a Lorentz-invariant object on the left-hand side, e.g., $F=G=\mathbf{1}, F=\gamma^{\mu}$ and $G=\gamma_{\mu}$, and so on.

The case of interest for us is $F=\mathcal{O}_{L}^{\alpha}, G=\mathcal{O}_{L \alpha}$. We can show that

$$
\begin{align*}
\mathcal{O}_{L}^{\alpha} 1 \mathcal{O}_{L \alpha} & =\gamma^{\alpha}\left(1-\gamma^{5}\right) \gamma_{\alpha}\left(1-\gamma^{5}\right)=\gamma^{\alpha} \gamma_{\alpha}\left(1+\gamma^{5}\right)\left(1-\gamma^{5}\right)=0, \\
\mathcal{O}_{L}^{\alpha} \gamma^{\mu} \mathcal{O}_{L \alpha} & =\gamma^{\alpha}\left(1-\gamma^{5}\right) \gamma^{\mu} \gamma_{\alpha}\left(1-\gamma^{5}\right)=2 \gamma^{\alpha} \gamma^{\mu} \gamma_{\alpha}\left(1-\gamma^{5}\right)=-4 \gamma^{\mu}\left(1-\gamma^{5}\right)=-4 \mathcal{O}_{L}^{\mu}, \\
\mathcal{O}_{L}^{\alpha} \sigma^{\mu \nu} \mathcal{O}_{L \alpha} & =\gamma^{\alpha}\left(1-\gamma^{5}\right) \sigma^{\mu \nu} \gamma_{\alpha}\left(1-\gamma^{5}\right)=\gamma^{\alpha} \sigma^{\mu \nu} \gamma_{\alpha}\left(1+\gamma^{5}\right)\left(1-\gamma^{5}\right)=0,  \tag{2.72}\\
\mathcal{O}_{L}^{\alpha} i \gamma^{5} \gamma^{\mu} \mathcal{O}_{L \alpha} & =-i \gamma^{5} \mathcal{O}_{L}^{\alpha} \gamma^{\mu} \mathcal{O}_{L \alpha}=4 i \gamma^{5} \mathcal{O}_{L}^{\mu}=4 i \mathcal{O}_{L}^{\mu}, \\
\mathcal{O}_{L}^{\alpha} \gamma^{5} \mathcal{O}_{L \alpha} & =-\gamma^{5} \mathcal{O}_{L}^{\alpha} \mathcal{O}_{L \alpha}=0,
\end{align*}
$$

where on the second line we used the identity

$$
\begin{equation*}
\gamma^{\alpha} \gamma^{\mu} \gamma_{\alpha}=\gamma^{\alpha}\left(2 \delta^{\mu}{ }_{\alpha}-\gamma_{\alpha} \gamma^{\mu}\right)=2 \gamma^{\mu}-4 \gamma^{\mu}=-2 \gamma^{\mu} . \tag{2.73}
\end{equation*}
$$

From this it follows that

$$
\begin{align*}
\left(\bar{a} \mathcal{O}_{L}^{\alpha} b\right)\left(\bar{c} \mathcal{O}_{L \alpha} d\right) & =\frac{1}{4}\left(-4\left(\bar{c} \gamma_{\mu} b\right)\left(\bar{a} \mathcal{O}_{L}^{\mu} d\right)+4\left(\bar{c} i \gamma^{5} \gamma_{\mu} b\right)\left(\bar{a} i \mathcal{O}_{L}^{\mu} d\right)\right) \\
& =-\left(\left(\bar{c} \gamma_{\mu} b\right)\left(\bar{a} \mathcal{O}_{L}^{\mu} d\right)+\left(\bar{c} \gamma_{\mu} \gamma^{5} b\right)\left(\bar{a} \mathcal{O}_{L}^{\mu} d\right)\right)=-\left(\bar{a} \mathcal{O}_{L}^{\mu} d\right)\left(\bar{c} \mathcal{O}_{L \mu} b\right) . \tag{2.74}
\end{align*}
$$

### 2.5 Appendix: phase-space integral

We compute here the phase-space integral $I_{\alpha \beta}(q)$ of Eq. (2.24) in the general case of final particles of masses $m_{1}$ and $m_{2}$. Using relativistic normalisation for the integration measure, let

$$
\begin{equation*}
\mathcal{I}_{\alpha \beta}(q)=\int d \Omega_{q_{1}} \int d \Omega_{q_{2}}(2 \pi)^{4} \delta^{(4)}\left(q-q_{1}-q_{2}\right) q_{1 \alpha} q_{2 \beta}, \tag{2.75}
\end{equation*}
$$

where

$$
\begin{equation*}
d \Omega_{q_{i}}=\frac{d^{3} q_{i}}{(2 \pi)^{3} 2 q_{i}^{0}}, \quad q_{i}^{0}=\sqrt{\vec{q}_{i}^{2}+m_{i}^{2}} . \tag{2.76}
\end{equation*}
$$

Clearly, $\mathcal{I}_{\alpha \beta}$ and $I_{\alpha \beta}$ differ only by a numerical factor,

$$
\begin{equation*}
\mathcal{I}_{\alpha \beta}(q)=\frac{I_{\alpha \beta}(q)}{4(2 \pi)^{2}} . \tag{2.77}
\end{equation*}
$$

The four-vectors $q_{1,2}$ are timelike, and so is their sum. The delta function in Eq. (2.75) has therefore nonvanishing support inside the integration domain only if $q$ is timelike; since it also imposes $q^{2}=\left(q_{1}+q_{2}\right)^{2}$, and $\left(q_{1}+q_{2}\right)^{2} \geq\left(m_{1}+m_{2}\right)^{2}$, this support is further restricted by $q^{2} \geq\left(m_{1}+m_{2}\right)^{2}$. Lorentz invariance of the integration measure implies that $\mathcal{I}_{\alpha \beta}(q)$ must take the form

$$
\begin{equation*}
\mathcal{I}_{\alpha \beta}(q)=A\left(q^{2}\right) q^{2} \eta_{\alpha \beta}+B\left(q^{2}\right) q_{\alpha} q_{\beta} . \tag{2.78}
\end{equation*}
$$

In fact, $\mathcal{I}_{\alpha \beta}$ must be a rank- 2 Lorentz tensor, and $\eta_{\alpha \beta}$ and $q_{\alpha} q_{\beta}$ are the only two independent structures that can be constructed with a single four-vector. The quantities $A$ and $B$ are Lorentz scalars, that can depend only on $q^{2}$, as well as $m_{1,2}$. From dimensional analysis it follows that they must be dimensionless. In order to find out their values, we will compute $\eta^{\alpha \beta} \mathcal{I}_{\alpha \beta}$ and $q^{\alpha} q^{\beta} \mathcal{I}_{\alpha \beta}$, so dealing with scalar quantities which are easier to manipulate. We have

$$
\begin{align*}
\eta^{\alpha \beta} \mathcal{I}_{\alpha \beta}(q) & =\left(4 A\left(q^{2}\right)+B\left(q^{2}\right)\right) q^{2}=\int d \Omega_{q_{1}} \int d \Omega_{q_{2}}(2 \pi)^{4} \delta^{(4)}\left(q-q_{1}-q_{2}\right) q_{1} \cdot q_{2} \\
& =\int d \Omega_{q_{1}} \int d \Omega_{q_{2}}(2 \pi)^{4} \delta^{(4)}\left(q-q_{1}-q_{2}\right) \frac{1}{2}\left(q^{2}-m_{1}^{2}-m_{2}^{2}\right)  \tag{2.79}\\
& =\frac{1}{2}\left(q^{2}-m_{1}^{2}-m_{2}^{2}\right) C\left(q^{2}\right),
\end{align*}
$$

where

$$
\begin{equation*}
C\left(q^{2}\right)=\int d \Omega_{q_{1}} \int d \Omega_{q_{2}}(2 \pi)^{4} \delta^{(4)}\left(q-q_{1}-q_{2}\right) \tag{2.80}
\end{equation*}
$$

By the same arguments used above, this is a Lorentz-invariant quantity with support in $q^{2} \geq$ $\left(m_{1}+m_{2}\right)^{2}$. To derive Eq. (2.79) we have used the fact that due to the delta function, inside the integral one can identify $q^{2}=\left(q_{1}+q_{2}\right)^{2}=m_{1}^{2}+m_{2}^{2}+2 q_{1} \cdot q_{2}$. Furthermore

$$
\begin{align*}
q^{\alpha} q^{\beta} \mathcal{I}_{\alpha \beta}(q) & =\left(A\left(q^{2}\right)+B\left(q^{2}\right)\right)\left(q^{2}\right)^{2}=\int d \Omega_{q_{1}} \int d \Omega_{q_{2}}(2 \pi)^{4} \delta^{(4)}\left(q-q_{1}-q_{2}\right) q \cdot q_{1} q \cdot q_{2} \\
& =\frac{1}{2}\left(q^{2}+m_{1}^{2}-m_{2}^{2}\right) \frac{1}{2}\left(q^{2}+m_{2}^{2}-m_{1}^{2}\right) C\left(q^{2}\right)  \tag{2.81}\\
& =\frac{1}{4}\left(\left(q^{2}\right)^{2}-\left(m_{1}^{2}-m_{2}^{2}\right)^{2}\right) C\left(q^{2}\right),
\end{align*}
$$

having used $m_{2}^{2}=q_{2}^{2}=\left(q-q_{1}\right)^{2}=q^{2}+m_{1}^{2}-2 q \cdot q_{1}$, that holds for the arguments of the integrand thanks to the delta function, and a similar relation with the roles of $q_{1}$ and $q_{2}$ interchanged. In conclusion,

$$
\begin{align*}
4 A\left(q^{2}\right)+B\left(q^{2}\right) & =\frac{1}{2}\left(1-\frac{m_{1}^{2}+m_{2}^{2}}{q^{2}}\right) C\left(q^{2}\right), \\
A\left(q^{2}\right)+B\left(q^{2}\right) & =\frac{1}{4}\left[1-\left(\frac{m_{1}^{2}-m_{2}^{2}}{q^{2}}\right)^{2}\right] C\left(q^{2}\right) . \tag{2.82}
\end{align*}
$$

Subtracting the second equation from the first and dividing by 3 we find

$$
\begin{align*}
A\left(q^{2}\right) & =\frac{C\left(q^{2}\right)}{6}\left[1-\frac{m_{1}^{2}+m_{2}^{2}}{q^{2}}-\frac{1}{2}+\frac{1}{2}\left(\frac{m_{1}^{2}-m_{2}^{2}}{q^{2}}\right)^{2}\right] \\
& =\frac{C\left(q^{2}\right)}{12}\left[1-\frac{2\left(m_{1}^{2}+m_{2}^{2}\right)}{q^{2}}+\left(\frac{m_{1}^{2}+m_{2}^{2}}{q^{2}}\right)^{2}-\left(\frac{2 m_{1} m_{2}}{q^{2}}\right)^{2}\right]  \tag{2.83}\\
& =\frac{C\left(q^{2}\right)}{12}\left[\left(1-\frac{m_{1}^{2}+m_{2}^{2}}{q^{2}}\right)^{2}-\left(\frac{2 m_{1} m_{2}}{q^{2}}\right)^{2}\right] \\
& =\frac{C\left(q^{2}\right)}{12}\left[1-\frac{\left(m_{1}+m_{2}\right)^{2}}{q^{2}}\right]\left[1-\frac{\left(m_{1}-m_{2}\right)^{2}}{q^{2}}\right] .
\end{align*}
$$

The value of $B$ can be similarly determined, but it is more convenient instead to rearrange Eq. (2.78) as follows,

$$
\begin{equation*}
\mathcal{I}_{\alpha \beta}(q)=A\left(q^{2}\right)\left(q^{2} \eta_{\alpha \beta}-q_{\alpha} q_{\beta}\right)+\left(A\left(q^{2}\right)+B\left(q^{2}\right)\right) q_{\alpha} q_{\beta}, \tag{2.84}
\end{equation*}
$$

obtaining directly from Eqs. (2.82) and 2.83)

$$
\begin{align*}
\mathcal{I}_{\alpha \beta}(q)= & \frac{C\left(q^{2}\right)}{12}\left[1-\frac{\left(m_{1}+m_{2}\right)^{2}}{q^{2}}\right]\left[1-\frac{\left(m_{1}-m_{2}\right)^{2}}{q^{2}}\right]\left(q^{2} \eta_{\alpha \beta}-q_{\alpha} q_{\beta}\right) \\
& +\frac{C\left(q^{2}\right)}{4}\left[1-\left(\frac{m_{1}^{2}-m_{2}^{2}}{q^{2}}\right)^{2}\right] q_{\alpha} q_{\beta} . \tag{2.85}
\end{align*}
$$

We are left with the calculation of $C\left(q^{2}\right)$. Since it is Lorentz-invariant and supported in the timelike domain $q^{2} \geq\left(m_{1}+m_{2}\right)^{2}$, we can compute it in the most convenient frame, which in this case is the "rest frame" $q_{R}=\left(q_{R}^{0}, \vec{q}_{R}=\overrightarrow{0}\right)$, where $q_{R}^{0}=\sqrt{q^{2}} \geq m_{1}+m_{2}$. We then find

$$
\begin{align*}
C\left(q^{2}\right) & =\int \frac{d^{3} q_{1}}{(2 \pi)^{3} 2 q_{1}^{0}} \int \frac{d^{3} q_{2}}{(2 \pi)^{3} 2 q_{2}^{0}}(2 \pi)^{4} \delta\left(q_{R}^{0}-q_{1}^{0}-q_{2}^{0}\right) \delta^{(3)}\left(\vec{q}_{1}+\vec{q}_{2}\right)  \tag{2.86}\\
& =\frac{1}{4(2 \pi)^{2}} \int \frac{d^{3} q_{1}}{(2 \pi)^{3} q_{1}^{0} q_{2}^{0}} \delta\left(q^{0}-q_{1}^{0}-q_{2}^{0}\right),
\end{align*}
$$

where now $q_{1}^{0}=\sqrt{\vec{q}_{1}^{2}+m_{1}^{2}}$ and $q_{2}^{0}=\sqrt{\vec{q}_{1}^{2}+m_{2}^{2}}$. Going over to polar coordinates,

$$
\begin{align*}
C\left(q^{2}\right) & =\frac{1}{4(2 \pi)^{2}} \int d \Omega \int_{0}^{\infty} \frac{d x x^{2}}{\sqrt{m_{1}^{2}+x^{2}} \sqrt{m_{2}^{2}+x^{2}}} \delta\left(q_{R}^{0}-\sqrt{m_{1}^{2}+x^{2}}-\sqrt{m_{2}^{2}+x^{2}}\right) \\
& =\frac{1}{4 \pi} \int_{0}^{\infty} \frac{d x x^{2}}{\sqrt{m_{1}^{2}+x^{2}} \sqrt{m_{2}^{2}+x^{2}}}\left(\frac{x}{\sqrt{m_{1}^{2}+x^{2}}}+\frac{x}{\sqrt{m_{2}^{2}+x^{2}}}\right)^{-1} \delta\left(x-x_{*}\right) \tag{2.87}
\end{align*}
$$

where $x_{*}$ is the unique positive solution of the equation $q_{R}^{0}=\sqrt{m_{1}^{2}+x^{2}}+\sqrt{m_{2}^{2}+x^{2}}$, to be determined below. The integral in Eq. (2.87) is now trivial, and we find

$$
\begin{equation*}
C\left(q^{2}\right)=\frac{1}{4 \pi} \int_{0}^{\infty} \frac{d x x}{\sqrt{m_{1}^{2}+x^{2}}+\sqrt{m_{2}^{2}+x^{2}}} \delta\left(x-x_{*}\right)=\frac{x_{*}}{4 \pi q_{R}^{0}}=\frac{x_{*}}{4 \pi \sqrt{q^{2}}} . \tag{2.88}
\end{equation*}
$$

Our last task is to find $x_{*}$. Here is the derivation ${ }^{23}$

$$
\begin{align*}
q_{R}^{0}-\sqrt{m_{1}^{2}+x^{2}} & =\sqrt{m_{2}^{2}+x^{2}}, \\
\left(q_{R}^{0}\right)^{2}+m_{1}^{2}+x^{2}-2 q_{R}^{0} \sqrt{m_{1}^{2}+x^{2}} & =m_{2}^{2}+x^{2}, \\
\left(q_{R}^{0}\right)^{2}+m_{1}^{2}-m_{2}^{2} & =2 q_{R}^{0} \sqrt{m_{1}^{2}+x^{2}},  \tag{2.89}\\
\left(\left(q_{R}^{0}\right)^{2}+m_{1}^{2}-m_{2}^{2}\right)^{2} & =4\left(q_{R}^{0}\right)^{2}\left(m_{1}^{2}+x^{2}\right), \\
\left(\left(q_{R}^{0}\right)^{2}+m_{1}^{2}-m_{2}^{2}-2 q_{R}^{0} m_{1}\right)\left(\left(q_{R}^{0}\right)^{2}+m_{1}^{2}-m_{2}^{2}+2 q_{R}^{0} m_{1}\right) & =4\left(q_{R}^{0}\right)^{2} x^{2}, \\
\left(\left(q_{R}^{0}-m_{1}\right)^{2}-m_{2}^{2}\right)\left(\left(q_{R}^{0}+m_{1}\right)^{2}-m_{2}^{2}\right) & =4\left(q_{R}^{0}\right)^{2} x^{2} .
\end{align*}
$$

Simple manipulations show that the left-hand side equals

$$
\begin{align*}
& \left(\left(q_{R}^{0}-m_{1}\right)^{2}-m_{2}^{2}\right)\left(\left(q_{R}^{0}+m_{1}\right)^{2}-m_{2}^{2}\right) \\
& =\left(q_{R}^{0}-m_{1}-m_{2}\right)\left(q_{R}^{0}-m_{1}+m_{2}\right)\left(q_{R}^{0}+m_{1}-m_{2}\right)\left(q_{R}^{0}+m_{1}+m_{2}\right)  \tag{2.90}\\
& =\left(\left(q_{R}^{0}\right)^{2}-\left(m_{1}+m_{2}\right)^{2}\right)\left(\left(q_{R}^{0}\right)^{2}-\left(m_{1}-m_{2}\right)^{2}\right) \\
& =\left(q^{2}-\left(m_{1}+m_{2}\right)^{2}\right)\left(q^{2}-\left(m_{1}-m_{2}\right)^{2}\right)
\end{align*}
$$

where in the last passage we made Lorentz invariance manifest. We can now solve Eq. (2.89) for $x^{2}$, obtaining

$$
\begin{align*}
x_{*}^{2} & =\frac{\left(q^{2}-\left(m_{1}+m_{2}\right)^{2}\right)\left(q^{2}-\left(m_{1}-m_{2}\right)^{2}\right)}{4 q^{2}} \\
& =\frac{q^{2}}{4}\left(1-\frac{\left(m_{1}+m_{2}\right)^{2}}{q^{2}}\right)\left(1-\frac{\left(m_{1}-m_{2}\right)^{2}}{q^{2}}\right) \tag{2.91}
\end{align*}
$$

that has real solutions if and only if $q^{2} \geq\left(m_{1}+m_{2}\right)^{2}{ }^{24}$ Taking the positive solution for $x_{*}$ we conclude

$$
\begin{equation*}
C\left(q^{2}\right)=\frac{1}{4 \pi} \sqrt{\frac{x_{*}^{2}}{q^{2}}}=\frac{1}{8 \pi} \sqrt{1-\frac{\left(m_{1}+m_{2}\right)^{2}}{q^{2}}} \sqrt{1-\frac{\left(m_{1}-m_{2}\right)^{2}}{q^{2}}} \tag{2.92}
\end{equation*}
$$

if $q^{2} \geq\left(m_{1}+m_{2}\right)^{2}$, and $C\left(q^{2}\right)=0$ otherwise. Setting $M=m_{1}+m_{2}$ and $\mu=m_{1}-m_{2}$, we then have

$$
\begin{align*}
\mathcal{I}_{\alpha \beta}(q)=\frac{1}{32 \pi} \sqrt{1-\frac{M^{2}}{q^{2}}} \sqrt{1-\frac{\mu^{2}}{q^{2}}}\{ & \frac{1}{3}\left(1-\frac{M^{2}}{q^{2}}\right)\left(1-\frac{\mu^{2}}{q^{2}}\right)\left(q^{2} \eta_{\alpha \beta}-q_{\alpha} q_{\beta}\right)  \tag{2.93}\\
& \left.+\left(1-\frac{M^{2}}{q^{2}} \frac{\mu^{2}}{q^{2}}\right) q_{\alpha} q_{\beta}\right\}
\end{align*}
$$

[^15]Let us discuss a few simple cases. When $m_{1}=m_{2}=m$, we have $\mu=0$ and so

$$
\begin{equation*}
\mathcal{I}_{\alpha \beta}(q)=\frac{1}{32 \pi} \sqrt{1-\frac{4 m^{2}}{q^{2}}}\left\{\frac{1}{3}\left(1-\frac{4 m^{2}}{q^{2}}\right)\left(q^{2} \eta_{\alpha \beta}-q_{\alpha} q_{\beta}\right)+q_{\alpha} q_{\beta}\right\} . \tag{2.94}
\end{equation*}
$$

If furthermore $m=0$ we recover Eq. (2.24,

$$
\begin{equation*}
\mathcal{I}_{\alpha \beta}(q)=\frac{1}{32 \pi}\left\{\frac{1}{3}\left(q^{2} \eta_{\alpha \beta}-q_{\alpha} q_{\beta}\right)+q_{\alpha} q_{\beta}\right\}=\frac{1}{96 \pi}\left\{q^{2} \eta_{\alpha \beta}+2 q_{\alpha} q_{\beta}\right\} . \tag{2.95}
\end{equation*}
$$

If instead $m_{1}=0, m_{2}=m$, we have $M=|\mu|=m$ and so

$$
\begin{gather*}
\mathcal{I}_{\alpha \beta}(q)=\frac{1}{96 \pi}\left(1-\frac{m^{2}}{q^{2}}\right)\left\{q^{2} \eta_{\alpha \beta}+2 q_{\alpha} q_{\beta}-\frac{2 m^{2}}{q^{2}}\left(q^{2} \eta_{\alpha \beta}-q_{\alpha} q_{\beta}\right)\right. \\
\left.+\left(\frac{m^{2}}{q^{2}}\right)^{2}\left(q^{2} \eta_{\alpha \beta}-4 q_{\alpha} q_{\beta}\right)\right\} \tag{2.96}
\end{gather*}
$$

## 3 Strangeness-conserving semileptonic processes

In semileptonic processes where the strangeness is conserved, the relevant term of the Lagrangian is (we ignore heavy quarks $b$ and $t$ )

$$
\begin{equation*}
\delta \mathscr{L}=-\frac{G}{\sqrt{2}} \cos \theta_{C}\left(\bar{u} \mathcal{O}_{L}^{\alpha} d\right) J_{l \alpha}+\text { h.c. } \tag{3.1}
\end{equation*}
$$

For initial/final hadronic states $\left|h_{i, f}\right\rangle$ and initial/final leptonic states $\left|\ell_{i, f}\right\rangle$, the relevant matrix elements are (in lowest-order perturbation theory)

$$
\begin{align*}
\mathcal{M}_{\mathrm{fi}} & =-\frac{G}{\sqrt{2}} \cos \theta_{C}\left(H^{\alpha} L_{\alpha}+\tilde{H}^{\alpha} \tilde{L}_{\alpha}\right), & & \\
H^{\alpha} & =\left\langle h_{f}\right|\left(\bar{u} \mathcal{O}_{L}^{\alpha} d\right)(0)\left|h_{i}\right\rangle, & & L^{\alpha}=\left\langle\ell_{f}\right| J_{l}^{\alpha}(0)\left|\ell_{i}\right\rangle  \tag{3.2}\\
\tilde{H}^{\alpha} & =\left\langle h_{f}\right|\left(\bar{u} \mathcal{O}_{L}^{\alpha} d\right)(0)^{\dagger}\left|h_{i}\right\rangle, & & \tilde{L}^{\alpha}=\left\langle\ell_{f}\right| J_{l}^{\alpha}(0)^{\dagger}\left|\ell_{i}\right\rangle,
\end{align*}
$$

having used Eq. 1.60 . Depending on the quantum numbers of the states $\left|h_{i, f}\right\rangle$, either one or the other term in Eq. (3.2) will only be nonzero. In fact, the current $\bar{u} \mathcal{O}_{L}^{\alpha} d$ has electric charge $Q=1$ (also isospin $I_{3}=1$, and hypercharge $Y=0$ ) while its Hermitian conjugate $\bar{d} \mathcal{O}_{L}^{\alpha} u$ has $Q=-1$ (isospin $I_{3}=-1$, and hypercharge $Y=0$ ), so they cannot both have a nonzero matrix element with given initial and final states. We will simplify the notation in the following and drop the unnecessary tilde.

To lowest perturbative order, the hadronic states $\left|h_{i, f}\right\rangle$ are determined by strong interactions alone, with corrections suppressed by powers of the weak coupling constant. However, at low momentum transfer the matrix elements $H^{\alpha}$ of interest cannot be studied using perturbative QCD, since its low-energy dynamics is inherently nonperturbative. Nonetheless, a lot can be said about them based simply on symmetries. Decay amplitudes are in fact determined by translation and Lorentz invariance up to a few functions of the transferred momentum squared. These can be studied using nonperturbative techniques, e.g., numerical calculations on the lattice, or, more pragmatically, treated as phenomenological parameters that can be determined experimentally. The relevant parameters for different processes are furthermore related, to a certain degree of approximation, by the known (approximate) symmetries of the strong interactions, namely isospin and (to a lesser extent) flavour $\mathrm{SU}(3)$ symmetry.

### 3.1 Isotopic spin (isospin) invariance

If we organise the up and down quark into an isospin doublet,

$$
\begin{equation*}
q=\binom{u}{d}, \tag{3.3}
\end{equation*}
$$

then the two relevant currents are part of the isovector triplet

$$
\begin{equation*}
\mathcal{J}_{a}^{\mu}=\bar{q} \mathcal{O}_{L}^{\mu} \frac{\tau_{a}}{2} q=\bar{q} \gamma^{\mu} \frac{\tau_{a}}{2} q-\bar{q} \gamma^{\mu} \gamma^{5} \frac{\tau_{a}}{2} q=V_{a}^{\mu}-A_{a}^{\mu} \tag{3.4}
\end{equation*}
$$

which is the sum of a vector-isovector and an axial vector-isovector current. Here $\tau_{a}$ denotes the Pauli matrices (the usage of $\tau_{a}$ instead of $\sigma_{a}$ is standard in this context). In particular, the charged weak current $\bar{u} \mathcal{O}_{L}^{\mu} d$ and its conjugate $\bar{d} \mathcal{O}_{L}^{\mu} u$ correspond to $\mathcal{J}_{ \pm}^{\mu}=\bar{q} \mathcal{O}_{L}^{\mu} \frac{\tau_{ \pm}}{2} q$, where $\tau_{ \pm}=\tau_{1} \pm i \tau_{2}$. Notice that $\left(V_{+}^{\mu}\right)^{\dagger}=q^{\dagger} \gamma^{\mu \dagger} \gamma^{0} \tau_{+}^{\dagger} q=q^{\dagger} \gamma^{0} \gamma^{\mu} \tau_{-} q=V_{-}^{\mu}$.

Conservation of the vector current The vector part $V_{+}^{\mu}$ of the charged current and its Hermitean conjugate $V_{-}^{\mu}$ are components of the same isotriplet as the isotriplet part of the electromagnetic current. Stated more simply, the electromagnetic current receives a contribution from $V_{3}^{\mu}$, since

$$
\begin{equation*}
V_{\mathrm{em}}^{\mu}=\frac{2}{3} \bar{u} \gamma^{\mu} u-\frac{1}{3} \bar{d} \gamma^{\mu} d=\frac{1}{2}\left(\bar{u} \gamma^{\mu} u-\bar{d} \gamma^{\mu} d\right)+\frac{1}{6}\left(\bar{u} \gamma^{\mu} u+\bar{d} \gamma^{\mu} d\right)=V_{3}^{\mu}+S^{\mu}, \tag{3.5}
\end{equation*}
$$

where $S^{\mu}$ is a vector-isoscalar current. A first consequence of this is that, since $V_{\mathrm{em}}^{\mu}$ is conserved, $\partial_{\mu} V_{\mathrm{em}}^{\mu}=0$, the vector current must be conserved as well in the limit of exact isospin invariance. This implies first of all that $V_{3}^{\mu}$ and $S^{\mu}$ must be conserved separately, due their different transformation properties under isospin rotations; and that the whole isotriplet $V_{a}^{\mu}$ must be conserved, since its components are related to each other by isospin transformations. Conservation of a current $J^{\mu}$ has an important and well-known consequence on its matrix elements between momentum eigenstates. Using translation invariance, we have in general that

$$
\begin{align*}
& \left\langle\vec{p}^{\prime}\right| \partial_{\mu} J^{\mu}(x)|\vec{p}\rangle=\partial_{\mu}\left\langle\vec{p}^{\prime}\right| J^{\mu}(x)|\vec{p}\rangle=\partial_{\mu}\left\langle\vec{p}^{\prime}\right| e^{i P \cdot x} J^{\mu}(0) e^{-i P \cdot x}|\vec{p}\rangle \\
& =\partial_{\mu} e^{i\left(p^{\prime}-p\right) \cdot x}\left\langle\vec{p}^{\prime}\right| J^{\mu}(0)|\vec{p}\rangle=\partial_{\mu} e^{-i q \cdot x}\left\langle\vec{p}^{\prime}\right| J^{\mu}(0)|\vec{p}\rangle=-i e^{-i q \cdot x} q_{\mu}\left\langle\vec{p}^{\prime}\right| J^{\mu}(0)|\vec{p}\rangle, \tag{3.6}
\end{align*}
$$

and setting $x=0$,

$$
\begin{equation*}
\left\langle\vec{p}^{\prime}\right| \partial_{\mu} J^{\mu}(0)|\vec{p}\rangle=-i q_{\mu}\left\langle\vec{p}^{\prime}\right| J^{\mu}(0)|\vec{p}\rangle . \tag{3.7}
\end{equation*}
$$

Conservation of the current, $\partial_{\mu} J^{\mu}(x)=0$, implies that the matrix elements of $J^{\mu}(0)$ are transverse to $q=p-p^{\prime}$,

$$
\begin{equation*}
q_{\mu}\left\langle\vec{p}^{\prime}\right| J^{\mu}(0)|\vec{p}\rangle=0 \tag{3.8}
\end{equation*}
$$

Isospin selection rules - Wigner-Eckart theorem A more direct consequence of isospin invariance is that matrix elements of the current $\mathcal{J}_{a}^{\mu}$ can be nonzero only between initial and final states with total isospin $I_{i, f}$ differing by $\Delta I=I_{f}-I_{i}=0, \pm 1$. This follows from the usual composition rules of $\mathrm{SU}(2)$ representations. In particular, the matrix elements of the vectorisovector current between states $A$ and $B$ belonging to the same isospin multiplet, $I_{f}=I_{i}=I$, must have the form

$$
\begin{equation*}
\langle A| V_{a}^{\mu}|B\rangle=C_{(I)}^{\mu}\left(T_{a}^{(I)}\right)_{A B}, \tag{3.9}
\end{equation*}
$$

with $T_{a}^{(I)}$ the generators in the representation $\mathbf{R}=\mathbf{2 I}+\mathbf{1}$ of dimension $R=2 I+1$ corresponding to a multiplet of total isospin $I{ }^{25}$ The reason is that under an isospin rotation of $A$ and $B$ these matrix elements must transform like an isovector, resulting from the composition of representations $\overline{\mathbf{R}} \otimes \mathbf{R}=\mathbf{R} \otimes \mathbf{R}=\oplus_{d=1}^{2 R} \mathbf{d}$. Since each representation appears only once in the decomposition, there is a single tensorial structure that can be formed, and that is provided by $T_{a}^{(I)}$. Similarly, for the vector-isoscalar current one finds

$$
\begin{equation*}
\langle A| S^{\mu}|B\rangle=\tilde{C}_{(I)}^{\mu} \delta_{A B}^{(I)} \tag{3.10}
\end{equation*}
$$

since the $(2 I+1)$-dimensional identity matrix is the only scalar structure that can be formed out of $\overline{\mathbf{R}} \otimes \mathbf{R}$. These are particular instances of the Wigner-Eckart theorem.

[^16]We can now combine these results with Eq. (3.5) to find a useful relation. Take $A$ and $B$ to be $I_{3}$ eigenstates in a multiplet of isospin $I$, and consider the matrix elements of the vector currents $V_{+}^{\mu}, V_{3}^{\mu}$, and $S^{\mu}$. We find

$$
\begin{align*}
\left\langle I I_{3}^{\prime}\right| V_{+}^{\mu}\left|I I_{3}\right\rangle & =C_{(I)}^{\mu}\left(T_{+}^{(I)}\right)_{I_{3}^{\prime} I_{3}}=C_{(I)}^{\mu} \sqrt{I(I+1)-I_{3}\left(I_{3}+1\right)} \delta_{I_{3}^{\prime} I_{3}+1}, \\
\left\langle I I_{3}^{\prime}\right| V_{3}^{\mu}\left|I I_{3}\right\rangle & =C_{(I)}^{\mu}\left(T_{3}^{(I)}\right)_{I_{3}^{\prime} I_{3}}=C_{(I)}^{\mu} I_{3} \delta_{I_{3}^{\prime} I_{3}},  \tag{3.11}\\
\left\langle I I_{3}^{\prime}\right| S^{\mu}\left|I I_{3}\right\rangle & =\tilde{C}_{(I)}^{\mu} \delta_{I_{3}^{\prime} I_{3}} .
\end{align*}
$$

Here we assumed the Condon-Shortley convention on isospin eigenstates. If we subtract the diagonal matrix elements of $V_{3}^{\mu}$ with isospin $I_{3}+1$ and $I_{3}$ we find

$$
\begin{align*}
C_{(I)}^{\mu} & =\left\langle I I_{3}+1\right| V_{3}^{\mu}\left|I I_{3}+1\right\rangle-\left\langle I I_{3}\right| V_{3}^{\mu}\left|I I_{3}\right\rangle \\
& =\left\langle I I_{3}+1\right| V_{\mathrm{em}}^{\mu}+S^{\mu}\left|I I_{3}+1\right\rangle-\left\langle I I_{3}\right| V_{\mathrm{em}}^{\mu}+S^{\mu}\left|I I_{3}\right\rangle  \tag{3.12}\\
& =\left\langle I I_{3}+1\right| V_{\mathrm{em}}^{\mu}\left|I I_{3}+1\right\rangle-\left\langle I I_{3}\right| V_{\mathrm{em}}^{\mu}\left|I I_{3}\right\rangle
\end{align*}
$$

Combining this with the first equation in Eq. (3.11) we obtain

$$
\begin{align*}
\left\langle I I_{3}+1\right| V_{+}^{\mu}\left|I I_{3}\right\rangle & =\sqrt{I(I+1)-I_{3}\left(I_{3}+1\right)}\left[\left\langle I I_{3}+1\right| V_{3}^{\mu}\left|I I_{3}+1\right\rangle-\left\langle I I_{3}\right| V_{3}^{\mu}\left|I I_{3}\right\rangle\right] \\
& =\sqrt{I(I+1)-I_{3}\left(I_{3}+1\right)}\left[\left\langle I I_{3}+1\right| V_{\mathrm{em}}^{\mu}\left|I I_{3}+1\right\rangle-\left\langle I I_{3}\right| V_{\mathrm{em}}^{\mu}\left|I I_{3}\right\rangle\right] . \tag{3.13}
\end{align*}
$$

Weak charge The Noether charges associated with the vector-isovector current read

$$
\begin{equation*}
T_{a}=\int d^{3} x V_{a}^{0}(x) \tag{3.14}
\end{equation*}
$$

Here $x^{0}$ is arbitrary due to conservation, so we take $x^{0}=0$. For the matrix element between isospin and momentum eigenstates $A$ and $B$, with respectively isospin $I^{\prime}, I_{3}^{\prime}$ and $I, I_{3}$ and momenta $\vec{p}^{\prime}$ and $\vec{p}$, we have

$$
\begin{align*}
\langle A| T_{a}|B\rangle & =\int d^{3} x\langle A| V_{a}^{0}(x)|B\rangle=\int d^{3} x e^{i\left(\vec{p}-\vec{p}^{\prime}\right) \cdot x}\langle A| V_{a}^{0}(0)|B\rangle  \tag{3.15}\\
& =(2 \pi)^{3} \delta^{(3)}(\vec{q})\langle A| V_{a}^{0}(0)|B\rangle,
\end{align*}
$$

with $\vec{q}=\vec{p}-\vec{p}^{\prime}$, and so for the + component, with the usual relativistic normalisation of states, we find

$$
\begin{equation*}
\int d^{3} x\langle A| V_{+}^{0}(x)|B\rangle=\langle A| T_{+}|B\rangle=\delta_{I^{\prime} I} \delta_{I_{3}^{\prime} I_{3}+1} \sqrt{I(I+1)-I_{3}\left(I_{3}+1\right)}(2 \pi)^{3} 2 p^{0} \delta^{(3)}(\vec{q}) \tag{3.16}
\end{equation*}
$$

Comparing with Eq. 3.15 we conclude that

$$
\begin{equation*}
\left.\langle A| V_{+}^{0}(0)|B\rangle\right|_{\vec{q}=0}=2 p^{0} \delta_{I^{\prime} I} \delta_{I_{3}^{\prime} I_{3}+1} \sqrt{I(I+1)-I_{3}\left(I_{3}+1\right)} \equiv 2 p^{0} \delta_{I^{\prime} I} \delta_{I_{3}^{\prime} I_{3}+1} Q_{W}\left(I, I_{3}\right) . \tag{3.17}
\end{equation*}
$$

An analogous derivation leads to the following result for the - component of the vector current,

$$
\begin{equation*}
\left.\langle A| V_{-}^{0}(0)|B\rangle\right|_{\vec{q}=0}=2 p^{0} \delta_{I^{\prime} I} \delta_{I_{3}^{\prime} I_{3}-1} \sqrt{I(I+1)-I_{3}\left(I_{3}-1\right)}=2 p^{0} \delta_{I^{\prime} I} \delta_{I_{3}^{\prime} I_{3}-1} Q_{W}\left(I, I_{3}^{\prime}\right) . \tag{3.18}
\end{equation*}
$$

In the case of transitions where the initial and final state belong to the same isospin multiplet, the amplitude in the static limit $\vec{q}=0$ is determined entirely by the weak charge, $Q_{W}\left(I, I_{3}\right)$. (Notice that in this case also $q^{0}=0$, since states in the same multiplets have the same mass.) Analogously, for the electromagnetic current we have

$$
\begin{equation*}
\int d^{3} x\langle A| V_{\mathrm{em}}^{0}(x)|B\rangle=\langle A| Q_{\mathrm{em}}|B\rangle=\delta_{Q^{\prime} Q} Q(2 \pi)^{3} 2 p^{0} \delta^{(3)}(\vec{q}), \tag{3.19}
\end{equation*}
$$

from which it follows that

$$
\begin{equation*}
\left.\langle A| V_{\mathrm{em}}^{0}(0)|B\rangle\right|_{\vec{q}=0}=2 p^{0} \delta_{Q^{\prime} Q} Q \tag{3.20}
\end{equation*}
$$

This is not in contradiction with Eq. (3.13): in fact, combining the two equations one finds $Q\left(I_{3}+1\right)-Q\left(I_{3}\right)=1$, which implies that $Q\left(I_{3}\right)$ is a linear function of $I_{3}$; but this follows directly from Eq. (3.5). (In fact, we know that $Q=I_{3}+\frac{1}{2} Y$, the Gell-Mann-Nishijima relation.)

Axial current and chiral symmetry If quarks were massless, the $\mathrm{SU}(2)$ isospin symmetry could actually be extended to a chiral isospin symmetry $\mathrm{SU}(2)_{L} \otimes \mathrm{SU}(2)_{R}$, with the two factors acting independently on the two chiralities of the quark. In that case the corresponding chiral currents would be conserved, and therefore both the vector and the axial current would be exactly conserved, as they are the sum and the difference of the chiral ones. In the real world the light quarks have small but finite masses, which leads to conservation of the axial current being only partial,${ }^{26}$ the current divergence being proportional to the light quark masses.

On the other hand, even in the massless quark limit the vector and axial part of the chiral symmetry are realised in different ways: while the vacuum is invariant under a vector isospin rotation, it is not under an axial one. Chiral symmetry is therefore spontaneously broken, and being a continuous symmetry it generates massless bosons through the Goldstone mechanism. These are nothing but the pions, whose nonzero mass is due to the explicit but soft breaking of chiral symmetry due to the small light quark masses, and which would vanish in the massless limit.

Partial conservation of the axial current (PCAC) was correctly guessed before the discovery of quarks and of QCD. If the axial part of the chiral symmetry is spontaneously broken, then Goldstone bosons $\pi_{a}$ are generated, one for each broken generator. Such bosons are coupled to the axial current,

$$
\begin{equation*}
\langle 0| A_{a}^{\mu}(0)\left|\pi_{b}\right\rangle=i p^{\mu} f_{a b} \tag{3.21}
\end{equation*}
$$

with $f_{a b}$ some constants. The form of the right-hand side is dictated by Lorentz invariance. If the vector part of the symmetry is not broken, then isospin invariance implies

$$
\begin{equation*}
\langle 0| A_{a}^{\mu}(0)\left|\pi_{b}\right\rangle=i p^{\mu} f_{\pi} \delta_{a b} \tag{3.22}
\end{equation*}
$$

The rea ${ }^{27}$ quantity $f_{\pi}$ is the pion decay constant, for reasons that will become clear soon, and

[^17]having used the transformation property
$$
\Theta\left|\pi_{a}(\vec{p})\right\rangle=\eta_{T} \eta_{P} \eta_{C}\left|\bar{\pi}_{a}(\vec{p})\right\rangle=\left|\pi_{a}(\vec{p})\right\rangle
$$
it has dimensions of a mass. Taking the divergence on both sides [see Eq. (3.7)]
\[

$$
\begin{equation*}
\langle 0| \partial_{\mu} A_{a}^{\mu}(0)\left|\pi_{b}\right\rangle=(-i) p_{\mu} i p^{\mu} f_{\pi} \delta_{a b}=m_{\pi}^{2} f_{\pi} \delta_{a b} \tag{3.23}
\end{equation*}
$$

\]

The PCAC hypothesis is the generalisation of Eq. (3.23) to an operator relation,

$$
\begin{equation*}
\partial_{\mu} A_{a}^{\mu}(x)=f_{\pi} m_{\pi}^{2} \phi_{a}(x), \tag{3.24}
\end{equation*}
$$

with $\phi_{a}$ the pion fields (of mass dimension 1), normalised to have amplitude 1 between the vacuum and the one-particle states,

$$
\begin{equation*}
\langle 0| \phi_{a}(0)\left|\pi_{b}\right\rangle=\delta_{a b} \tag{3.25}
\end{equation*}
$$

Here the states $\left|\pi_{a}\right\rangle$ are related to the physical pion states $\left|\pi^{0}\right\rangle$ and $\left|\pi^{ \pm}\right\rangle$as $\left|\pi^{0}\right\rangle=\left|\pi_{3}\right\rangle$ and

$$
\begin{equation*}
\left|\pi^{ \pm}\right\rangle=\frac{1}{\sqrt{2}} I_{ \pm}\left|\pi^{0}\right\rangle=\frac{\left|\pi_{1}\right\rangle \pm i\left|\pi_{2}\right\rangle}{\sqrt{2}} \tag{3.26}
\end{equation*}
$$

and the fields $\phi_{a}(x)$ are related to the neutral pion field $\pi^{0}(x)=\pi^{0}(x)^{\dagger}$ and to the charged pion fields $\pi^{+}(x)$ and $\pi^{-}(x)=\pi^{+}(x)^{\dagger}$, normalised such that

$$
\begin{equation*}
\left\langle\pi^{0}\right| \pi^{0}(0)|0\rangle=1, \quad\left\langle\pi^{ \pm}\right| \pi^{ \pm}(0)|0\rangle=1, \tag{3.27}
\end{equation*}
$$

as $\pi^{0}=\pi_{3}$ and $\sqrt{2} \pi^{ \pm}=\phi_{1} \mp i \phi_{2}$. Indeed, taking the complex conjugate

$$
\begin{equation*}
\langle 0| \sqrt{2} \pi^{\mp}(0)\left|\pi^{ \pm}\right\rangle=\frac{1}{\sqrt{2}}\langle 0|\left(\phi_{1}(0) \mp i \phi_{2}(0)\right)\left(\left|\pi_{1}\right\rangle \pm i\left|\pi_{2}\right\rangle\right)=\sqrt{2}, \tag{3.28}
\end{equation*}
$$

as required. These fields form an isotriplet: by construction the effect of a unitary isospin transformation $U$ on $\left|\pi_{a}\right\rangle$ is $U\left|\pi_{a}\right\rangle=\mathcal{U}_{b a}^{3}\left|\pi_{b}\right\rangle$, with $\mathcal{U}^{3}$ the representative of the transformation in the adjoint (triplet) representation, $\mathrm{s}^{28}$

$$
\begin{equation*}
\left\langle\pi_{b}\right| U \phi_{a}(0) U^{\dagger}|0\rangle=\left(\mathcal{U}^{\mathbf{3} \dagger}\right)_{c b}^{*}\left\langle\pi_{c}\right| \phi_{a}(0)|0\rangle=\mathcal{U}_{b a}^{\mathbf{3}} \Longrightarrow U \phi_{a}(0) U^{\dagger}=\sum_{c} \mathcal{U}_{c a}^{\mathbf{3}} \phi_{c}(0) . \tag{3.29}
\end{equation*}
$$

While $\partial_{\mu} A_{a}^{\mu}(x)$ is trivially "a" pion field, as it can create pion states out of the vacuum having the right quantum numbers, Eq. (3.24) tells us that pions can be excited using a total divergence. This has nontrivial consequences in the form of low-energy theorems (see Chapter 5 in Ref. [10]).

From a modern perspective, the PCAC relation Eq. (3.24) is the transcription in terms of an effective mesonic field of the Ward identity for the axial current,

$$
\begin{equation*}
\partial_{\mu} A_{a}^{\mu}(x)=2 m_{u d} P_{a}(x), \tag{3.30}
\end{equation*}
$$

where $P_{a}=\bar{q} \gamma^{5} \frac{\tau_{a}}{2} q$ is the pseudoscalar density, and $m_{u d}$ is the light quark mass in the isospin limit. The identity Eq. (3.30) is an exact result in the isospin limit of QCD. Comparison with Eq. (3.24) shows that the pion mass square is proportional to the light-quark mass (in the limit of small mass where the other states excited by $P_{a}$ can be neglected), a relation known as the Gell-Mann-Oakes-Renner relation.
Notice that the states $\left|\pi_{a}\right\rangle$ are $C$ eigenstates, and that the (arbitrary) residual phase $\eta_{T}$ for the $T$ transformation is chosen so that $\eta_{T} \eta_{P} \eta_{C}=1$, where $\eta_{P}$ and $\eta_{C}$ are the intrinsic parity and charge-conjugation phases. From the equation above it follows that $f_{\pi}^{*}=f_{\pi}$.
${ }^{28}$ This treatment is appropriate for free pion fields, like, e.g., the ones appearing in the interaction picture, that only create one-particle states out of the vacuum. For the fully interacting fields, one should start from the isospin invariant action, from which the symmetry generators are obtained via Noether's theorem. Their action on the pion states, defined as the lightest states created by the interacting fields, follows then from the transformation properties of the fields.

### 3.2 Pion decays

Let us apply the general results of the previous subsection to concrete cases.

Leptonic decays of charged pions We begin with the decays

$$
\begin{equation*}
\pi^{+} \rightarrow \ell^{+} \nu_{\ell}, \quad \pi^{-} \rightarrow \ell^{-} \bar{\nu}_{\ell} . \tag{3.31}
\end{equation*}
$$

Since pions are pseudoscalars, a fully leptonic decay of a pion can only be mediated by the axial current, i.e., $\langle 0| V_{\mp}^{\mu}\left|\pi^{ \pm}\right\rangle=0$. This matrix element is in fact an axial vector under Lorentz transformations, but no axial vector is available. The axial current matrix element is instead a vector, and Lorentz invariance dictates it to be of the form [see Eqs. (3.22), (3.26) and (3.28)]

$$
\begin{equation*}
\langle 0| A_{\mp}^{\mu}\left|\pi^{ \pm}\right\rangle=i \sqrt{2} f_{\pi} p^{\mu}, \tag{3.32}
\end{equation*}
$$

with real constant $f_{\pi}$ of mass dimension 1 . It should now be clear why $f_{\pi}$ is called the "pion decay constant". The value of $f_{\pi}$ is the same for both charged pions, even away from the isospin limit: in fact, $C P$ symmetry is sufficient to show that this is the case; $C P T$ symmetry implies furthermore that $f_{\pi}$ is real. ${ }^{29}$ The choice of signs in Eq. (3.32) follows from isospin conservation, or, from a quark model perspective, from the fact that $\pi^{+}=\bar{d} u$ and $\pi^{-}=\bar{u} d$, so that the currents $\bar{d} \mathcal{O}_{L}^{\mu} u$ and $\bar{u} \mathcal{O}_{L}^{\mu} d$ are respectively needed to annihilate them ${ }^{30}$ These are respectively coupled to $\bar{\nu}_{\ell} \mathcal{O}_{L}^{\mu} \ell$ and $\bar{\ell} \mathcal{O}_{L}^{\mu} \nu_{\ell}$.

Let us focus on $\pi^{+} \rightarrow \ell^{+} \nu_{\ell}$ for definiteness: $C P$ symmetry implies that the same width is

[^18]where $x_{P}=\left(x^{0},-\vec{x}\right)$ and $\mathcal{C}=i \gamma^{2} \gamma^{0}=-\mathcal{C}^{\dagger}$ is such that $\mathcal{C}^{\dagger} \gamma^{\mu} \mathcal{C}=-\left(\gamma^{\mu}\right)^{T}$. For the charged pions and the axial-vector current one has
\[

$$
\begin{aligned}
\langle 0| \Theta^{\dagger} A_{+}^{\mu}(0) \Theta\left|\pi^{+}(\vec{p})\right\rangle & =-\langle 0| A_{-}^{\mu}(0)\left|\pi^{+}(\vec{p})\right\rangle=-i f_{\pi^{+}} p^{\mu}=\langle 0| A_{+}^{\mu}(0)\left|\pi^{-}(-\vec{p})\right\rangle^{*}=\left(i f_{\pi^{-}} p^{\mu}\right)^{*}, \\
\langle 0|(C P)^{\dagger} A_{+}^{\mu}(0) C P\left|\pi^{+}(\vec{p})\right\rangle & =-\eta^{\mu \mu}\langle 0| A_{-}^{\mu}(0)\left|\pi^{+}(\vec{p})\right\rangle=-i f_{\pi^{+}} p_{P}^{\mu}=\eta_{C} \eta_{P}\langle 0| A_{+}^{\mu}(0)\left|\pi^{-}(-\vec{p})\right\rangle=i f_{\pi^{-}} p_{P}^{\mu} \eta_{C} \eta_{P},
\end{aligned}
$$
\]

with $p_{P}^{\mu}=\left(p^{0},-\vec{p}\right)$. In the equation above we made use of

$$
\begin{aligned}
& \gamma^{5} \gamma^{0}\left[\gamma^{\mu} \gamma^{5}\left(\tau_{1}+i \tau_{2}\right)\right]^{\dagger} \gamma^{0} \gamma^{5}=\gamma^{5} \gamma^{0} \gamma^{5} \gamma^{0} \gamma^{\mu} \gamma^{0} \gamma^{0} \gamma^{5}\left(\tau_{1}-i \tau_{2}\right)=-\gamma^{\mu} \gamma^{5}\left(\tau_{1}-i \tau_{2}\right) \\
& \gamma^{0} \mathcal{C}^{\dagger}\left[\gamma^{\mu} \gamma^{5}\left(\tau_{1}+i \tau_{2}\right)\right]^{T} \mathcal{C} \gamma^{0}=\gamma^{0} \mathcal{C}^{\dagger} \gamma^{5} \gamma^{\mu T} \mathcal{C} \gamma^{0}\left(\tau_{1}-i \tau_{2}\right)=-\gamma^{0} \gamma^{5} \gamma^{\mu} \gamma^{0}\left(\tau_{1}-i \tau_{2}\right)=-\eta^{\mu \mu} \gamma^{\mu} \gamma^{5}\left(\tau_{1}-i \tau_{2}\right)
\end{aligned}
$$

The exact $C P T$ symmetry implies $f_{\pi^{+}}=f_{\pi^{-}}^{*}$, so that they might differ only by a phase. Since $\eta_{P}=-1$ and $\eta_{C}=1$ for pions, $C P$ symmetry further implies $f_{\pi^{+}}=f_{\pi^{-}}=f_{\pi^{+}}^{*} . C P$ is a good symmetry as long as the heaviest quarks can be ignored, so this relation is expected to be very accurate. At the present level of approximation, where hadronic matrix elements are computed considering only strong interactions, $C P$ is exact, and so is $T$. One could then use directly $T$ invariance and the transformation property $A_{ \pm}^{\mu}(x) \rightarrow \eta^{\mu \mu} A_{ \pm}^{\mu}\left(x_{T}\right)$ to show that $f_{\pi^{+}}=f_{\pi^{-}}=f_{\pi}$ is real,

$$
i f_{\pi} \eta^{\mu \mu} p^{\mu}=\eta^{\mu \mu}\langle 0| A_{+}^{\mu}(0)\left|\pi^{+}(\vec{p})\right\rangle=\langle 0| T^{\dagger} A_{+}^{\mu}(0) T\left|\pi^{+}(\vec{p})\right\rangle=\langle 0| A_{+}^{\mu}(0)\left|\pi^{+}(-\vec{p})\right\rangle^{*} \eta_{P} \eta_{C}=\left(-\eta_{P} \eta_{C}\right) i f_{\pi}^{*} \eta^{\mu \mu} p^{\mu}
$$

[^19]obtained for $\pi^{-} \rightarrow \ell^{-} \bar{\nu}_{\ell}$. We have for the decay amplitude
\[

$$
\begin{align*}
\mathcal{M}_{\mathrm{fi}} & =\frac{G}{\sqrt{2}} \cos \theta_{C} i \sqrt{2} f_{\pi} p_{(\pi)}^{\mu}\left\langle\ell^{+} \nu_{\ell}\right|\left(\bar{\nu}_{\ell} \mathcal{O}_{L \mu} \ell\right)(0)|0\rangle \\
& =i G \cos \theta_{C} f_{\pi} p_{(\pi)}^{\mu} \bar{u}_{(\nu)}\left(p_{(\nu)}\right) \gamma_{\mu}\left(1-\gamma^{5}\right) v_{(\ell)}\left(p_{(\ell)}\right) \\
& =i G \cos \theta_{C} f_{\pi} \bar{u}_{(\nu)}\left(p_{(\nu)}\right)\left(\not p_{(\nu)}+\not p_{(\ell)}\right)\left(1-\gamma^{5}\right) v_{(\ell)}\left(p_{(\ell)}\right)  \tag{3.33}\\
& =i G \cos \theta_{C} f_{\pi} \bar{u}_{(\nu)}\left(1+\gamma^{5}\right) \not p_{(\ell)} v_{(\ell)}\left(p_{(\ell)}\right) \\
& =-i G \cos \theta_{C} f_{\pi} m_{\ell} \bar{u}_{(\nu)}\left(1+\gamma^{5}\right) v_{(\ell)}\left(p_{(\ell)}\right),
\end{align*}
$$
\]

where we have used momentum conservation and the Dirac equation. Taking the absolute value square,

$$
\begin{equation*}
\left|\mathcal{M}_{\mathrm{f}}\right|^{2}=G^{2} \cos ^{2} \theta_{C} f_{\pi}^{2} m_{\ell}^{2} \bar{u}_{(\nu)}\left(1+\gamma^{5}\right) v_{(\ell)}\left(p_{(\ell)}\right) \bar{v}_{(\ell)}\left(p_{(\ell)}\right)\left(1-\gamma^{5}\right) u_{(\nu)} \tag{3.34}
\end{equation*}
$$

and summing over spins (recall Eq. 2.9) for the neutrino bispinors)

$$
\begin{align*}
& \left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle=G^{2} \cos ^{2} \theta_{C} f_{\pi}^{2} m_{\ell}^{2} \operatorname{tr}\left(1+\gamma^{5}\right)\left(\not p_{(\ell)}-m_{\ell}\right)\left(1-\gamma^{5}\right) \not{ }^{2}(\nu) \frac{1+\gamma^{5}}{2} \\
& =G^{2} \cos ^{2} \theta_{C} f_{\pi}^{2} m_{\ell}^{2} 2 \operatorname{tr}\left(\not{ }_{(\ell)}-m_{\ell}\right) \phi_{(\nu)}\left(1+\gamma^{5}\right)  \tag{3.35}\\
& =2 G^{2} \cos ^{2} \theta_{C} f_{\pi}^{2} m_{\ell}^{2} \operatorname{tr} \not{ }_{(\ell)} \not p_{(\nu)}=8 G^{2} \cos ^{2} \theta_{C} f_{\pi}^{2} m_{\ell}^{2} p_{(\ell)} \cdot p_{(\nu)} .
\end{align*}
$$

Taking the square of the momentum conservation relation, $p_{(\pi)}=p_{(\ell)}+p_{(\nu)}$, we find

$$
\begin{equation*}
m_{\pi}^{2}=m_{\ell}^{2}+2 p_{(\ell)} \cdot p_{(\nu)} \tag{3.36}
\end{equation*}
$$

and so

$$
\begin{equation*}
\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle=4 G^{2} \cos ^{2} \theta_{C} f_{\pi}^{2} m_{\ell}^{2}\left(m_{\pi}^{2}-m_{\ell}^{2}\right), \tag{3.37}
\end{equation*}
$$

which is a constant. The total decay width is found integrating the differential width,

$$
\begin{equation*}
\Gamma=\int d \Gamma=\int d \Phi^{(2)} \frac{\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle}{2 m_{\pi}}=\frac{\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle}{2 m_{\pi}} \int d \Phi^{(2)}=\frac{\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle}{2 m_{\pi}} \Phi^{(2)}, \tag{3.38}
\end{equation*}
$$

where

$$
\begin{equation*}
d \Phi^{(2)}=(2 \pi)^{4} \delta^{(4)}\left(p_{(\pi)}-p_{(\ell)}-p_{(\nu)}\right) \frac{d^{3} p_{(\ell)}}{(2 \pi)^{3} 2 E_{\ell}} \frac{d^{3} p_{(\nu)}}{(2 \pi)^{3} 2 E_{\nu}} \tag{3.39}
\end{equation*}
$$

with $E=p^{0}=\sqrt{\vec{p}^{2}+m^{2}}$ the particle energy. The two-body phase-space integral is Lorentz invariant, so is most conveniently obtained in the pion rest frame as follows (for generality we include a neutrino mass $m_{\nu}$ ):

$$
\begin{align*}
\Phi^{(2)} & =\int \frac{d^{3} p_{(\ell)}}{(2 \pi)^{3} 2 E_{\ell}} \int \frac{d^{3} p_{(\nu)}}{(2 \pi)^{3} 2 E_{\nu}}(2 \pi)^{4} \delta^{(4)}\left(p_{(\pi)}-p_{(\ell)}-p_{(\nu)}\right) \\
& =\frac{1}{(4 \pi)^{2}} \int \frac{d^{3} p_{(\ell)}}{E_{\ell}} \int \frac{d^{3} p_{(\nu)}}{E_{\nu}} \delta\left(m_{\pi}-E_{\ell}-E_{\nu}\right) \delta^{(3)}\left(\vec{p}_{(\ell)}+\vec{p}_{(\nu)}\right) \\
& =\frac{1}{(4 \pi)^{2}} \int \frac{d^{3} p_{(\ell)}}{E_{\ell} E_{\nu}} \delta\left(m_{\pi}-E_{\ell}-E_{\nu}\right)  \tag{3.40}\\
& =\frac{1}{4 \pi} \int_{0}^{\infty} \frac{d p p^{2}}{\sqrt{m_{\ell}^{2}+p^{2}} \sqrt{m_{\nu}^{2}+p^{2}}} \delta\left(m_{\pi}-\sqrt{m_{\ell}^{2}+p^{2}}-\sqrt{m_{\nu}^{2}+p^{2}}\right) .
\end{align*}
$$

The delta function can be recast as

$$
\begin{equation*}
\delta\left(m_{\pi}-\sqrt{m_{\ell}^{2}+p^{2}}-\sqrt{m_{\nu}^{2}+p^{2}}\right)=\left(\frac{p}{E_{\ell}}+\frac{p}{E_{\nu}}\right)^{-1} \delta\left(p-p_{*}\right)=\frac{E_{\ell} E_{\nu}}{p m_{\pi}} \delta\left(p-p_{*}\right) \tag{3.41}
\end{equation*}
$$

where $p_{*}$ is the magnitude of the spatial momentum of the final particles in the rest frame of the pion, obtained by solving the following equation:

$$
\begin{align*}
m_{\pi}-\sqrt{m_{\nu}^{2}+p^{2}} & =\sqrt{m_{\ell}^{2}+p^{2}} \\
m_{\pi}^{2}-2 m_{\pi} \sqrt{m_{\nu}^{2}+p^{2}}+m_{\nu}^{2}+p^{2} & =m_{\ell}^{2}+p^{2}  \tag{3.42}\\
m_{\pi}^{2}+m_{\nu}^{2}-m_{\ell}^{2} & =2 m_{\pi} \sqrt{m_{\nu}^{2}+p^{2}} \\
\left(m_{\pi}^{2}+m_{\nu}^{2}-m_{\ell}^{2}\right)^{2} & =4 m_{\pi}^{2}\left(m_{\nu}^{2}+p^{2}\right) .
\end{align*}
$$

Since at each step we are squaring positive quantities, we are always solving an equivalent equation. From the last line we finally find

$$
\begin{equation*}
p_{*}^{2}=\frac{\left(m_{\pi}^{2}+m_{\nu}^{2}-m_{\ell}^{2}\right)^{2}}{4 m_{\pi}^{2}}-m_{\nu}^{2}=\frac{\left[m_{\pi}^{2}-\left(m_{\nu}+m_{\ell}\right)^{2}\right]\left[m_{\pi}^{2}-\left(m_{\nu}-m_{\ell}\right)^{2}\right]}{4 m_{\pi}^{2}} . \tag{3.43}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\Phi^{(2)}=\frac{1}{4 \pi} \int_{0}^{\infty} \frac{d p p^{2}}{E_{\ell} E_{\nu}} \frac{E_{\ell} E_{\nu}}{p m_{\pi}} \delta\left(p-p_{*}\right)=\frac{p_{*}}{4 \pi m_{\pi}} . \tag{3.44}
\end{equation*}
$$

In the case at hand $m_{\nu}=0$, so

$$
\begin{equation*}
p_{*}^{2}=\left(\frac{m_{\pi}^{2}-m_{\ell}^{2}}{2 m_{\pi}}\right)^{2} \tag{3.45}
\end{equation*}
$$

and

$$
\begin{equation*}
\Phi^{(2)}=\frac{m_{\pi}^{2}-m_{\ell}^{2}}{8 \pi m_{\pi}^{2}} \tag{3.46}
\end{equation*}
$$

We are now ready to write the final result for the total width:

$$
\begin{align*}
\Gamma & =\frac{1}{2 m_{\pi}} 4 G^{2} \cos ^{2} \theta_{C} f_{\pi}^{2} m_{\ell}^{2}\left(m_{\pi}^{2}-m_{\ell}^{2}\right) \frac{m_{\pi}^{2}-m_{\ell}^{2}}{8 \pi m_{\pi}^{2}} \\
& =\frac{G^{2} \cos ^{2} \theta_{C} f_{\pi}^{2} m_{\ell}^{2}}{4 \pi m_{\pi}^{3}}\left(m_{\pi}^{2}-m_{\ell}^{2}\right)^{2}=\frac{G^{2} \cos ^{2} \theta_{C} f_{\pi}^{2}}{4 \pi} m_{\pi} m_{\ell}^{2}\left(1-\frac{m_{\ell}^{2}}{m_{\pi}^{2}}\right)^{2} \tag{3.47}
\end{align*}
$$

As a function of the lepton mass, $\Gamma$ is suppressed both near $m_{\pi}$, which is a threshold effect due to the limited available phase space, and near 0 . This suppression is due to the definite handedness of the current: for a very light lepton helicity is almost a good quantum number, and almost only left-handed leptons and right-handed antileptons appear. Since the pion has zero spin, the spins of the fermions in the final state must be opposite, and since also their spatial momenta are opposite this requires that the two fermions have the same helicity; being a lepton and an antilepton, this cannot happen if they are both massless. For this reason, despite the limited phase space available, the dominant decay mode is $\pi^{+} \rightarrow \mu^{+} \nu_{\mu}$, instead of $\pi^{+} \rightarrow e^{+} \nu_{e}$. In fact (using $m_{\pi^{ \pm}}=140 \mathrm{MeV}, m_{\mu}=106 \mathrm{MeV}, m_{e}=0.5 \mathrm{MeV}$ )

$$
\begin{equation*}
\frac{\Gamma_{\pi^{+} \rightarrow e^{+} \nu_{e}}}{\Gamma_{\pi^{+} \rightarrow \mu^{+} \nu_{\mu}}}=\left(\frac{m_{e}}{m_{\mu}}\right)^{2}\left(\frac{m_{\pi}^{2}-m_{e}^{2}}{m_{\pi}^{2}-m_{\mu}^{2}}\right)^{2} \simeq 1.2 \cdot 10^{-4} \tag{3.48}
\end{equation*}
$$

Pion beta decay Next we consider the three-body decay ("pion beta decay")

$$
\begin{equation*}
\pi^{+} \rightarrow \pi^{0} e^{+} \nu_{e} . \tag{3.49}
\end{equation*}
$$

(Why do we not consider also $\pi^{+} \rightarrow \pi^{0} \mu^{+} \nu_{\mu}$ ?) Let $p_{1}$ and $p_{2}$ be the initial and final pion momenta, and $p_{(e)}$ and $p_{(\nu)}$ the positron and neutrino momenta. The decay amplitude is

$$
\begin{align*}
\mathcal{M}_{\mathrm{fi}} & =-\frac{G}{\sqrt{2}} \cos \theta_{C}\left\langle\pi^{0}\right|\left(\bar{d} \mathcal{O}_{L}^{\mu} u\right)(0)\left|\pi^{+}\right\rangle\left\langle e^{+} \nu_{e}\right|\left(\bar{\nu}_{e} \mathcal{O}_{L \mu} e\right)(0)|0\rangle  \tag{3.50}\\
& =-\frac{G}{\sqrt{2}} \cos \theta_{C}\left\langle\pi^{0}\right| V_{-}^{\mu}\left|\pi^{+}\right\rangle \bar{u}_{(\nu)}\left(p_{(\nu)}\right) \gamma_{\mu}\left(1-\gamma^{5}\right) v_{(e)}\left(p_{(e)}\right)
\end{align*}
$$

since only the vector current can have a nonvanishing hadronic matrix element (no axial vector is available). This matrix element must be of the form

$$
\begin{equation*}
\left\langle\pi^{0}\right| V_{-}^{\mu}\left|\pi^{+}\right\rangle=f_{+}\left(q^{2}\right) p^{\mu}+f_{-}\left(q^{2}\right) q^{\mu} \tag{3.51}
\end{equation*}
$$

where $p=p_{1}+p_{2}, q=p_{1}-p_{2}$, and $f_{ \pm}$are dimensionless rea ${ }^{31}$ functions of $q^{2}{ }^{32}$ In the isospin limit, conservation of the vector current implies

$$
\begin{equation*}
0=\left.q_{\mu}\left\langle\pi^{0}\right| V_{-}^{\mu}\left|\pi^{+}\right\rangle\right|_{\text {iso }}=\left.q^{2} f_{-}\left(q^{2}\right)\right|_{\text {iso }}, \tag{3.52}
\end{equation*}
$$

so that $\left.f_{-}\right|_{\text {iso }} \equiv 0$. (Obviously, the subscript "iso" means that we take the isospin limit.) Of course, there must be a mass difference between the pions for the decay to take place. Nonetheless, if we expand in the symmetry-breaking parameter $\Delta \equiv m_{\pi^{+}}-m_{\pi^{0}}$ we find

$$
\begin{equation*}
f_{-}\left(q^{2}\right)=\left.f_{-}\left(q^{2}\right)\right|_{\text {iso }}+\left.\Delta f_{-}^{(1)}\left(q^{2}\right)\right|_{\text {iso }}+\ldots=\left.\Delta f_{-}^{(1)}\left(q^{2}\right)\right|_{\text {iso }}+\ldots, \tag{3.53}
\end{equation*}
$$

where $f_{-}^{(1)}=\partial f_{-} / \partial \Delta$. Furthermore, the transferred momentum must be small,

$$
\begin{align*}
q^{2} & =m_{\pi^{+}}^{2}+m_{\pi^{0}}^{2}-2 p_{1} \cdot p_{2} \leq m_{\pi^{+}}^{2}+m_{\pi^{0}}^{2}-\left[\left(m_{\pi^{+}}+m_{\pi^{0}}\right)^{2}-m_{\pi^{+}}^{2}-m_{\pi^{0}}^{2}\right] \\
& =m_{\pi^{+}}^{2}+m_{\pi^{0}}^{2}-2 m_{\pi^{+}} m_{\pi^{0}}=\Delta^{2}, \tag{3.54}
\end{align*}
$$

which reflects the fact that there is little phase space available due to the small mass difference. We thus see that the term $f_{-}\left(q^{2}\right) q^{\mu}$ in Eq. (3.51) is of order $\Delta^{2}$, and moreover that $f_{+}\left(q^{2}\right)=$ $f_{+}(0)+\mathcal{O}\left(\Delta^{2}\right)$. Since the transition is between states belonging to the same isospin multiplet, in the isospin limit we can use Eq. (3.18), telling us that in the static limit the amplitude is governed by the weak charge. Since $I=1, I_{3}=1, I_{3}^{\prime}=0$, we find

$$
\begin{equation*}
\left.\left.f_{+}\left(q^{2}\right)\right|_{\text {iso }} p^{0}\right|_{\vec{q}=0}=\left.f_{+}(0)\right|_{\text {iso }} 2 p_{1}^{0}=\left.2 p_{1}^{0} \sqrt{2} \Longrightarrow f_{+}(0)\right|_{\text {iso }}=\sqrt{2} . \tag{3.55}
\end{equation*}
$$

```
\({ }^{31}\) This is shown again using \(T\) invariance, which implies
\[
\begin{aligned}
& \eta^{\mu \mu}\left\langle\pi^{0}\left(\vec{p}_{2}\right)\right| V_{-}^{\mu}(0)\left|\pi^{+}\left(\vec{p}_{1}\right)\right\rangle=\left\langle\pi^{0}\left(\vec{p}_{2}\right)\right| T^{\dagger} V_{-}^{\mu}(0) T\left|\pi^{+}\left(\vec{p}_{1}\right)\right\rangle=\left\langle\pi^{0}\left(-\vec{p}_{2}\right)\right| V_{-}^{\mu}(0)\left|\pi^{+}\left(-\vec{p}_{1}\right)\right\rangle^{*}\left(\eta_{P}^{0} \eta_{C}^{0}\right)^{*} \eta_{P}^{+} \eta_{C}^{+} \\
& =\eta^{\mu \mu}\left\langle\pi^{0}\left(\vec{p}_{2}\right)\right| V_{-}^{\mu}(0)\left|\pi^{+}\left(\vec{p}_{1}\right)\right\rangle^{*} \Longrightarrow f_{ \pm}=f_{ \pm}^{*},
\end{aligned}
\]
```

having used the properties of the explicit form, Eq. 3.51), and having made the usual choice $\eta_{C}^{ \pm}=\eta_{C}^{0}$ for the charge-conjugation phase of the charged pions (which is automatic in the isospin limit as $\pi^{ \pm}$and $\pi^{0}$ are in the same triplet).
${ }^{32}$ In general they must be scalar functions of $p$ and $q$, so functions of $q^{2}, q \cdot p=m_{\pi^{+}}^{2}-m_{\pi^{0}}^{2}$ and $p^{2}=q^{2}+4 q \cdot p$, and therefore can be written as functions of $q^{2}$ and the pion masses.

One can show that corrections due to isospin breaking are of order $\Delta^{2}$, and so we can write

$$
\begin{equation*}
\left\langle\pi^{0}\right| V_{-}^{\mu}\left|\pi^{+}\right\rangle=\left.f_{+}(0)\right|_{\text {iso }} p^{\mu}+\mathcal{O}\left(\Delta^{2}\right)=\sqrt{2} p^{\mu}+\mathcal{O}\left(\Delta^{2}\right) . \tag{3.56}
\end{equation*}
$$

Neglecting the mass difference in the matrix element thus gives the correct result up to $\mathcal{O}\left(\Delta^{2}\right)$.
The fact that corrections to $\left.f_{+}\right|_{\text {iso }}$ are of second order in the symmetry-breaking parameter is a consequence of the analogue of the Ademollo-Gatto theorem (see Section 4.2) in the case at hand. Let $I_{3}$ and $I_{ \pm}$denote the isospin generators in the unperturbed limit of exact isospin symmetry, and treat the symmetry-breaking term in the Hamiltonian as a perturbation. Let this term be proportional to some small parameter $\delta$. In the perturbed theory the energy eigenstates are generally mixtures of the unpertubed energy and isospin eigenstates, but since $I_{3}$ is still an exact symmetry only states with the same eigenvalue of $I_{3}$ can mix. Moreover, for small $\delta$ one can still associate the perturbed states uniquely with the unperturbed ones. We write then $\left|I I_{3}\right\rangle_{\delta}$ for the perturbed states that are exact eigenstates of $I_{3}$, and that as $\delta \rightarrow 0$ become the $\vec{I}^{2}$ and $I_{3}$ eigenstates $\left|I I_{3}\right\rangle_{0}=\left|I I_{3}\right\rangle_{\text {iso }^{\prime}}$. Starting now from the commutation relation $\left[I_{+}, I_{-}\right]=2 I_{3}$, and taking its expectation value on a hadronic state $\left|I I_{3}\right\rangle_{\delta}$ we find

$$
\begin{align*}
& 2_{\delta}\left\langle I I_{3}\right| I_{3}\left|I I_{3}\right\rangle_{\delta}=2 I_{3}={ }_{\delta}\left\langle I I_{3}\right|\left[I_{+}, I_{-}\right]\left|I I_{3}\right\rangle_{\delta} \\
& =\sum_{n}\left\langle I I_{3}\right| I_{+}|n\rangle_{\delta \delta}\langle n| I_{-}\left|I I_{3}\right\rangle_{\delta}-{ }_{\delta}\left\langle I I_{3}\right| I_{-}|n\rangle_{\delta \delta}\langle n| I_{+}\left|I I_{3}\right\rangle_{\delta}  \tag{3.57}\\
& \left.=\sum_{n}\left|{ }_{\delta}\langle n| I_{-}\right| I I_{3}\right\rangle\left._{\delta}\right|^{2}-\left.\left.\right|_{\delta}\langle n| I_{+}\left|I I_{3}\right\rangle_{\delta}\right|^{2},
\end{align*}
$$

having inserted a complete set of states. Among the states $|n\rangle_{\delta}$ we now separate those corresponding to the same isomultiplet as $\left|I I_{3}\right\rangle_{\delta}$, and write

$$
\begin{align*}
2 I_{3}= & \left.\left|{ }_{\delta}\left\langle I I_{3}-1\right| I_{-}\right| I I_{3}\right\rangle\left._{\delta}\right|^{2}-\left.\left.\right|_{\delta}\left\langle I I_{3}+1\right| I_{+}\left|I I_{3}\right\rangle_{\delta}\right|^{2} \\
& \left.+\sum_{n}^{\prime}\left|\delta\langle n| I_{-}\right| I I_{3}\right\rangle\left._{\delta}\right|^{2}-\left.\left.\right|_{\delta}\langle n| I_{+}\left|I I_{3}\right\rangle_{\delta}\right|^{2}, \tag{3.58}
\end{align*}
$$

where the sum extends now only on states corresponding to multiplets other than that of the state of interest. Since the symmetry-breaking term in the Hamiltonian is proportional to $\delta$, this sum must be at least of order $\delta^{2}$, since the matrix elements themselves are at least of order $\delta$ (they vanish in the isospin limit). On the other hand, in the isospin limit

$$
\begin{align*}
& \left.\left.\right|_{\text {iso }}\left\langle I I_{3}-1\right| I_{-}\left|I I_{3}\right\rangle_{\text {iso }}\right|^{2}-\left.\left.\right|_{\text {iso }}\left\langle I I_{3}+1\right| I_{+}\left|I I_{3}\right\rangle_{\text {iso }}\right|^{2} \\
& \quad=\left[I(I+1)-I_{3}\left(I_{3}-1\right)\right]-\left[I(I+1)-I_{3}\left(I_{3}+1\right)\right]=2 I_{3}, \tag{3.59}
\end{align*}
$$

and so

$$
\begin{align*}
& \left.\left|\delta\left\langle I I_{3}-1\right| I_{-}\right| I I_{3}\right\rangle\left._{\delta}\right|^{2}-\left.\left.\right|_{\delta}\left\langle I I_{3}+1\right| I_{+}\left|I I_{3}\right\rangle_{\delta}\right|^{2}  \tag{3.60}\\
& =\left.\left.\right|_{\text {iso }}\left\langle I I_{3}-1\right| I_{-}\left|I I_{3}\right\rangle_{\text {iso }}\right|^{2}-\left.\left.\right|_{\text {iso }}\left\langle I I_{3}+1\right| I_{+}\left|I I_{3}\right\rangle_{\text {iso }}\right|^{2}+\mathcal{O}\left(\delta^{2}\right)=2 I_{3}+\mathcal{O}\left(\delta^{2}\right) .
\end{align*}
$$

Applied to the states $\left|\pi^{+}\right\rangle=-|11\rangle$ and $\left|\pi^{0}\right\rangle=|10\rangle$ this gives

$$
\begin{equation*}
\left.\left|{ }_{\delta}\left\langle\pi^{0}\right| I_{-}\right| \pi^{+}\right\rangle\left._{\delta}\right|^{2}=|\sqrt{2}|^{2}+\mathcal{O}\left(\delta^{2}\right) . \tag{3.61}
\end{equation*}
$$

Notice that here we used states with unit normalisation, rather than the relativistically normalised ones. Adapting the proof is straightforward, and requires only a multiplicative factor $2 p^{0}$. Including this factor, we have $\left.\left(2 p^{0}\right)^{-1}\left\langle\pi^{0}\right| V_{-}^{0}\left|\pi^{+}\right\rangle\right|_{\vec{q}=0}=\left.\left(2 p^{0}\right)^{-1}\left\langle\pi^{0}\right| I_{-}\left|\pi^{+}\right\rangle\right|_{\vec{q}=0}=f_{+}(0)=\sqrt{2}+C$, with $C$ real (since $f_{+}$is real) and vanishing in the isospin limit. Then from Eq. (3.61) $\mathcal{O}\left(\delta^{2}\right)=f_{+}(0)^{2}-2=$ $(\sqrt{2}+C)^{2}-2=2 \sqrt{2} C+C^{2}$, and so $C=\mathcal{O}\left(\delta^{2}\right)$. For the meson masses, one finds instead that the leading order correction in perturbation theory is of order $\delta$, as it comes from the diagonalisation of the symmetry-breaking part of the Hamiltonian, restricted to the given (degenerate) multiplet, and so $\Delta \propto \delta$. In conclusion, $f_{+}(0)=\sqrt{2}+\mathcal{O}\left(\Delta^{2}\right)$, which is what we wanted to show.

To next-to-leading order in $\Delta$ we thus have

$$
\begin{align*}
\mathcal{M}_{\mathrm{fi}} & =-\frac{G}{\sqrt{2}} \cos \theta_{C} \sqrt{2} p^{\mu} \bar{u}_{(\nu)}\left(p_{(\nu)}\right) \gamma_{\mu}\left(1-\gamma^{5}\right) v_{(e)}\left(p_{(e)}\right) \\
& =-G \cos \theta_{C} p^{\mu} \bar{u}_{(\nu)}\left(p_{(\nu)}\right) \gamma_{\mu}\left(1-\gamma^{5}\right) v_{(e)}\left(p_{(e)}\right),  \tag{3.62}\\
\left|\mathcal{M}_{\mathrm{fi}}\right|^{2} & =G^{2} \cos ^{2} \theta_{C} p^{\mu} p^{\nu} \bar{u}_{(\nu)}\left(p_{(\nu)}\right) \gamma_{\mu}\left(1-\gamma^{5}\right) v_{(e)}\left(p_{(e)}\right) \bar{v}_{(e)}\left(p_{(e)}\right) \gamma_{\nu}\left(1-\gamma^{5}\right) u_{(\nu)}\left(p_{(\nu)}\right),
\end{align*}
$$

and summing over spins

$$
\begin{align*}
\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle & =G^{2} \cos ^{2} \theta_{C} p^{\mu} p^{\nu} \operatorname{tr} \gamma_{\mu}\left(1-\gamma^{5}\right)\left(\not p_{(e)}-m_{e}\right) \gamma_{\nu}\left(1-\gamma^{5}\right) \not p_{(\nu)} \\
& =2 G^{2} \cos ^{2} \theta_{C} p^{\mu} p^{\nu} \operatorname{tr}\left(1+\gamma^{5}\right) \gamma_{\mu}\left(\not{ }_{(e)}-m_{e}\right) \gamma_{\nu} \not p_{(\nu)} \\
& =2 G^{2} \cos ^{2} \theta_{C} p^{\mu} p^{\nu} \operatorname{tr}\left(1+\gamma^{5}\right) \gamma_{\mu} \not p_{(e)} \gamma_{\nu} \not p_{(\nu)}  \tag{3.63}\\
& =8 G^{2} \cos ^{2} \theta_{C} p^{\mu} p^{\nu}\left[p_{(e) \mu} p_{(\nu) \nu}+p_{(e) \nu} p_{(\nu) \mu}-\eta_{\mu \nu}\left(p_{(e)} \cdot p_{(\nu)}\right)-i \varepsilon_{\mu \alpha \nu \beta} p_{(e)}^{\alpha} p_{(\nu)}^{\beta}\right] \\
& =8 G^{2} \cos ^{2} \theta_{C}\left[2\left(p \cdot p_{(e)}\right)\left(p \cdot p_{(\nu)}\right)-p^{2}\left(p_{(e)} \cdot p_{(\nu)}\right)\right] .
\end{align*}
$$

The differential decay width is then

$$
\begin{equation*}
d \Gamma=\frac{4 G^{2} \cos ^{2} \theta_{C}}{m_{\pi^{+}}}\left[2\left(p \cdot p_{(e)}\right)\left(p \cdot p_{(\nu)}\right)-p^{2}\left(p_{(e)} \cdot p_{(\nu)}\right)\right] d \Phi^{(3)} \tag{3.64}
\end{equation*}
$$

where we recall that the term in square brackets is correct to leading and first subleading order in $\Delta$. In order to get the total width we write

$$
\begin{equation*}
\Gamma=\frac{4 G^{2} \cos ^{2} \theta_{C}}{m_{\pi^{+}}} \int \frac{d^{3} p_{2}}{(2 \pi)^{3} 2 E_{2}} p^{\mu} p^{\nu}\left[2 \mathcal{I}_{\mu \nu}(q)-\eta_{\mu \nu} \mathcal{I}_{\alpha}^{\alpha}(q)\right] \tag{3.65}
\end{equation*}
$$

where, since $q$ is timelike (see below), we can use the general result Eq. 2.93) for $\mathcal{I}_{\mu \nu}(q)$, which in the case at hand reads

$$
\begin{equation*}
\mathcal{I}_{\alpha \beta}(q)=\frac{1}{32 \pi}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)\left\{\frac{1}{3}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)^{2}\left(q^{2} \eta_{\alpha \beta}-q_{\alpha} q_{\beta}\right)+\left[1-\left(\frac{m_{e}^{2}}{q^{2}}\right)^{2}\right] q_{\alpha} q_{\beta}\right\} . \tag{3.66}
\end{equation*}
$$

The contraction $\mathcal{I}^{\alpha}{ }_{\alpha}$ can be computed straightforwardly,

$$
\begin{equation*}
\mathcal{I}^{\alpha}{ }_{\alpha}(q)=\frac{q^{2}}{32 \pi}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)\left\{\left(1-\frac{m_{e}^{2}}{q^{2}}\right)^{2}+\left[1-\left(\frac{m_{e}^{2}}{q^{2}}\right)^{2}\right]\right\}=\frac{q^{2}}{16 \pi}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)^{2} . \tag{3.67}
\end{equation*}
$$

Contracting the tensorial structures in Eq. (3.65) with $p^{\mu} p^{\nu}$ we find

$$
\begin{align*}
p^{\mu} p^{\nu} \mathcal{I}_{\mu \nu}(q)=\frac{1}{32 \pi}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)\{ & \frac{1}{3}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)^{2}\left(p^{2} q^{2}-(p \cdot q)^{2}\right) \\
& \left.+\left[1-\left(\frac{m_{e}^{2}}{q^{2}}\right)^{2}\right](p \cdot q)^{2}\right\},  \tag{3.68}\\
p^{\mu} p^{\nu} \eta_{\mu \nu} \mathcal{I}^{\alpha}{ }_{\alpha}(q)=\frac{p^{2} q^{2}}{16 \pi}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)^{2} . &
\end{align*}
$$

We then plug this into Eq. (3.65) to get

$$
\begin{align*}
\mathcal{Q}= & p^{\mu} p^{\nu}\left[2 \mathcal{I}_{\mu \nu}(q)-\eta_{\mu \nu} \mathcal{I}_{\alpha}^{\alpha}(q)\right] \\
= & \frac{1}{16 \pi}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)\left\{\left[1-\left(\frac{m_{e}^{2}}{q^{2}}\right)^{2}-\frac{1}{3}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)^{2}\right](p \cdot q)^{2}\right. \\
& \left.+\left[\frac{1}{3}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)^{2}-\left(1-\frac{m_{e}^{2}}{q^{2}}\right)\right] p^{2} q^{2}\right\} \\
= & \frac{1}{16 \pi}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)\left\{\left[\frac{2}{3}+\frac{2}{3} \frac{m_{e}^{2}}{q^{2}}-\frac{4}{3}\left(\frac{m_{e}^{2}}{q^{2}}\right)^{2}\right](p \cdot q)^{2}\right.  \tag{3.69}\\
& \left.\quad+\left[-\frac{2}{3}+\frac{1}{3} \frac{m_{e}^{2}}{q^{2}}+\frac{1}{3}\left(\frac{m_{e}^{2}}{q^{2}}\right)^{2}\right] p^{2} q^{2}\right\} \\
= & \frac{1}{48 \pi}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)\left\{2(p \cdot q)^{2}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)\left(1+\frac{2 m_{e}^{2}}{q^{2}}\right)-p^{2} q^{2}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)\left(2+\frac{m_{e}^{2}}{q^{2}}\right)\right\} \\
= & \frac{1}{24 \pi}\left(1-\frac{m_{e}^{2}}{q^{2}}\right)^{2}\left\{(p \cdot q)^{2}\left(1+\frac{2 m_{e}^{2}}{q^{2}}\right)-p^{2} q^{2}\left(1+\frac{m_{e}^{2}}{2 q^{2}}\right)\right\} .
\end{align*}
$$

The various Lorentz invariants read $\left(2 m \equiv m_{\pi^{+}}+m_{\pi^{0}}\right)$

$$
\begin{align*}
q^{2} & =m_{\pi^{+}}^{2}+m_{\pi^{0}}^{2}-2 m_{\pi^{+}} E_{2} \\
p^{2} & =m_{\pi^{+}}^{2}+m_{\pi^{0}}^{2}+2 m_{\pi^{+}} E_{2}=q^{2}+4 m_{\pi^{+}} E_{2}  \tag{3.70}\\
q \cdot p & =m_{\pi^{+}}^{2}-m_{\pi^{0}}^{2}=2 m \Delta
\end{align*}
$$

Notice that $q^{2}$ must be positive, hence $q^{\mu}$ is a timelike vector, and furthemore $q^{2}$ is bounded from above:

$$
\begin{equation*}
q^{2}=\left(p_{(e)}+p_{(\nu)}\right)^{2} \geq m_{e}^{2}, \quad q^{2} \leq m_{\pi^{+}}^{2}+m_{\pi^{0}}^{2}-2 m_{\pi^{+}} m_{\pi^{0}}=\Delta^{2} \tag{3.71}
\end{equation*}
$$

We still have to integrate over the neutral pion momentum. To this end, taking into account that Eq. 3.69 depends only on $E_{2}$ and not on the angular variables, we can replace

$$
\begin{equation*}
\int \frac{d^{3} p_{2}}{(2 \pi)^{3} 2 E_{2}} \rightarrow \frac{4 \pi}{2(2 \pi)^{3}} \int_{0}^{p_{2 \max }} \frac{d p_{2} p_{2}^{2}}{E_{2}}=\frac{1}{(2 \pi)^{2}} \int_{E_{2 \min }}^{E_{2 \max }} d E_{2} \sqrt{E_{2}^{2}-m_{\pi^{0}}^{2}} \tag{3.72}
\end{equation*}
$$

We further relate (notice that here $p_{2}$ denotes the magnitude of the spatial momentum of the neutral pion, not its four-momentum)

$$
\begin{align*}
E_{2} & =\frac{m_{\pi^{+}}^{2}+m_{\pi^{0}}^{2}-q^{2}}{2 m_{\pi^{+}}} \\
p_{2}^{2} & =E_{2}^{2}-m_{\pi^{0}}^{2}=\left(\frac{m_{\pi^{+}}^{2}+m_{\pi^{0}}^{2}-q^{2}}{2 m_{\pi^{+}}}\right)^{2}-m_{\pi^{0}}^{2}  \tag{3.73}\\
& =\frac{\left[m_{\pi^{+}}^{2}-\left(m_{\pi^{0}}+\omega\right)^{2}\right]\left[m_{\pi^{+}}^{2}-\left(m_{\pi^{0}}-\omega\right)^{2}\right]}{4 m_{\pi^{+}}^{2}} \\
p^{2} & =2\left(m_{\pi^{+}}^{2}+m_{\pi^{0}}^{2}\right)-\omega^{2}
\end{align*}
$$

where $\omega^{2}=q^{2}$ and we have used Eq. (3.43). We can further manipulate the expression for $p_{2}^{2}$ and $p^{2}$ to get

$$
\begin{align*}
p_{2}^{2} & =\frac{\left[m_{\pi^{+}}-m_{\pi^{0}}-\omega\right]\left[m_{\pi^{+}}+m_{\pi^{0}}+\omega\right]\left[m_{\pi^{+}}-m_{\pi^{0}}+\omega\right]\left[m_{\pi^{+}}^{2}+m_{\pi^{0}}-\omega\right]}{4 m_{\pi^{+}}^{2}} \\
& =\frac{[\Delta-\omega][2 m+\omega][\Delta+\omega][2 m-\omega]}{4 m_{\pi^{+}}^{2}}=\frac{\left[\Delta^{2}-\omega^{2}\right]\left[4 m^{2}-\omega^{2}\right]}{4 m_{\pi^{+}}^{2}},  \tag{3.74}\\
p^{2} & =4\left(m^{2}+\left(\frac{\Delta}{2}\right)^{2}\right)-\omega^{2}=4 m^{2}+\Delta^{2}-\omega^{2} .
\end{align*}
$$

The most convenient integration variable is $\omega$. We only have to determine the integration range and the Jacobian:

$$
\begin{equation*}
\omega_{\min }=\sqrt{q_{\min }^{2}}=m_{e}, \quad \omega_{\max }=\sqrt{q_{\max }^{2}}=\Delta, \quad 2 \omega d \omega=-2 m_{\pi^{+}} d E_{2} \tag{3.75}
\end{equation*}
$$

The integration measure Eq. (3.72) becomes

$$
\begin{equation*}
\int \frac{d^{3} p_{2}}{(2 \pi)^{3} 2 E_{2}} \rightarrow \frac{1}{(2 \pi)^{2} 2 m_{\pi^{+}}^{2}} \int_{m_{e}}^{\Delta} d \omega \omega \sqrt{\left[\Delta^{2}-\omega^{2}\right]\left[4 m^{2}-\omega^{2}\right]} \tag{3.76}
\end{equation*}
$$

while the integrand becomes

$$
\begin{equation*}
\mathcal{Q}=\frac{1}{24 \pi}\left(1-\frac{m_{e}^{2}}{\omega^{2}}\right)^{2}\left\{4 m^{2} \Delta^{2}\left(1+\frac{2 m_{e}^{2}}{\omega^{2}}\right)-\left(4 m^{2}+\Delta^{2}-\omega^{2}\right) \omega^{2}\left(1+\frac{m_{e}^{2}}{2 \omega^{2}}\right)\right\} \tag{3.77}
\end{equation*}
$$

It is convenient to rescale $\omega \rightarrow \Delta \omega$ and write

$$
\begin{align*}
& \int \frac{d^{3} p_{2}}{(2 \pi)^{3} 2 E_{2}} \rightarrow \frac{\Delta^{3} 2 m}{(2 \pi)^{2} 2 m_{\pi^{+}}^{2}} \int_{\frac{m_{e}}{\Delta}}^{1} d \omega \omega \sqrt{\left[1-\omega^{2}\right]\left[1-\frac{\Delta^{2}}{4 m^{2}} \omega^{2}\right]} \\
& \mathcal{Q}=\frac{\Delta^{2} m^{2}}{6 \pi}\left(1-\frac{m_{e}^{2}}{\Delta^{2} \omega^{2}}\right)^{2}\left\{\left(1+\frac{2 m_{e}^{2}}{\Delta^{2} \omega^{2}}\right)-\left(1+\frac{\Delta^{2}}{4 m^{2}}\left(1-\omega^{2}\right)\right) \omega^{2}\left(1+\frac{m_{e}^{2}}{2 \Delta^{2} \omega^{2}}\right)\right\}  \tag{3.78}\\
& \\
& \\
& =\frac{\Delta^{2} m^{2}}{6 \pi} \tilde{\mathcal{Q}}
\end{align*}
$$

All in all we have

$$
\begin{align*}
\Gamma & =\frac{4 G^{2} \cos ^{2} \theta_{C}}{m_{\pi^{+}}} \frac{\Delta^{3} 2 m}{(2 \pi)^{2} 2 m_{\pi^{+}}^{2}} \frac{\Delta^{2} m^{2}}{6 \pi} \int_{\frac{m_{e}}{\Delta}}^{1} d \omega \omega \sqrt{\left[1-\omega^{2}\right]\left[1-\frac{\Delta^{2}}{4 m^{2}} \omega^{2}\right]} \tilde{\mathcal{Q}}(\omega) \\
& =\frac{G^{2} \cos ^{2} \theta_{C} \Delta^{5}}{6 \pi^{3}}\left(\frac{m}{m_{\pi^{+}}}\right)^{3} \int_{\frac{m_{e}}{\Delta}}^{1} d \omega \omega \sqrt{\left[1-\omega^{2}\right]\left[1-\frac{\Delta^{2}}{4 m^{2}} \omega^{2}\right]} \tilde{\mathcal{Q}}(\omega) \tag{3.79}
\end{align*}
$$

We now set $\varepsilon=m_{e}^{2} / \Delta^{2}$ and drop orders of $\Delta / m$ higher than $(\Delta / m)^{1}$ to get

$$
\begin{align*}
\Gamma & =\frac{G^{2} \cos ^{2} \theta_{C} \Delta^{5}}{6 \pi^{3}}\left(1-\frac{3 \Delta}{2 m}\right) \int_{\sqrt{\varepsilon}}^{1} d \omega \omega \sqrt{1-\omega^{2}} \\
& \times\left(1-\frac{\varepsilon}{\omega^{2}}\right)^{2}\left\{\left(1+\frac{2 \varepsilon}{\omega^{2}}\right)-\omega^{2}\left(1+\frac{\varepsilon}{2 \omega^{2}}\right)\right\}  \tag{3.80}\\
= & \frac{G^{2} \cos ^{2} \theta_{C} \Delta^{5}}{6 \pi^{3}}\left(1-\frac{3 \Delta}{2 m}\right) K(\varepsilon) .
\end{align*}
$$

The calculation of $K(\varepsilon)$ is particularly annoying. Let us change variables to $z=\omega^{2}$ to get

$$
\begin{equation*}
K(\varepsilon)=\frac{1}{2} \int_{\varepsilon}^{1} d z \sqrt{1-z}\left(1-\frac{\varepsilon}{z}\right)^{2}\left[\left(1+\frac{2 \varepsilon}{z}\right)-z\left(1+\frac{\varepsilon}{2 z}\right)\right] . \tag{3.81}
\end{equation*}
$$

Expanding the polynomial part of the integrand we obtain

$$
\begin{equation*}
\left(1-\frac{\varepsilon}{z}\right)^{2}\left[\left(1+\frac{2 \varepsilon}{z}\right)-z\left(1+\frac{\varepsilon}{2 z}\right)\right]=1-z+\frac{3}{2} \varepsilon-\varepsilon^{2}\left(3+\frac{\varepsilon}{2}\right) \frac{1}{z^{2}}+2 \varepsilon^{3} \frac{1}{z^{3}}, \tag{3.82}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
K(\varepsilon)=\frac{1}{2} \int_{\varepsilon}^{1} d z \sqrt{1-z}\left[1-z+\frac{3}{2} \varepsilon-\varepsilon^{2}\left(3+\frac{\varepsilon}{2}\right) \frac{1}{z^{2}}+2 \varepsilon^{3} \frac{1}{z^{3}}\right] . \tag{3.83}
\end{equation*}
$$

The exact evaluation is left for later (see Appendix 3.5). For the time being we will be content with finding the lowest order approximation. This requires some care: if the integrand were regular at zero, it would suffice to expand $K(\varepsilon)=K(0)+\varepsilon K^{\prime}(0)+\ldots$, but $K^{\prime}(0)$ does not exist: among the various contributions there is that coming from deriving the integral with respect to its lower integration limit, which is minus times the integrand, which blows up at the origin. To circumvent this problem we isolate the most singular part of the integrand, and write

$$
\begin{align*}
\int_{\varepsilon}^{1} d z \sqrt{1-z} \frac{1}{z^{n}} & =\int_{\varepsilon}^{1} d z \frac{1}{z^{n}}-\int_{\varepsilon}^{1} d z(1-\sqrt{1-z}) \frac{1}{z^{n}} \\
& =-\left.\frac{1}{n-1} \frac{1}{z^{n-1}}\right|_{\varepsilon} ^{1}-\int_{\varepsilon}^{1} d z(1-\sqrt{1-z}) \frac{1}{z^{n}}  \tag{3.84}\\
& =\frac{1}{n-1} \frac{1}{\varepsilon^{n-1}}+\text { less singular } .
\end{align*}
$$

Integrating exactly the first two terms and retaining only the leading contributions of the rest we find

$$
\begin{equation*}
K(\varepsilon) \simeq \frac{1}{2}\left[\frac{2}{5}(1-\varepsilon)^{\frac{5}{2}}+\varepsilon(1-\varepsilon)^{\frac{3}{2}}-3 \varepsilon^{2} \frac{1}{\varepsilon}+2 \varepsilon^{3} \frac{1}{2 \varepsilon^{2}}\right] \simeq \frac{1}{5}(1-5 \varepsilon) \tag{3.85}
\end{equation*}
$$

Shoving this into Eq. 3.80 we finally get

$$
\begin{equation*}
\Gamma=\frac{G^{2} \cos ^{2} \theta_{C} \Delta^{5}}{6 \pi^{3}}\left(1-\frac{3 \Delta}{2 m}\right) \frac{1}{5}(1-5 \varepsilon)=\frac{G^{2} \cos ^{2} \theta_{C} \Delta^{5}}{30 \pi^{3}}\left(1-\frac{3 \Delta}{2 m}-5 \frac{m_{e}^{2}}{\Delta^{2}}\right), \tag{3.86}
\end{equation*}
$$

which is correct to $\mathcal{O}\left(\frac{\Delta}{m}\right)$ and $\mathcal{O}\left(\frac{m_{e}^{2}}{\Delta^{2}}\right)$.

### 3.3 Neutron beta decay

We move to one of the most important decay processes governed by weak interactions, namely the beta decay of the neutron,

$$
\begin{equation*}
n \longrightarrow p+e^{-}+\bar{\nu}_{e} \tag{3.87}
\end{equation*}
$$

This process is the basic process behind all nuclear beta decays. In turn, the fundamental process behind it at the quark level is

$$
\begin{equation*}
d \longrightarrow u+e^{-}+\bar{\nu}_{e}, \tag{3.88}
\end{equation*}
$$

which turns the neutron, $n=(u d d)$ in terms of quarks, into a proton. $p=(u u d)$. This is clearly the same fundamental process behind $\pi^{-} \rightarrow \pi^{0}+e^{-}+\bar{\nu}_{e}$. The relevant term in the Lagrangian is thus

$$
\begin{equation*}
-\frac{G}{\sqrt{2}} \cos \theta_{C}\left(\bar{u} \mathcal{O}_{L}^{\alpha} d\right)\left(\bar{e} \mathcal{O}_{L \alpha} \nu_{e}\right), \tag{3.89}
\end{equation*}
$$

and the decay amplitude reads

$$
\begin{equation*}
\mathcal{M}_{\mathrm{fi}}=-\frac{G}{\sqrt{2}} \cos \theta_{C} H^{\alpha} \bar{u}_{e}\left(p_{(e)}\right) \gamma_{\alpha}\left(1-\gamma^{5}\right) v_{e}\left(p_{(\nu)}\right) \tag{3.90}
\end{equation*}
$$

where

$$
\begin{align*}
H^{\alpha} & =\langle p|\left(\bar{u} \mathcal{O}_{L}^{\alpha} d\right)(0)|n\rangle=V_{+}^{\alpha}-A_{+}^{\alpha}, \\
V_{+}^{\alpha} & =\langle p|\left(\bar{u} \gamma^{\alpha} d\right)(0)|n\rangle,  \tag{3.91}\\
A_{+}^{\alpha} & =\langle p|\left(\bar{u} \gamma^{\alpha} \gamma^{5} d\right)(0)|n\rangle .
\end{align*}
$$

The subscript + refers to the fact that these are the + components of isovector currents. While it is very difficult to obtain the hadronic matrix elements of Eq. (3.91) from first principles (and this requires techniques beyond perturbation theory), it is nevertheless possible to constrain their form by means of symmetry considerations. In fact, Lorentz invariance imposes

$$
\begin{align*}
\langle p| V_{+}^{\mu}|n\rangle & =\bar{u}_{p}\left(p_{p}, s_{p}\right)\left(f_{1}\left(q^{2}\right) \gamma^{\mu}+f_{2}\left(q^{2}\right) i \sigma^{\mu \nu} \frac{q_{\nu}}{2 m}+f_{3}\left(q^{2}\right) \frac{q^{\mu}}{2 m}\right) u_{n}\left(p_{n}, s_{n}\right) \\
& =\bar{u}_{p}\left(p_{p}, s_{p}\right) M^{\mu}(q) u_{n}\left(p_{n}, s_{n}\right) \\
\langle p| A_{+}^{\mu}|n\rangle & =\bar{u}_{p}\left(p_{p}, s_{p}\right)\left(g_{1}\left(q^{2}\right) \gamma^{\mu}+g_{2}\left(q^{2}\right) i \sigma^{\mu \nu} \frac{q_{\nu}}{2 m}+g_{3}\left(q^{2}\right) \frac{q^{\mu}}{2 m}\right) \gamma^{5} u_{n}\left(p_{n}, s_{n}\right)  \tag{3.92}\\
& =\bar{u}_{p}\left(p_{p}, s_{p}\right) M_{5}^{\mu}(q) u_{n}\left(p_{n}, s_{n}\right),
\end{align*}
$$

where $q=p_{n}-p_{p}, m=\frac{1}{2}\left(m_{p}+m_{n}\right)$, and $f_{i}, g_{i}$ are real dimensionless functions of $q^{2}$ called form factors (see Appendix 3.6 for details). In fact, these are the most general linearly independent structures one can build out of $\mathrm{D}=\left(\frac{1}{2}, 0\right) \oplus\left(0, \frac{1}{2}\right)$ and its conjugate $\overline{\mathrm{D}}$ that transforms respectively like a vector and an axial vector under Lorentz transformations. The form factors can be determined experimentally by means of (anti)neutrino-nucleon scattering, since the same hadronic matrix elements appear there. For the same reason, the matrix elements of the electromagnetic current must read

$$
\begin{align*}
& \langle p| V_{\mathrm{em}}^{\mu}|p\rangle=\bar{u}_{p}\left(p_{p}^{\prime}, s_{p}^{\prime}\right)\left(f_{p 1}\left(q^{2}\right) \gamma^{\mu}+f_{p 2}\left(q^{2}\right) i \sigma^{\mu \nu} \frac{q_{\nu}}{2 m_{p}}+f_{p 3}\left(q^{2}\right) \frac{q^{\mu}}{2 m_{p}}\right) u_{p}\left(p_{p}, s_{p}\right), \\
& \langle n| V_{\mathrm{em}}^{\mu}|n\rangle=\bar{u}_{n}\left(p_{n}^{\prime}, s_{n}^{\prime}\right)\left(f_{n 1}\left(q^{2}\right) \gamma^{\mu}+f_{n 2}\left(q^{2}\right) i \sigma^{\mu \nu} \frac{q_{\nu}}{2 m_{n}}+f_{n 3}\left(q^{2}\right) \frac{q^{\mu}}{2 m_{n}}\right) u_{n}\left(p_{n}, s_{n}\right), \tag{3.93}
\end{align*}
$$

where $f_{p i}$ and $f_{n i}$ are the electromagnetic form factors of the proton and the neutron, that can be studied by means of electron-nucleon scattering.

We can further constrain the hadronic matrix element by recalling that the electromagnetic current is conserved, which implies that the matrix elements of $V_{\mathrm{em}}^{\mu}$ are transverse, i.e., $q_{\mu} V_{\mathrm{em}}^{\mu}=0$. Since $\bar{u}\left(p^{\prime}, s^{\prime}\right) q u(p, s)=\bar{u}\left(p^{\prime}, s^{\prime}\right)\left(\not p^{\prime}-\not p\right) u(p, s)=\bar{u}\left(p^{\prime}, s^{\prime}\right)(m-m) u(p, s)=0$, and $\sigma^{\mu \nu}$ is antisymmetric, this implies that $f_{p 3}\left(q^{2}\right)=f_{n 3}\left(q^{2}\right)=0$. In the case of exact isospin invariance, this implies conservation of the vector current (the isovector and isoscalar part of
the electromagnetic current must be conserved separately), $q_{\mu} V_{+}^{\mu}=0$. This implies that also $f_{3}\left(q^{2}\right)=0$. Alternatively, one can use the behaviour of the vector current under $G$-parity to drop this term in the isospin limit (see Appendix 3.7). There is actually more: applied to the nucleon isospin doublet ( $I=\frac{1}{2}$ ) relevant to neutron decay, Eq. (3.13) implies

$$
\begin{equation*}
\langle p| V_{+}^{\mu}|n\rangle=\langle p| V_{\mathrm{em}}^{\mu}|p\rangle-\langle n| V_{\mathrm{em}}^{\mu}|n\rangle, \tag{3.94}
\end{equation*}
$$

and so in the isospin limit

$$
\begin{equation*}
f_{i}\left(q^{2}\right)=f_{p i}\left(q^{2}\right)-f_{n i}\left(q^{2}\right), \quad i=1,2,3 \tag{3.95}
\end{equation*}
$$

These relations are particularly useful in the limit $q \rightarrow 0$. Recalling Eq. 3.19),

$$
\begin{align*}
\left.\langle p| V_{\mathrm{em}}^{0}(0)|p\rangle\right|_{\vec{q}=0} & =f_{p 1}(0) \bar{u}_{p}\left(p, s_{p}^{\prime}\right) \gamma^{0} u_{p}\left(p, s_{p}\right)=f_{p 1}(0) 2 p^{0} \delta_{s_{p}^{\prime}, s_{p}} \\
& =2 p^{0} Q_{p} \delta_{s_{s^{\prime}}, s_{p}}=2 p^{0} \delta_{s_{p}^{\prime}, s_{p}},  \tag{3.96}\\
\left.\langle n| V_{\mathrm{em}}^{0}(0)|n\rangle\right|_{\vec{q}=0} & =f_{n 1}(0) \bar{u}_{n}\left(p, s_{n}^{\prime}\right) \gamma^{0} u_{n}\left(p, s_{n}\right)=f_{n 1}(0) 2 p^{0} \delta_{s_{n}^{\prime}, s_{n}} \\
& =2 p^{0} Q_{n} \delta_{s_{n}^{\prime}, s_{n}}=0,
\end{align*}
$$

it follows that

$$
\begin{equation*}
f_{1}(0)=f_{p 1}(0)-f_{n 1}(0)=Q_{p}-Q_{n}=1 . \tag{3.97}
\end{equation*}
$$

Alternatively, since the $n \rightarrow p$ transition is a $\Delta I=0$, strangeness-conserving transition, at zero transferred momentum it is governed by the weak charge $Q_{W}\left(\frac{1}{2},-\frac{1}{2}\right)=1$, i.e., recalling Eq. (3.17),

$$
\begin{align*}
\left.\langle p| V_{+}^{0}(0)|n\rangle\right|_{\vec{q}=0} & =f_{1}(0) \bar{u}_{p}\left(p, s_{p}\right) \gamma^{0} u_{n}\left(p, s_{n}\right)=f_{1}(0) 2 p^{0} \delta_{s_{p}, s_{n}} \\
& =2 p^{0} \delta_{s_{p}, s_{n}} \sqrt{\frac{3}{4}+\frac{1}{4}}=2 p^{0} \delta_{s_{p}, s_{n}}, \tag{3.98}
\end{align*}
$$

we again find $f_{1}(0)=1$. Yet another way to find this result is to consider the decay process at the quark level, and compute the matrix elements of the current between the appropriate quark states. Schematically,

$$
\begin{equation*}
\left\langle u \frac{u d-d u}{\sqrt{2}}\right| \bar{u} d\left|d \frac{u d-d u}{\sqrt{2}}\right\rangle=\left(\left\langle\frac{u d-d u}{\sqrt{2}}\right|-2\left\langle\frac{u d}{\sqrt{2}}\right|\right)\left(\left|\frac{u d-d u}{\sqrt{2}}\right\rangle-2\left|\frac{d u}{\sqrt{2}}\right\rangle\right)=1-2 \frac{1}{2}+2 \frac{1}{2}=1 . \tag{3.99}
\end{equation*}
$$

The form factors $f_{p, n 2}(0)$ at zero momentum are instead related to the response of the particles to an external static magnetic field, and are known as the anomalous magnetic moments of the proton and the neutron. These are well known experimentally:

$$
\begin{equation*}
f_{p 2}(0)=1.79, \quad f_{n 2}(0)=-1.91 \tag{3.100}
\end{equation*}
$$

from which it follows that

$$
\begin{equation*}
f_{2}(0)=3.7 . \tag{3.101}
\end{equation*}
$$

This terms is known as "weak magnetism". The third term is known as "effective scalar" since it can be expressed as a scalar through

$$
\begin{equation*}
q_{\mu} \bar{u}_{\nu} \gamma^{\mu}\left(1-\gamma^{5}\right) u_{e}=\bar{u}_{\nu}\left(q_{\nu}+q_{e}\right)\left(1-\gamma^{5}\right) u_{e}=\bar{u}_{\nu}\left(1+\gamma^{5}\right) \phi_{e} u_{e}=m_{e} \bar{u}_{\nu}\left(1+\gamma^{5}\right) u_{e} . \tag{3.102}
\end{equation*}
$$

All in all then

$$
\begin{equation*}
\langle p| V_{+}^{\mu}|n\rangle=\bar{u}_{p}\left(p_{p}, s_{p}\right)\left(\gamma^{\mu}+i \sigma^{\mu \nu} q_{\nu} \frac{f_{2}(0)}{2 m}\right) u_{n}\left(p_{n}, s_{n}\right) . \tag{3.103}
\end{equation*}
$$

A more precise treatment is as follows. Near the exact isospin point, expand in the isospin-breaking parameter $\Delta=m_{n}-m_{p}$. The transferred momentum satisfies $q^{2}=\left(p_{(e)}+p_{(\nu)}\right)^{2} \geq m_{e}^{2}$ and $q^{2}=$ $m_{n}^{2}+m_{p}^{2}-2 m_{n} E_{p} \leq\left(m_{n}-m_{p}\right)^{2}=\Delta^{2}$. In the isospin limit $\Delta=0$, and expanding in $\Delta, f_{i}=$ $f_{i}^{(0)}+\Delta f_{i}^{(1)}+\ldots$ we find from current conservation in the isospin limit that $f_{3}^{(0)}\left(q^{2}\right)=0$, while for $f_{1,2}^{(0)}$ one finds nonzero values. Since $q=\mathcal{O}(\Delta)$, we find that the $f_{1}$ term is $\mathcal{O}(1)$, the $f_{2} q$ term is $\mathcal{O}(\Delta)$, and the $f_{3} q$ term is $\mathcal{O}\left(\Delta^{2}\right)$, so if we work to order $\Delta$ we can neglect it. We can furthermore neglect deviations from $f_{1,2}(0)$ since they also are $\mathcal{O}\left(\Delta^{2}\right)$.

We now proceed with the axial vector matrix element. In the isospin limit, the second term ("weak electrism") is forbidden by $G$-parity, since it is $G$-even while the axial current is $G$-odd. Since we are working at order $\Delta$ we can therefore drop it. We then approximate again the form factors with their values at zero, and obtain

$$
\begin{equation*}
\langle p| A_{+}^{\mu}|n\rangle=\bar{u}_{p}\left(p_{p}, s_{p}\right)\left(\gamma^{\mu} g_{1}(0)+q^{\mu} \frac{g_{3}(0)}{2 m}\right) \gamma^{5} u_{n}\left(p_{n}, s_{n}\right) \tag{3.104}
\end{equation*}
$$

What can we say about the axial form factors? The PCAC hypothesis, Eq. (3.24), implies

$$
\begin{equation*}
-i q_{\mu}\langle p| A_{+}^{\mu}(0)|n\rangle=f_{\pi} m_{\pi}^{2}\langle p| \phi_{+}(0)|n\rangle \tag{3.105}
\end{equation*}
$$

and using the known pole structure of the nucleon matrix element (see Appendix 3.8, in particular Eq. 3.216) we find

$$
\begin{equation*}
-i q_{\mu}\langle p| A_{+}^{\mu}(0)|n\rangle=2 i f_{\pi} m_{\pi}^{2} \frac{g_{\pi N N}\left(q^{2}\right)}{m_{\pi}^{2}-q^{2}} \bar{u}_{p}\left(p_{p}, s_{p}\right) \gamma^{5} u_{n}\left(p_{n}, s_{n}\right) \tag{3.106}
\end{equation*}
$$

with $g_{\pi N N}\left(q^{2}\right)$ the pion-nucleon-nucleon vertex function. On the other hand, using the general structure of the axial current matrix element we find

$$
\begin{align*}
-\bar{u}_{p}\left(p_{p}, s_{p}\right)\left(\not q g_{1}\left(q^{2}\right)+q^{2} \frac{g_{3}\left(q^{2}\right)}{2 m}\right) \gamma^{5} u_{n}\left(p_{n}, s_{n}\right) & =2 f_{\pi} m_{\pi}^{2} \frac{g_{\pi N N}\left(q^{2}\right)}{m_{\pi}^{2}-q^{2}} \bar{u}_{p}\left(p_{p}, s_{p}\right) \gamma^{5} u_{n}\left(p_{n}, s_{n}\right) \\
\left(m_{p}+m_{n}\right) g_{1}\left(q^{2}\right)-q^{2} \frac{g_{3}\left(q^{2}\right)}{2 m} & =2 f_{\pi} m_{\pi}^{2} \frac{g_{\pi N N}\left(q^{2}\right)}{m_{\pi}^{2}-q^{2}} \tag{3.107}
\end{align*}
$$

Setting $q^{2}=0$ we find $\left(m_{p}=m_{n}=m\right)$

$$
\begin{equation*}
m g_{1}(0)=f_{\pi} g_{\pi N N}(0) \tag{3.108}
\end{equation*}
$$

The quantity $g_{1}(0)$ is known experimentally, comparing Eq. (3.104) with measurements of neutron $\beta$ decay, and is $g_{1}(0) \simeq 1.267$. On the other hand, from the experimental study of nucleonnucleon scattering one can obtain (under the assumption that pion exchange dominates the amplitude) the physical value $g=g_{\pi N N}\left(m_{\pi}^{2}\right)$ of the pion-nucleon coupling, with $g \simeq 13.169$. If we further assume that $g_{\pi N N}$ does not vary much between zero and the pion mass squared, then we obtain the Goldberger-Treiman relation

$$
\begin{equation*}
m g_{1}(0)=f_{\pi} g \tag{3.109}
\end{equation*}
$$

which is satisfied within $2 \div 3 \%$.

Is it worth noting that in the limit of massless pions the axial current becomes exactly conserved, and so we obtain the relation

$$
\begin{equation*}
2 m g_{1}\left(q^{2}\right)-q^{2} \frac{g_{3}\left(q^{2}\right)}{2 m}=0 \Longrightarrow \frac{g_{3}\left(q^{2}\right)}{2 m}=\frac{2 m g_{1}\left(q^{2}\right)}{q^{2}} \tag{3.110}
\end{equation*}
$$

which implies the presence of a pole at zero in $g_{3}$. This massless pole is a consequence of the spontaneous breaking of chiral symmetry, and indicates the presence of massless particles in the theory, i.e., the pions.

There is another, more phenomenological way to see the origin of the pole. Since nucleons are not elementary, their coupling to the leptonic weak current is not pointlike, but rather a "blob" taking into account the internal structure. Contributions to this blob run over the possible ways in which the final products can be obtained from the neutron, and include a pointlike neutron-proton-leptons four-fermion interaction, and a pointlike neutron-proton-pion interaction with the pion subsequently decaying into a lepton pair. The latter involves the coupling to a leptonic current as discussed in the previous subsections. The effective Lagrangian is

$$
\begin{equation*}
\mathscr{L}^{\mathrm{eff}}=-\frac{G \cos \theta}{\sqrt{2}}\left[\bar{N} \tau_{+} \gamma^{\mu}\left(1-g_{A} \gamma^{5}\right) N \bar{e} \gamma_{\mu}\left(1-\gamma^{5}\right) \nu+\text { h.c. }\right]+i g \bar{N} \tau_{a} \gamma^{5} N \phi_{a}, \tag{3.111}
\end{equation*}
$$

with $N$ the nucleon doublet of fields and $g_{A}=g_{1}(0)$. We then have to sum two contributions, corresponding respectively to the $g_{1}$ and the $g_{3}$ terms in the decay amplitude $i \mathcal{M}_{\mathrm{fi}}$,

$$
\begin{align*}
& g_{1} \text { term }: \frac{-i G}{\sqrt{2}} g_{A} u_{p} \gamma^{\mu} \gamma^{5} u_{n},  \tag{3.112}\\
& g_{3} \text { term }: i(i g \sqrt{2}) u_{p} \gamma^{5} u_{n} \frac{i}{q^{2}-m_{\pi}^{2}} \frac{-i G}{\sqrt{2}}\left(i \sqrt{2} f_{\pi} q_{\mu}\right)=\frac{2 f_{\pi} g q_{\mu}}{m_{\pi}^{2}-q^{2}} \frac{-i G}{\sqrt{2}},
\end{align*}
$$

that have to be contracted with the leptonic current $\bar{u}_{(e)} \mathcal{O}_{L \mu} v_{(\nu)}$.
To leading order we can then set set $q^{2}=0$ and just keep the terms proportional to $f_{1}(0)=$ $g_{V}=1$ and $g_{1}(0)=g_{A}$, obtaining for the decay amplitude

$$
\begin{equation*}
\mathcal{M}_{\mathrm{fi}}=-\frac{G}{\sqrt{2}} \cos \theta_{C} \bar{u}_{p} \gamma^{\mu}\left(1-\alpha \gamma^{5}\right) u_{n} \bar{u}_{e} \gamma_{\mu}\left(1-\gamma^{5}\right) v_{\nu_{e}} \tag{3.113}
\end{equation*}
$$

where $\alpha=g_{1}(0) / f_{1}(0)=g_{A} / g_{V}$. This is the starting point to derive phenomenological consequences. Let us include a polarisation for the neutron and the electron, with corresponding polarisation vectors

$$
\begin{equation*}
s_{n}=\left(0, \vec{\eta}_{n}\right), \quad s_{e}=\left(\frac{\vec{\eta}_{e} \cdot \vec{k}_{e}}{m_{e}}, \vec{\eta}_{e}+\frac{\vec{k}_{e}\left(\vec{k}_{e} \cdot \vec{\eta}_{e}\right)}{m_{e}\left(E_{e}+m_{e}\right)}\right) . \tag{3.114}
\end{equation*}
$$

Summing over the unobserved proton spin we find

$$
\begin{align*}
\sum_{s_{p}}\left|\mathcal{M}_{\mathrm{fi}}\right|^{2}=\frac{G^{2} \cos ^{2} \theta_{C}}{2} & \operatorname{tr} \gamma^{\mu}\left(1-\alpha \gamma^{5}\right)\left(\not p+m_{n}\right) \frac{1+\gamma^{5} \phi_{n}}{2} \gamma^{\nu}\left(1-\alpha \gamma^{5}\right)\left(\not p^{\prime}+m_{p}\right)  \tag{3.115}\\
\times & \operatorname{tr} \gamma_{\mu}\left(1-\gamma^{5}\right) \not k_{(\nu)} \gamma_{\nu}\left(1-\gamma^{5}\right)\left(\not k_{e}+m_{e}\right) \frac{1+\gamma^{5} \phi_{e}}{2} .
\end{align*}
$$

We already know how to deal with the second trace, which is equal to

$$
\begin{align*}
& \operatorname{tr} \gamma_{\mu}\left(1-\gamma^{5}\right) \not k_{(\nu)} \gamma_{\nu}\left(1-\gamma^{5}\right)\left(\not k_{e}+m_{e}\right) \frac{1+\gamma^{5} \phi_{e}}{2}=\operatorname{tr}\left(1+\gamma^{5}\right) \gamma_{\mu} k_{(\nu)} \gamma_{\nu} \tilde{k}_{e}  \tag{3.116}\\
& =4\left\{k_{(\nu) \mu} \tilde{k}_{(e) \mu}+k_{(\nu) \nu} \tilde{k}_{(e) \mu}-\eta_{\mu \nu} k_{(\nu)} \cdot \tilde{k}_{(e)}-i \epsilon_{\mu \alpha \nu \beta} k_{(\nu)}^{\alpha} \tilde{k}_{(e)}^{\beta}\right\},
\end{align*}
$$

with $\tilde{k}_{e}=k_{e}-m_{e} s_{e}$. For the first trace we have instead, taking into account that the trace of an odd number of gamma matrices vanishes,

$$
\begin{align*}
& \operatorname{tr} \gamma^{\mu}\left(1-\alpha \gamma^{5}\right)\left(\not p+m_{n}\right)\left(1+\gamma^{5} \phi_{n}\right) \gamma^{\nu}\left(1-\alpha \gamma^{5}\right)\left(\not p^{\prime}+m_{p}\right) \\
& =\operatorname{tr}\left(1+\alpha \gamma^{5}\right)^{2} \gamma^{\mu}\left(\not p+\gamma^{5} m_{n} \phi_{n}\right) \gamma^{\nu} \not p^{\prime}+\operatorname{tr}\left(1-\left(\alpha \gamma^{5}\right)^{2}\right) \gamma^{\mu}\left(m_{n}-\gamma^{5} \not p \phi_{n}\right) \gamma^{\nu} m_{p} \\
& =\operatorname{tr}\left(1+\alpha^{2}+2 \alpha \gamma^{5}\right) \gamma^{\mu}\left(\not p+\gamma^{5} m_{n} \phi_{n}\right) \gamma^{\nu} \not p^{\prime}+\left(1-\alpha^{2}\right) \operatorname{tr} \gamma^{\mu}\left(m_{n}-\gamma^{5} \not p \phi_{n}\right) \gamma^{\nu} m_{p}  \tag{3.117}\\
& =\operatorname{tr} \gamma^{\mu}\left[\left(1+\alpha^{2}\right) \not p-2 \alpha m_{n} \phi_{n}\right] \gamma^{\nu} \not p^{\prime}-\operatorname{tr} \gamma^{5} \gamma^{\mu}\left[\left(1+\alpha^{2}\right) m_{n} \phi_{n}-2 \alpha \not p\right] \gamma^{\nu} \not p^{\prime} \\
& \quad+\left(1-\alpha^{2}\right) m_{n} m_{p} \operatorname{tr} \gamma^{\mu} \gamma^{\nu}+\left(1-\alpha^{2}\right) m_{p} \operatorname{tr} \gamma^{5} \gamma^{\mu} \not p \phi_{n} \gamma^{\nu} .
\end{align*}
$$

Since the electron and neutrino energies and momenta (as well as the electron mass) are of order $\Delta=m_{n}-m_{p}$, Eq. (3.116) is of order $\Delta^{2}$. We will retain only the leading order in Eq. (3.117), which means in practice setting $m_{n}=m_{p}=m$ and $p^{\prime}=p$. This simplifies the calculation, leading to

$$
\begin{align*}
& \operatorname{tr} \gamma^{\mu}\left(1-\alpha \gamma^{5}\right)\left(\not p+m_{n}\right)\left(1+\gamma^{5} \phi_{n}\right) \gamma^{\nu}\left(1-\alpha \gamma^{5}\right)\left(p^{\prime}+m_{p}\right) \rightarrow \\
& \quad 4\left\{\left(1+\alpha^{2}\right)\left(2 p^{\mu} p^{\nu}-m^{2} \eta^{\mu \nu}\right)-2 \alpha m\left(s_{n}^{\mu} p^{\nu}+s_{n}^{\nu} p^{\mu}\right)+i m\left(1+\alpha^{2}\right) \epsilon^{\mu \alpha^{\prime} \nu \beta^{\prime}} s_{n \alpha^{\prime}} p_{\beta^{\prime}}\right. \\
& \left.\quad+\left(1-\alpha^{2}\right)\left[m^{2} \eta^{\mu \nu}-i m \epsilon^{\mu \alpha^{\prime} \beta^{\prime} \nu} p_{\alpha^{\prime}} s_{n \beta^{\prime}}\right]\right\}  \tag{3.118}\\
& =\quad 8\left\{\left(1+\alpha^{2}\right) p^{\mu} p^{\nu}-\alpha^{2} m^{2} \eta^{\mu \nu}-\alpha m\left(s_{n}^{\mu} p^{\nu}+s_{n}^{\nu} p^{\mu}\right)+i \alpha^{2} m \epsilon^{\mu \alpha^{\prime} \nu \beta^{\prime}} s_{n \alpha^{\prime}} p_{\beta^{\prime}}\right\} .
\end{align*}
$$

Contracting Eq. (3.116) and Eq. (3.118) and including the missing factors we find

$$
\begin{align*}
& \sum_{s_{p}}\left|\mathcal{M}_{\mathrm{f}}\right|^{2}=\frac{G^{2} \cos ^{2} \theta_{C}}{2} 4^{2}\left\{\left(1+\alpha^{2}\right) p^{\mu} p^{\nu}-\alpha^{2} m^{2} \eta^{\mu \nu}-\alpha m\left(s_{n}^{\mu} p^{\nu}+s_{n}^{\nu} p^{\mu}\right)+i \alpha^{2} m \epsilon^{\mu \alpha^{\prime} \nu \beta^{\prime}} s_{n \alpha^{\prime}} p_{\beta^{\prime}}\right\} \\
& \times\left\{k_{(\nu) \mu} \tilde{k}_{(e) \mu}+k_{(\nu) \nu} \tilde{k}_{(e) \mu}-\eta_{\mu \nu} k_{(\nu)} \cdot \tilde{k}_{(e)}-i \epsilon_{\mu \alpha \nu \beta} k_{(\nu)}^{\alpha} \tilde{k}_{(e)}^{\beta}\right\} \\
&=8 G^{2} \cos ^{2} \theta_{C}\left\{2\left(1+\alpha^{2}\right) p \cdot k_{(\nu)} p \cdot \tilde{k}_{(e)}-\left(1-\alpha^{2}\right) m^{2} k_{(\nu)} \cdot \tilde{k}_{(e)}\right. \\
&\left.+2 m\left[\left(\alpha^{2}-\alpha\right) s_{n} \cdot \tilde{k}_{(e)} p \cdot k_{(\nu)}-\left(\alpha^{2}+\alpha\right) s_{n} \cdot k_{(\nu)} p \cdot \tilde{k}_{(e)}\right]\right\} . \tag{3.119}
\end{align*}
$$

Let us consider now several cases.

Unpolarised neutron, no spin measurement In this case we sum over electron spin and average over neutron spin. The result is obtained replacing $s_{n} \rightarrow 0, \tilde{k}_{(e)} \rightarrow k_{(e)}$, and including a factor of $4 / 2=2$ in the decay width. We find

$$
\begin{align*}
d \Gamma & =\frac{8 G^{2} \cos ^{2} \theta_{C}}{m}\left\{2\left(1+\alpha^{2}\right) m^{2} E_{e} E_{\nu}-\left(1-\alpha^{2}\right) m^{2}\left(E_{e} E_{\nu}-\vec{k}_{(\nu)} \cdot \vec{k}_{(e)}\right)\right\} d \Phi^{(3)}  \tag{3.120}\\
& =8 G^{2} \cos ^{2} \theta_{C} m E_{e} E_{\nu}\left\{\left(1+3 \alpha^{2}\right)+\left(1-\alpha^{2}\right) \beta_{e} \cos \theta\right\} d \Phi^{(3)},
\end{align*}
$$

where $\beta_{e}=\frac{\left|\vec{k}_{(e)}\right|}{E_{e}}$ is the electron velocity and $\theta$ the relative angle between the electron and neutrino trajectories. The phase space element reads

$$
\begin{align*}
d \Phi^{(3)} & =\frac{1}{8(2 \pi)^{5}} \frac{d^{3} k_{e}}{E_{e}} \frac{d^{3} k_{\nu}}{E_{\nu}} \frac{d^{3} p^{\prime}}{E_{p}} \delta^{(4)}\left(p-p^{\prime}-k_{(\nu)}-k_{(e)}\right) \\
& =\frac{1}{8(2 \pi)^{5}} \frac{d^{3} k_{e}}{E_{e}} \frac{d^{3} k_{\nu}}{E_{\nu}} \frac{1}{E_{p}} \delta\left(m_{n}-E_{p}-E_{\nu}-E_{e}\right) \tag{3.121}
\end{align*}
$$

where $E_{p}^{2}=m_{p}^{2}+\left(\vec{k}_{\nu}+\vec{k}_{e}\right)^{2}$. Since

$$
\begin{align*}
\left(p-p^{\prime}\right)^{2} & =\left(k_{(\nu)}+k_{(e)}\right)^{2}
\end{aligned} \geq m_{e}^{2}, ~ \begin{aligned}
m_{n}^{2}+m_{p}^{2}-2 m_{n} E_{p} & \geq m_{e}^{2} \\
\left(m_{n}-m_{p}\right)^{2}-m_{e}^{2} & \geq 2 m_{n}\left(E_{p}-m_{p}\right) \tag{3.122}
\end{align*}
$$

we have

$$
\begin{align*}
& E_{p}-m_{p} \leq \frac{\Delta^{2}-m_{e}^{2}}{2 m_{n}}=\mathcal{O}\left(\frac{\Delta^{2}}{m}, \frac{m_{e}^{2}}{m}\right)  \tag{3.123}\\
& E_{\nu}+E_{e}=m_{n}-m_{p}-\left(E_{p}-m_{p}\right)=\Delta+\mathcal{O}\left(\frac{\Delta^{2}}{m}, \frac{m_{e}^{2}}{m}\right)
\end{align*}
$$

To leading order in $\Delta$ we can then replace $E_{p}$ with the proton mass, neglecting the proton recoil $\left|\vec{p}^{\prime}\right|$,

$$
\begin{equation*}
\vec{p}^{\prime 2}=E_{p}^{2}-m_{p}^{2}=\left(E_{p}-m_{p}\right)\left(E_{p}-m_{p}+2 m_{p}\right)=2 m_{p} \cdot \mathcal{O}\left(\frac{\Delta^{2}}{m}, \frac{m_{e}^{2}}{m}\right) . \tag{3.124}
\end{equation*}
$$

We find

$$
\begin{align*}
d \Phi^{(3)} & \simeq \frac{1}{8(2 \pi)^{5}} \frac{d^{3} k_{e}}{E_{e}} \frac{d^{3} k_{\nu}}{E_{\nu}} \frac{1}{m} \delta\left(E_{\nu}+E_{e}-\Delta\right) \\
& =\frac{1}{8(2 \pi)^{5} m E_{e} E_{\nu}} d E_{e} E_{e} \sqrt{E_{e}^{2}-m_{e}^{2}} d \Omega_{(e)} d E_{\nu} E_{\nu}^{2} d \Omega_{(\nu)} \delta\left(E_{\nu}+E_{e}-\Delta\right) \tag{3.125}
\end{align*}
$$

Since the overall orientation of the electron-neutrino pair can be integrated out trivially, from Eq. (3.120) we can read off the angular correlation $C(\cos \theta)$ between electron and neutrino (that can be measured indirectly by measuring the proton recoil, see below), namely

$$
\begin{equation*}
C(\cos \theta)=\left(\frac{d \Gamma}{d E_{e}}\right)^{-1} \frac{d \Gamma}{d E_{e} d \cos \theta}=1+\frac{1-\alpha^{2}}{1+3 \alpha^{2}} \beta_{e} \cos \theta \tag{3.126}
\end{equation*}
$$

This measures the distribution of electrons and neutrinos in the relative angle, for fixed electron energy. In order to find the energy distribution $\frac{1}{\Gamma} \frac{d \Gamma}{d E_{e}}$ of the electron we integrate over the angles and find

$$
\begin{align*}
d \Gamma & =8 G^{2} \cos ^{2} \theta_{C}\left(1+3 \alpha^{2}\right) \frac{1}{8(2 \pi)^{5}}(4 \pi)^{2} d E_{e} E_{e} \sqrt{E_{e}^{2}-m_{e}^{2}}\left(\Delta-E_{e}\right)^{2} \\
& =\frac{G^{2} \cos ^{2} \theta_{C}}{2 \pi^{3}}\left(1+3 \alpha^{2}\right) d E_{e} E_{e} \sqrt{E_{e}^{2}-m_{e}^{2}}\left(\Delta-E_{e}\right)^{2} . \tag{3.127}
\end{align*}
$$

Finally, the total decay width is

$$
\begin{align*}
\Gamma & =\frac{G^{2} \cos ^{2} \theta_{C}}{2 \pi^{3}}\left(1+3 \alpha^{2}\right) \int_{m_{e}}^{\Delta} d E_{e} E_{e} \sqrt{E_{e}^{2}-m_{e}^{2}}\left(\Delta-E_{e}\right)^{2}  \tag{3.128}\\
& =\frac{G^{2} \cos ^{2} \theta_{C}}{2 \pi^{3}}\left(1+3 \alpha^{2}\right) \Delta^{5} I\left(\frac{m_{e}}{\Delta}\right),
\end{align*}
$$

where

$$
\begin{equation*}
I(z)=\int_{z}^{1} d x x \sqrt{x^{2}-z^{2}}(1-x)^{2} \tag{3.129}
\end{equation*}
$$

The integral $I(z)$ can be calculated explicitly, and reads

$$
\begin{equation*}
I(z)=\frac{2}{3}\left\{-\frac{1}{5}\left(1-z^{2}\right)^{\frac{5}{2}}+\frac{3}{8} z^{4} \operatorname{arccosh} \frac{1}{z}+\frac{1}{4} \sqrt{1-z^{2}}\left[1-\frac{5}{2} z^{2}\right]\right\} \tag{3.130}
\end{equation*}
$$

Notice $I(0)=1 / 30$. For $m_{e}=0.51 \mathrm{MeV}$ and $\Delta=1.29 \mathrm{MeV}$ we have $z=0.395$, and $I(z) / I(0)=$ 0.47 , so

$$
\begin{equation*}
\Gamma=0.47 \frac{G^{2} \cos ^{2} \theta_{C}}{60 \pi^{3}}\left(1+3 \alpha^{2}\right) \Delta^{5} \tag{3.131}
\end{equation*}
$$

Unpolarised neutron, electron spin measurement In this case we have to average over the neutron spin and keep the electron spin:

$$
\begin{align*}
d \Gamma & =\frac{4 G^{2} \cos ^{2} \theta_{C}}{m}\left\{2\left(1+\alpha^{2}\right) p \cdot k_{(\nu)} p \cdot \tilde{k}_{(e)}-\left(1-\alpha^{2}\right) m^{2} k_{(\nu)} \cdot \tilde{k}_{(e)}\right\} d \Phi^{(3)} \\
& =\frac{4 G^{2} \cos ^{2} \theta_{C}}{m}\left\{2\left(1+\alpha^{2}\right) m^{2} E_{\nu} \tilde{k}_{(e)}^{0}-\left(1-\alpha^{2}\right) m^{2}\left(E_{\nu} \tilde{k}_{(e)}^{0}-\vec{k}_{\nu} \cdot \overrightarrow{\tilde{k}}_{e}\right)\right\} d \Phi^{(3)}  \tag{3.132}\\
& =4 G^{2} \cos ^{2} \theta_{C} m E_{\nu} E_{e}\left\{\left(1+3 \alpha^{2}\right) \frac{\tilde{k}_{(e)}^{0}}{E_{e}}+\left(1-\alpha^{2}\right) \frac{\vec{k}_{\nu}}{E_{\nu}} \cdot \frac{\overrightarrow{\tilde{k}}_{e}}{E_{e}}\right\} d \Phi^{(3)}
\end{align*}
$$

If we integrate over the neutrino momenta (including over the relative angle between the electron and neutrino trajectories), we can read off the electron polarisation $\vec{P}_{e}$ from

$$
\begin{equation*}
\frac{\tilde{k}_{(e)}^{0}}{E_{e}}=1-\frac{\vec{\eta}_{e} \cdot \vec{k}_{e}}{E_{e}} \tag{3.133}
\end{equation*}
$$

i.e., $\vec{P}_{e}=-\frac{\vec{k}_{e}}{E_{e}}$.

Polarised neutron In this case we sum over the electron spin, obtaining

$$
\begin{align*}
& d \Gamma=\frac{8 G^{2} \cos ^{2} \theta_{C}}{m}\left\{2\left(1+\alpha^{2}\right) p \cdot k_{(\nu)} p \cdot k_{(e)}-\left(1-\alpha^{2}\right) m^{2} k_{(\nu)} \cdot k_{(e)}\right. \\
&\left.+2 m\left[\left(\alpha^{2}-\alpha\right) s_{n} \cdot k_{(e)} p \cdot k_{(\nu)}-\left(\alpha^{2}+\alpha\right) s_{n} \cdot k_{(\nu)} p \cdot k_{(e)}\right]\right\} d \Phi^{(3)} \\
&=8 G^{2} \cos ^{2} \theta_{C} m E_{e} E_{\nu}\left\{2\left(1+\alpha^{2}\right)-\left(1-\alpha^{2}\right)\left(1-\beta_{e} \cos \theta\right)\right.  \tag{3.134}\\
&\left.-2\left[\left(\alpha^{2}-\alpha\right) \vec{\eta}_{n} \cdot \frac{\vec{k}_{e}}{E_{e}}-\left(\alpha^{2}+\alpha\right) \vec{\eta}_{n} \cdot \frac{\vec{k}_{\nu}}{E_{\nu}}\right]\right\} d^{(3)} \Phi .
\end{align*}
$$

Integrating over everything but the direction of the electron or that of the neutrino, we are left with

$$
\begin{align*}
& \left(\frac{d \Gamma}{d E_{e}}\right)^{-1} \frac{d \Gamma}{d E_{e} d \cos \theta_{e}}=1-\frac{2\left(\alpha^{2}-\alpha\right)}{1+3 \alpha^{2}} \frac{\vec{\eta}_{n} \cdot \vec{k}_{e}}{E_{e}}=1-\frac{2\left(\alpha^{2}-\alpha\right)}{1+3 \alpha^{2}} \frac{\left|\vec{k}_{e}\right| \cos \theta_{e}}{E_{e}} \quad(e \text { correlation }) \\
& \left(\frac{d \Gamma}{d E_{e}}\right)^{-1} \frac{d \Gamma}{d E_{e} d \cos \theta_{\nu}}=1+\frac{2\left(\alpha^{2}+\alpha\right)}{1+3 \alpha^{2}} \frac{\vec{\eta}_{n} \cdot \vec{k}_{\nu}}{E_{\nu}}=1+\frac{2\left(\alpha^{2}+\alpha\right)}{1+3 \alpha^{2}} \frac{E_{\nu} \cos \theta_{\nu}}{E_{\nu}} \quad\left(\bar{\nu}_{e} \text { correlation }\right) \tag{3.135}
\end{align*}
$$



Figure 6: Fermi function

Summary of free neutron decay Let us collect the results concerning free neutron decay when no measurement is made on the electron spin. The differential decay width in the static approximation $\left(q^{2} \simeq 0\right)$ reads

$$
\begin{align*}
d \Gamma= & \frac{G^{2} \cos ^{2} \theta_{C}}{2 \pi^{3}}\left(1+3 \alpha^{2}\right)\left\{1+\frac{1-\alpha^{2}}{1+3 \alpha^{2}} \vec{\beta}_{e} \cdot \vec{n}_{\nu}-2\left[\frac{\alpha^{2}-\alpha}{1+3 \alpha^{2}} \vec{\beta}_{e} \cdot \vec{\eta}_{n}-\frac{\alpha^{2}+\alpha}{1+3 \alpha^{2}} \vec{n}_{\nu} \cdot \vec{\eta}_{n}\right]\right\}  \tag{3.136}\\
& \times \frac{d \Omega_{e}}{4 \pi} \frac{d \Omega_{\nu}}{4 \pi} d E_{e} E_{e} \sqrt{E_{e}^{2}-m_{e}^{2}}\left(\Delta-E_{e}\right)^{2} .
\end{align*}
$$

The coefficients of $\vec{\beta}_{e} \cdot \vec{n}_{\nu}, \vec{\beta}_{e} \cdot \vec{\eta}_{n}$ and $\vec{n}_{\nu} \cdot \vec{\eta}_{n}$ give the angular correlation between the momenta of the electron and the neutrino, between the neutron polarisation and the electron momentum, and between the neutron polarisation and the neutrino momentum, respectively. From the electronneutrino angular correlation, measured from the proton recoil, $\vec{p}_{p}^{2}=2 m_{p}\left(E_{\nu}^{2}+\vec{k}_{e}^{2}+2 E_{\nu}\left|\vec{k}_{e}\right| \cos \theta\right)$, one can obtain $|\alpha|=\left|g_{A} / g_{V}\right|$, which since $g_{V} \simeq 1$ yields the axial charge. The sign of $\alpha$ is obtained instead from the angular correlation of the electron with the neutron spin in polarised neutron decay. Combining the result for $\alpha^{2}$ with the experimental values of the neutron lifetime and of the Fermi constant $G$ obtained from muon decay, one can then determine $\left|\cos \theta_{C}\right|$.

The energy spectrum of the electron is given by the function

$$
\begin{equation*}
F\left(x, W_{0}\right)=x \sqrt{x^{2}-1}\left(W_{0}-x\right)^{2}, \quad x=\frac{E_{e}}{m_{e}}, \quad W_{0}=\frac{\Delta}{m_{e}}, \tag{3.137}
\end{equation*}
$$

up to constant factors. This is the Fermi function for the energy spectrum in the case of free neutron decay, and provides a first approximation of the energy spectrum in the case of nuclear $\beta^{-}$and $\beta^{+}$decays. The inclusion of effects due to the form factor $f_{2}(0)$, i.e., due to weak magnetism, leads to a modification of the Fermi spectrum to $F\left(x, W_{0}\right) \rightarrow F\left(x, W_{0}\right)(1 \pm \varepsilon x)$ for $\beta^{\mp}$ decays. Such effects are hard to detect in neutron decay since the vector current is dominated by the weak charge contribution, but they become the leading contribution from the vector current in decay processes involving the transition between nuclei belonging to different isomultiplets, since in this case $f_{1}(0)$ vanishes. A good example is provided by the transitions between the isotriplet $\left({ }^{12} \mathrm{~B},{ }^{12} \mathrm{C}^{*},{ }^{12} \mathrm{~N}\right)$ and the isosinglet ${ }^{12} \mathrm{C}$, which are respectively a $\beta^{-}, \gamma$, and $\beta^{+}$transitions. The electromagnetic transition ${ }^{12} \mathrm{C}^{*} \rightarrow{ }^{12} \mathrm{C}$ is a magnetic dipole transition governed by
the electromagnetic form factor $f_{2}^{\mathrm{em}}(0)$, which since $\Delta I=1$ contains only a contribution from the isovector current and none from the isoscalar. This belongs to the same isotriplet as the charge weak currents governing the $\beta^{\mp}$ decays of $\left({ }^{12} \mathrm{~B}\right.$ and $\left.{ }^{12} \mathrm{~N}\right)$, and so the weak form factor $f_{2}(0)$ can be obtained from $f_{2}^{\mathrm{em}}(0)$ as determined by an experimental study of ${ }^{12} \mathrm{C}^{*} \rightarrow{ }^{12} \mathrm{C}$, and the value of $\varepsilon$ in the modified spectrum can be predicted. The result is in agreement with experiments (see Ref. [11]).

Fermi and Gamow-Teller nuclear transitions The vector and axial vector current matrix element between proton and neutron read, in the non-relativistic limit

$$
\begin{align*}
& \langle p| V_{+}^{\mu}|n\rangle=g_{V} \bar{u}_{p} \gamma^{\mu} u_{n} \simeq g_{V} \delta^{\mu}{ }_{0} u_{p}^{\dagger} u_{n}=2 m g_{V} \delta^{\mu}{ }_{0} \delta_{s_{p} s_{n}} \\
& \langle p| A_{+}^{\mu}|n\rangle=g_{A} \bar{u}_{p} \gamma^{\mu} \gamma^{5} u_{n} \simeq g_{A} \sum_{j=1}^{3} \delta^{\mu}{ }_{j} u_{p}^{\dagger} \sigma_{j} u_{n}=2 m g_{A} \sum_{j=1}^{3} \delta_{j}^{\mu}\left(\sigma_{j}\right)_{s_{p} s_{n}} \tag{3.138}
\end{align*}
$$

Only the temporal component of the vector current and the spatial components of the axial current contribute. In the beta decay of a $n$-nucleon nucleus $N \rightarrow N^{\prime}$, one has instead for the relevant components

$$
\begin{align*}
\left\langle N^{\prime}\right| V_{+}^{0}|N\rangle & =2 m g_{V} \int d^{3 n} p \Psi_{N^{\prime}}^{f^{\prime} s^{\prime}}(\vec{p})^{*}\left(\sum_{i} \tau_{+}^{(i)} \mathbf{1}_{S}^{(i)} \prod_{j \neq i} \mathbf{1}_{F}^{(j)} \mathbf{1}_{S}^{(j)}\right)_{f^{\prime} s^{\prime}, f s} \Psi_{N}^{f s}(\vec{p}) \equiv g_{V}\langle 1\rangle  \tag{3.139}\\
\left\langle N^{\prime}\right| \vec{A}_{+}|N\rangle & =2 m g_{A} \int d^{3 n} p \Psi_{N^{\prime}}^{f^{\prime} s^{\prime}}(\vec{p})^{*}\left(\sum_{i} \tau_{+}^{(i)} \vec{\sigma}^{(i)} \prod_{j \neq i} \mathbf{1}_{F}^{(j)} \mathbf{1}_{S}^{(j)}\right)_{f^{\prime} s^{\prime}, f s} \Psi_{N}^{f s}(\vec{p}) \equiv g_{A}\langle\vec{\sigma}\rangle
\end{align*}
$$

where $\Psi_{N}$ and $\Psi_{N^{\prime}}$ are the wave functions of the initial and final nuclei, carrying collective isospin and indices $f=f_{1} \ldots f_{n}$ and $s=s_{1} \ldots s_{n}$, as well as a dependence on the momenta $\vec{p}_{1}, \ldots, \vec{p}_{n}$. The index $i$ runs over the $n$ nucleons in $N$. Since the isospin matrix $\tau_{+}^{(i)}$ has non-zero matrix element only if nucleon $i$ is in a neutron state in $N$ and in a proton state in $N^{\prime}$, the sum over $i$ (together with the antisymmetrisation of the wave functions) covers all possible ways in which the transition takes place through the beta decay of a neutron of $N$ into a proton of $N^{\prime}$. Transitions for which $\langle 1\rangle \neq 0$ and $\langle\vec{\sigma}\rangle=0$ are called Fermi transitions, those for which $\langle 1\rangle=0$ and $\langle\vec{\sigma}\rangle \neq 0$ are called Gamow-Teller transitions, and those for which both terms are nonzero are called mixed transitions. Denoting

$$
\begin{equation*}
X^{\mu} \equiv\left(\left\langle N^{\prime}\right| V^{0}|N\rangle,-\left\langle N^{\prime}\right| \vec{A}|N\rangle\right) \tag{3.140}
\end{equation*}
$$

we have for the decay amplitude squared

$$
\begin{equation*}
\left|\mathcal{M}_{\mathrm{f}}\right|^{2} \propto X^{\mu} X^{\nu *}\left(k_{(e) \mu} k_{(\nu) \nu}+k_{(e) \nu} k_{(\nu) \mu}-\eta_{\mu \nu} k_{(e)} \cdot k_{(\nu)}-i \epsilon_{\mu \alpha \nu \beta} k_{(\nu)}^{\alpha} k_{(e)}^{\beta}\right) \tag{3.141}
\end{equation*}
$$

where we have summed over the electron spin. If we are interested in the correlation between the electron and neutrino trajectories, we need to keep only terms that are either independent of the spatial momenta or that depend on both of them, while terms depending only on one of the momenta drop out after integration over the overall orientation of the final products. We
find

$$
\begin{align*}
\left.\left|\mathcal{M}_{\mathrm{f}}\right|^{2}\right|_{\text {relevant }} \propto & X^{0} X^{* 0} 2 E_{e} E_{\nu}+X^{i} X^{j *}\left(\vec{k}_{e i} \vec{k}_{\nu j}+\vec{k}_{\nu i} \vec{k}_{e j}\right)-X \cdot X^{*}\left(E_{e} E_{\nu}-\vec{k}_{e} \cdot \vec{k}_{\nu}\right) \\
& -i\left(X^{0} X^{j *}-X^{j} X^{0 *}\right) \varepsilon_{0 j k l} \vec{k}_{\nu}^{k} \vec{k}_{e}^{l} \\
= & {\left[2\left(g_{V}|\langle 1\rangle|\right)^{2}-\left(g_{V}|\langle 1\rangle|\right)^{2}+\left(g_{A}|\langle\vec{\sigma}\rangle|\right)^{2}\right] E_{e} E_{\nu}+\left(X^{i} X^{j *}+X^{j} X^{i *}\right) \vec{k}_{e i} \vec{k}_{\nu j} } \\
& +X \cdot X^{*} \vec{k}_{e} \cdot \vec{k}_{\nu}+i\left(X^{0} \vec{X}^{*}-\vec{X} X^{0 *}\right) \cdot \vec{k}_{\nu} \wedge \vec{k}_{e} \\
= & {\left[\kappa_{V}^{2}+\kappa_{A}^{2}\right] E_{e} E_{\nu}+\left(X^{i} X^{j *}+X^{j} X^{i *}\right) \vec{k}_{e i} \vec{k}_{\nu j}+X \cdot X^{*} \vec{k}_{e} \cdot \vec{k}_{\nu} } \\
& +i\left(X^{0} \vec{X}^{*}-\vec{X} X^{0 *}\right) \cdot \vec{k}_{\nu} \wedge \vec{k}_{e}, \tag{3.142}
\end{align*}
$$

where

$$
\begin{equation*}
\kappa_{V}^{2}=\left(g_{V}|\langle 1\rangle|\right)^{2}, \quad \kappa_{A}^{2}=\left(g_{A}|\langle\vec{\sigma}\rangle|\right)^{2} . \tag{3.143}
\end{equation*}
$$

Furthermore, we are generally not interested in the spin of the final nucleus, and we do not know the polarisation of the initial nucleus, so we sum over the final spin and average over the initial one. Up to a constant factor this amounts to average over all the spins, and since the result must be invariant under spin rotations it has to boil down to the replacement

$$
\begin{equation*}
X^{i} X^{* j} \underset{\text { average }}{\longrightarrow} \frac{1}{3} \delta^{i j} \vec{X} \cdot \vec{X}^{*} . \tag{3.144}
\end{equation*}
$$

We then find

$$
\begin{align*}
\left.\left|\mathcal{M}_{\mathrm{f}}\right|^{2}\right|_{\text {relevant }} & \propto\left(\kappa_{V}^{2}+\kappa_{A}^{2}\right) E_{e} E_{\nu}+\left(\kappa_{V}^{2}-\frac{1}{3} \kappa_{A}^{2}\right) \vec{k}_{e} \cdot \vec{k}_{\nu} \\
& =\left(\kappa_{V}^{2}+\kappa_{A}^{2}\right) E_{e} E_{\nu}\left(1+\frac{\kappa_{V}^{2}-\frac{1}{3} \kappa_{A}^{2}}{\kappa_{V}^{2}+\kappa_{A}^{2}} \beta_{e} \cos \theta\right) \propto 1-\xi \beta_{e} \cos \theta \tag{3.145}
\end{align*}
$$

For a Fermi transition the electron-neutrino correlation coefficient is $\xi=-1$, while for a GamowTeller transition it is $\xi=\frac{1}{3}$. For the decay of a free neutron, where both the vector and axial current contribute, we have instead

$$
\begin{equation*}
\xi=\frac{\alpha^{2}-1}{3 \alpha^{2}+1} \simeq 0.1 \tag{3.146}
\end{equation*}
$$

### 3.4 Hyperon decays

We conclude this section discussing the beta decay of the hyperons $\Sigma^{ \pm}$,

$$
\begin{equation*}
\Sigma^{+} \rightarrow \Lambda e^{+} \nu_{e}, \quad \Sigma^{-} \rightarrow \Lambda e^{-} \bar{\nu}_{e} . \tag{3.147}
\end{equation*}
$$

Since $m_{\Sigma^{+}}=1.1894 \mathrm{GeV}, m_{\Sigma^{-}}=1.1974 \mathrm{GeV}$, and $m_{\Lambda}=1.1157 \mathrm{GeV}$, we have that $\Delta_{+}=$ $m_{\Sigma^{+}}-m_{\Lambda}=73.7 \mathrm{MeV}$ and $\Delta_{-}=m_{\Sigma^{-}}-m_{\Lambda}=81.7 \mathrm{MeV}$, so decay into muons is forbidden. The quark content of the hyperons is $\Sigma^{+}=(u u s), \Sigma^{-}=(d d s)$, and $\Lambda=(u d s)$, so strangeness is conserved in these processes. The relevant currents are then $\bar{d} \mathcal{O}_{L}^{\mu} u$ and its conjugate $\bar{u} \mathcal{O}_{L}^{\mu} d$, respectively. We can write the relevant matrix elements as ( $2 M=m_{\Lambda}+m_{\Sigma^{+}}$)

$$
\begin{align*}
& \langle\Lambda| V_{-}^{\mu}\left|\Sigma^{+}\right\rangle=\langle\Lambda| \bar{d} \gamma^{\mu} u\left|\Sigma^{+}\right\rangle=\bar{u}_{\Lambda}\left(f_{1} \gamma^{\mu}+i \frac{f_{2}}{2 M} \sigma^{\mu \nu} q_{\nu}+f_{3} \frac{q^{\mu}}{2 M}\right) u_{\Sigma^{+}}  \tag{3.148}\\
& \langle\Lambda| A_{-}^{\mu}\left|\Sigma^{+}\right\rangle=\langle\Lambda| \bar{d} \gamma^{\mu} \gamma^{5} u\left|\Sigma^{+}\right\rangle=\bar{u}_{\Lambda}\left(g_{1} \gamma^{\mu}+i \frac{g_{2}}{2 M} \sigma^{\mu \nu} q_{\nu}+g_{3} \frac{q^{\mu}}{2 M}\right) \gamma^{5} u_{\Sigma^{+}}
\end{align*}
$$

and similarly for the decay of $\Sigma^{-}$. Due to the small mass difference we can take the form factors at $q^{2}=0$, with corrections of order $q^{2}=\mathcal{O}\left(\Delta_{+}^{2}\right)$. In the isospin limit, $\left.f_{1}(0)\right|_{\text {iso }}=0$ since $\Sigma^{ \pm}$and $\Lambda$ belong to different isomultiplets, so $f_{1}(0)$ is of the order of the isospin breaking parameter, $\mathcal{O}\left(m_{\Sigma^{+}}-m_{\Sigma^{-}}\right)=\mathcal{O}\left(\Delta_{+}-\Delta_{-}\right)$, after symmetry-breaking effects are included. Current conservation in the isospin limit also implies

$$
\begin{equation*}
0=\bar{u}_{\Lambda}\left(\left.f_{1}\left(q^{2}\right)\right|_{\text {iso }} q+\left.f_{3}\left(q^{2}\right)\right|_{\text {iso }} \frac{q^{2}}{2 M}\right) u_{\Sigma^{+}}=\bar{u}_{\Lambda}\left(\left.f_{1}\left(q^{2}\right)\right|_{\text {iso }} \Delta_{+}+\left.f_{3}\left(q^{2}\right)\right|_{\text {iso }} \frac{q^{2}}{2 M}\right) u_{\Sigma^{+}} \tag{3.149}
\end{equation*}
$$

and for $\left.f_{1}\left(q^{2}\right)\right|_{\text {iso }} / q^{2}=\mathcal{O}(1)$ near zero we find that $\left.f_{3}(0)\right|_{\text {iso }} /(2 M)=\mathcal{O}\left(\Delta_{+}\right)$, hence $f_{3}(0) /(2 M)=$ $\Delta_{+} \mathcal{O}\left(\Delta_{+}-\Delta_{-}\right)$. Finally, in the isospin limit the form factor $f_{2}$ can be related to the corresponding term in the electromagnetic decay $\Sigma^{0} \rightarrow \Lambda \gamma$, and is some number of order one, so that the weak magnetism contribution is of order $\Delta_{+}$. All in all, the vector current is suppressed with respect to the axial current, as long as $g_{1}(0) \neq 0$, so it is the latter that dominates the decay. In the isospin limit the relevant matrix element has the same value for $\Sigma^{+}$and $\Sigma^{-}$, so the ratio of the two decay widths is determined uniquely by the available phase space. Since this is proportional to $\Delta_{ \pm}^{5}$, as one can see by adapting the result obtained for the neutron, we find

$$
\begin{equation*}
\frac{\Gamma\left(\Sigma^{+} \rightarrow \Lambda e^{+} \nu_{e}\right)}{\Gamma\left(\Sigma^{-} \rightarrow \Lambda e^{-} \bar{\nu}_{e}\right)}=\left(\frac{\Delta_{+}}{\Delta_{-}}\right)^{5} \simeq 0.6 \tag{3.150}
\end{equation*}
$$

This agrees with experiments.

### 3.5 Appendix: evaluation of $K(\varepsilon)$

We want to compute the integral defined in Eq. (3.83), reported here for convenience,

$$
\begin{equation*}
K(\varepsilon)=\frac{1}{2} \int_{\varepsilon}^{1} d z \sqrt{1-z}\left[1-z+\frac{3}{2} \varepsilon-\varepsilon^{2}\left(3+\frac{\varepsilon}{2}\right) \frac{1}{z^{2}}+2 \varepsilon^{3} \frac{1}{z^{3}}\right] . \tag{3.151}
\end{equation*}
$$

Changing integration variable to $z \rightarrow 1-z$ this reads

$$
\begin{equation*}
K(\varepsilon)=\frac{1}{2} \int_{0}^{1-\varepsilon} d z \sqrt{z}\left[z+\frac{3}{2} \varepsilon-\varepsilon^{2}\left(3+\frac{\varepsilon}{2}\right) \frac{1}{(1-z)^{2}}+2 \varepsilon^{3} \frac{1}{(1-z)^{3}}\right] \tag{3.152}
\end{equation*}
$$

Integration of the first two terms is straightforward. For the other two terms it is convenient to write

$$
\begin{align*}
\frac{1}{2} \int_{0}^{1-\varepsilon} d z \sqrt{z} \frac{1}{(1-z)^{n}} & =\int_{0}^{\sqrt{1-\varepsilon}} d x x^{2} \frac{1}{\left(1-x^{2}\right)^{n}} \\
& =\int_{0}^{\sqrt{1-\varepsilon}} d x\left[\frac{1}{\left(1-x^{2}\right)^{n}}-\frac{1}{\left(1-x^{2}\right)^{n-1}}\right] \tag{3.153}
\end{align*}
$$

We only need $n=2,3$. For $n>0$ these integrals can be treated by elementary methods by replacing

$$
\begin{equation*}
\frac{1}{\left(1-x^{2}\right)^{n}}=\frac{1}{(1-x)^{n}(1+x)^{n}}=\sum_{j=1}^{n} A_{j}\left(\frac{1}{(1-x)^{j}}+\frac{1}{(1+x)^{j}}\right) \tag{3.154}
\end{equation*}
$$

and finding the appropriate $A_{j}$ (the same coefficient appears for both pole terms due to the $x \rightarrow-x$ symmetry). We have

$$
\begin{align*}
\frac{1}{\left(1-x^{2}\right)} & =A_{1}\left(\frac{1}{(1-x)}+\frac{1}{(1+x)}\right)=\frac{2 A_{1}}{\left(1-x^{2}\right)}, \\
\frac{1}{\left(1-x^{2}\right)^{2}} & =\frac{2 A_{1}}{\left(1-x^{2}\right)}+\frac{2 A_{2}\left(1+x^{2}\right)}{\left(1-x^{2}\right)^{2}}=\frac{2 A_{1}\left(1-x^{2}\right)+2 A_{2}\left(1+x^{2}\right)}{\left(1-x^{2}\right)^{2}} \\
\frac{1}{\left(1-x^{2}\right)^{3}} & =\frac{2 A_{1}}{\left(1-x^{2}\right)}+\frac{2 A_{2}\left(1+x^{2}\right)}{\left(1-x^{2}\right)^{2}}+\frac{2 A_{3}\left(1+3 x^{2}\right)}{\left(1-x^{2}\right)^{3}}  \tag{3.155}\\
& =\frac{2 A_{1}\left(1-2 x^{2}+x^{4}\right)+2 A_{2}\left(1-x^{4}\right)+2 A_{3}\left(1+3 x^{2}\right)}{\left(1-x^{2}\right)^{3}}
\end{align*}
$$

which are solved by

$$
\begin{array}{ll}
n=1: & A_{1}=\frac{1}{2}, \\
n=2: & A_{1}=A_{2}=\frac{1}{4},  \tag{3.156}\\
n=3: & A_{1}=A_{2}=\frac{3}{16}, \quad A_{3}=\frac{1}{8} .
\end{array}
$$

One can then integrate Eq. (3.154) to get

$$
\begin{equation*}
\int d x \frac{1}{\left(1-x^{2}\right)^{n}}=A_{1} \log \frac{1+x}{1-x}+\sum_{j=2}^{n} \frac{A_{j}}{j-1}\left(\frac{1}{(1-x)^{j-1}}-\frac{1}{(1+x)^{j-1}}\right) \tag{3.157}
\end{equation*}
$$

and in the cases of interest

$$
\begin{align*}
\int d x \frac{1}{\left(1-x^{2}\right)} & =\frac{1}{2} \log \frac{1+x}{1-x}, \\
\int d x \frac{1}{\left(1-x^{2}\right)^{2}} & =\frac{1}{4} \log \frac{1+x}{1-x}+\frac{1}{4}\left(\frac{1}{(1-x)}-\frac{1}{(1+x)}\right)=\frac{1}{4} \log \frac{1+x}{1-x}+\frac{1}{2} \frac{x}{\left(1-x^{2}\right)}, \\
\int d x \frac{1}{\left(1-x^{2}\right)^{3}} & =\frac{3}{16} \log \frac{1+x}{1-x}+\frac{3}{16}\left(\frac{1}{(1-x)}-\frac{1}{(1+x)}\right)+\frac{1}{16}\left(\frac{1}{(1-x)^{2}}-\frac{1}{(1+x)^{2}}\right) \\
& =\frac{3}{16} \log \frac{1+x}{1-x}+\frac{3}{8} \frac{x}{\left(1-x^{2}\right)}+\frac{1}{4} \frac{x}{\left(1-x^{2}\right)^{2}} . \tag{3.158}
\end{align*}
$$

We then have

$$
\begin{align*}
\frac{1}{2} \int_{0}^{1-\varepsilon} d z \sqrt{z} \frac{1}{(1-z)^{2}} & =\left[\frac{1}{2} \frac{x}{\left(1-x^{2}\right)}-\frac{1}{4} \log \frac{1+x}{1-x}\right]_{0}^{\sqrt{1-\varepsilon}}, \\
& =\frac{1}{2} \frac{\sqrt{1-\varepsilon}}{\varepsilon}-\frac{1}{2} \log \frac{1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}, \\
\frac{1}{2} \int_{0}^{1-\varepsilon} d z \sqrt{z} \frac{1}{(1-z)^{3}} & =\left[\frac{1}{4} \frac{x}{\left(1-x^{2}\right)^{2}}-\frac{1}{8} \frac{x}{\left(1-x^{2}\right)}-\frac{1}{16} \log \frac{1+x}{1-x}\right]_{0}^{\sqrt{1-\varepsilon}}  \tag{3.159}\\
& =\frac{1}{4} \frac{\sqrt{1-\varepsilon}}{\varepsilon^{2}}-\frac{1}{8} \frac{\sqrt{1-\varepsilon}}{\varepsilon}-\frac{1}{8} \log \frac{1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}
\end{align*}
$$

We can then combine these two contributions to get

$$
\begin{align*}
& \frac{1}{2} \int_{0}^{1-\varepsilon} d z \sqrt{z}\left[-\varepsilon^{2}\left(3+\frac{\varepsilon}{2}\right) \frac{1}{(1-z)^{2}}+2 \varepsilon^{3} \frac{1}{(1-z)^{3}}\right] \\
& =\left[-\varepsilon^{2}\left(3+\frac{\varepsilon}{2}\right)\left(\frac{1}{2} \frac{\sqrt{1-\varepsilon}}{\varepsilon}-\frac{1}{2} L\right)+2 \varepsilon^{3}\left(\frac{1}{4} \frac{\sqrt{1-\varepsilon}}{\varepsilon^{2}}-\frac{1}{8} \frac{\sqrt{1-\varepsilon}}{\varepsilon}-\frac{1}{8} L\right)\right]  \tag{3.160}\\
& =\frac{3 \varepsilon^{2}}{2} L-\frac{1}{2} \sqrt{1-\varepsilon}\left(2 \varepsilon+\varepsilon^{2}\right)
\end{align*}
$$

where $L=\log \frac{1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}$. The other two integrals are

$$
\begin{align*}
\frac{1}{2} \int_{0}^{1-\varepsilon} d z \sqrt{z} z & =\frac{1}{5}(1-\varepsilon)^{\frac{5}{2}}  \tag{3.161}\\
\frac{1}{2} \int_{0}^{1-\varepsilon} d z \sqrt{z} & =\frac{1}{2} \varepsilon(1-\varepsilon)^{\frac{3}{2}}
\end{align*}
$$

Putting everything together we find

$$
\begin{align*}
K(\varepsilon) & =\frac{1}{5}(1-\varepsilon)^{\frac{5}{2}}+\frac{1}{2} \varepsilon(1-\varepsilon)^{\frac{3}{2}}+\frac{3 \varepsilon^{2}}{2} L-\frac{1}{2} \sqrt{1-\varepsilon}\left(2 \varepsilon+\varepsilon^{2}\right) \\
& =\frac{1}{2} \sqrt{1-\varepsilon}\left\{\frac{2}{5}(1-\varepsilon)^{2}+\varepsilon(1-\varepsilon)-\left(2 \varepsilon+\varepsilon^{2}\right)\right\}+\frac{3 \varepsilon^{2}}{2} L \\
& =\frac{1}{2} \sqrt{1-\varepsilon}\left\{\frac{2}{5}-\frac{9}{5} \varepsilon-\frac{8}{5} \varepsilon^{2}\right\}+\frac{3 \varepsilon^{2}}{2} L  \tag{3.162}\\
& =\frac{1}{5}\left\{\sqrt{1-\varepsilon}\left[1-\frac{9}{2} \varepsilon-4 \varepsilon^{2}\right]+\frac{15 \varepsilon^{2}}{2} \log \frac{1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\right\} .
\end{align*}
$$

We can finally look for the lowest-order approximation in $\varepsilon$,

$$
\begin{equation*}
K(\varepsilon) \simeq \frac{1}{5}(1-5 \varepsilon) \tag{3.163}
\end{equation*}
$$

which agrees with Eq. (3.85).

### 3.6 Appendix: form factors

The matrix element of interest are of the form $\left\langle H^{\prime}\left(p^{\prime}, s^{\prime}\right)\right| V^{\mu}|H(p, s)\rangle,\left\langle H^{\prime}\left(p^{\prime}, s^{\prime}\right)\right| A^{\mu}|H(p, s)\rangle$, for momentum and spin component eigenstates of some spin- $\frac{1}{2}$ fermions $H$ and $H^{\prime}$, and for vector or axial-vector operators $V^{\mu}$ and $A^{\mu}$. In general, the matrix elements of any observable $\mathcal{O}$ can be written as

$$
\begin{equation*}
\left\langle H^{\prime}\left(p^{\prime}, s^{\prime}\right)\right| \mathcal{O}|H(p, s)\rangle=\bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) \Gamma_{\mathcal{O}}\left(p, p^{\prime}\right) u_{s}(\vec{p}) . \tag{3.164}
\end{equation*}
$$

To see this, notice that the left-hand side is in general a $2 \times 2$ matrix labelled by $s, s^{\prime}= \pm \frac{1}{2}$, and recall that

$$
\begin{equation*}
u_{s}(\vec{p})=\sqrt{p^{0}+m}\binom{\chi_{s}}{\frac{\vec{p} \cdot \overrightarrow{\vec{c}}}{p^{0}+m} \chi_{s}}, \tag{3.165}
\end{equation*}
$$

so writing $\Gamma_{\mathcal{O}}$ in block form

$$
\Gamma_{\mathcal{O}}\left(p, p^{\prime}\right)=\left(\begin{array}{ll}
M_{1}\left(p, p^{\prime}\right) & M_{2}\left(p, p^{\prime}\right)  \tag{3.166}\\
M_{3}\left(p, p^{\prime}\right) & M_{4}\left(p, p^{\prime}\right)
\end{array}\right)
$$

one has

$$
\begin{align*}
& \bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) \Gamma_{\mathcal{O}}\left(p, p^{\prime}\right) u_{s}(\vec{p}) \\
& =\sqrt{\left(p^{\prime 0}+m^{\prime}\right)\left(p^{0}+m\right)} \chi_{s^{\prime}}^{\dagger}\left(M_{1}\left(p, p^{\prime}\right)+M_{2}\left(p, p^{\prime}\right) \frac{\vec{p} \cdot \vec{\sigma}}{p^{0}+m}\right.  \tag{3.167}\\
& \\
& \\
& \left.\quad-\frac{\vec{p}^{\prime} \cdot \vec{\sigma}}{p^{0}+m^{\prime}} M_{3}\left(p, p^{\prime}\right)-\frac{\overrightarrow{p^{\prime}} \cdot \vec{\sigma}}{p^{00}+m^{\prime}} M_{4}\left(p, p^{\prime}\right) \frac{\vec{p} \cdot \vec{\sigma}}{p^{0}+m}\right) \chi_{s},
\end{align*}
$$

and any $2 \times 2$ matrix can be written this way (in fact this is redundant, as it suffices to choose $M_{1}$ appropriately and set $M_{2,3,4}$ to zero: only four complex parameters are needed). The form Eq. (3.164) is convenient in order to parameterise matrix elements with simple tranformation properties under Lorentz transformations. For $\mathcal{O}=V^{\mu}$ we have

$$
\begin{equation*}
\left\langle H^{\prime}\left(p^{\prime}, s^{\prime}\right)\right| V^{\mu}(0)|H(p, s)\rangle=\bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) \Gamma^{\mu}\left(p, p^{\prime}\right) u_{s}(\vec{p}), \tag{3.168}
\end{equation*}
$$

and Lorentz invariance dictates

$$
\begin{align*}
& \left\langle H^{\prime}\left(p^{\prime}, s^{\prime}\right)\right| U(\Lambda)^{\dagger} V^{\mu}(0) U(\Lambda)|H(p, s)\rangle \\
& =\Lambda_{\nu}^{\mu}\left\langle H^{\prime}\left(p^{\prime}, s^{\prime}\right)\right| V^{\nu}(0)|H(p, s)\rangle=\Lambda_{\nu}^{\mu} \bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) \Gamma^{\nu}\left(p, p^{\prime}\right) u_{s}(\vec{p}) \\
& =\sum_{\bar{s}, \bar{s}^{\prime}}\left\langle H^{\prime}\left(\Lambda p^{\prime}, \bar{s}^{\prime}\right)\right| V^{\mu}(0)|H(\Lambda p, \bar{s})\rangle \mathscr{D}_{\bar{s} s}^{\left(\frac{1}{2}\right)}(W(\Lambda)) \mathscr{D}_{\bar{s}^{\prime} s^{\prime}}^{\left(\frac{1}{2}\right)}(W(\Lambda))^{*}  \tag{3.169}\\
& =\bar{u}_{\bar{s}^{\prime}}\left(\Lambda \vec{p}^{\prime}\right) \Gamma^{\mu}\left(\Lambda p, \Lambda p^{\prime}\right) u_{\bar{s}}(\Lambda \vec{p}) \mathscr{D}_{\bar{s} s}^{\left(\frac{1}{2}\right)}(W(\Lambda)) \mathscr{D}_{\bar{s}^{\prime} s^{\prime}}^{\left(\frac{1}{2}\right)}(W(\Lambda))^{*} \\
& =\bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) S(\Lambda)^{-1} \Gamma^{\mu}\left(\Lambda p, \Lambda p^{\prime}\right) S(\Lambda) u_{s}(\vec{p})
\end{align*}
$$

The matrix $\Gamma^{\mu}\left(p, p^{\prime}\right)$ can always be written as a linear combination of the sixteen $4 \times 4$ matrices $\Gamma^{A}=1, i \gamma^{5}, \gamma^{\mu}, \gamma^{5} \gamma^{\mu}, \sigma^{\mu \nu}$, discussed above in Section 2.4, and the objects $\bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) \Gamma^{A} u_{s}(\vec{p})$ have simple transformation properties under Lorentz transformations. Still working in full generality,

$$
\begin{align*}
\bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) \Gamma^{\mu}\left(p, p^{\prime}\right) u_{s}(\vec{p})=\bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) & \left(c_{1}^{\mu}\left(p, p^{\prime}\right) \mathbf{1}++c_{2}^{\mu}\left(p, p^{\prime}\right) i \gamma^{5}+c_{3 \nu}^{\mu}\left(p, p^{\prime}\right) \gamma^{\nu}\right. \\
& \left.+c_{4 \nu}^{\mu}\left(p, p^{\prime}\right) \gamma^{5} \gamma^{\nu}+c_{5 \nu \rho}^{\mu}\left(p, p^{\prime}\right) \sigma^{\nu \rho}\right) u_{s}(\vec{p}) \tag{3.170}
\end{align*}
$$

The various terms correspond to a scalar, a pseudoscalar, a vector, an axial-vector, and a rank-2 tensor. The Lorentz transformation property Eq. 3.169, the availability only of two independent vectors that we take to be $P^{\mu}=p^{\mu}+p^{\prime}$ and $q^{\mu}=p^{\mu}-p^{\prime}$, and of a single independent Lorentz scalar (besides the particle masses), that we take to be $q^{2}$, and of no axial vector and no pseudoscalar then dictates

$$
\begin{align*}
c_{1}^{\mu}\left(p, p^{\prime}\right) & =c_{1,1}\left(q^{2}\right) P^{\mu}+c_{1,2}\left(q^{2}\right) q^{\mu}, \\
c_{2}^{\mu}\left(p, p^{\prime}\right) & =0, \\
c_{3 \nu}^{\mu}\left(p, p^{\prime}\right) & =c_{3,1}^{\mu}\left(q^{2}\right) \delta^{\mu}{ }_{\nu}+c_{3,2}\left(q^{2}\right) P^{\mu} P_{\nu}+c_{3,3}\left(q^{2}\right) P^{\mu} q_{\nu}+c_{3,4}\left(q^{2}\right) q^{\mu} P_{\nu}+c_{3,5}\left(q^{2}\right) q^{\mu} q_{\nu},  \tag{3.171}\\
c_{4 \nu}^{\mu}\left(p, p^{\prime}\right) & =0, \\
c_{5 \nu \rho}^{\mu}\left(p, p^{\prime}\right) & =\left(c_{5,1}\left(q^{2}\right) P^{\mu}+c_{5,2}\left(q^{2}\right) q^{\mu}\right)\left(P_{\nu} q_{\rho}-P_{\rho} q_{\nu}\right)+\left(c_{5,3}\left(q^{2}\right) P_{[\rho}+c_{5,4}\left(q^{2}\right) q_{[\rho}\right) \delta^{\mu}{ }_{\nu]} .
\end{align*}
$$

Moreover, since

$$
\begin{align*}
p_{\nu} \bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) \gamma^{\nu} u_{s}(\vec{p}) & =m \bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) u_{s}(\vec{p}), \\
p_{\nu}^{\prime} \bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) \gamma^{\nu} u_{s}(\vec{p}) & =m^{\prime} \bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) u_{s}(\vec{p}), \\
P_{\nu} q_{\rho}-P_{\rho} q_{\nu} & =\left(p_{\nu}+p_{\nu}^{\prime}\right)\left(p_{\rho}-p_{\rho}^{\prime}\right)-\left(p_{\rho}+p_{\rho}^{\prime}\right)\left(p_{\nu}-p_{\nu}^{\prime}\right) \\
& =2\left(p_{\nu}^{\prime} p_{\rho}-p_{\rho}^{\prime} p_{\nu}\right), \\
\left(P_{\nu} q_{\rho}-P_{\rho} q_{\nu}\right) \bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) \sigma^{\nu \rho} u_{s}(\vec{p}) & =-2 i\left(p_{\nu}^{\prime} p_{\rho}-p_{\rho}^{\prime} p_{\nu}\right) \bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) \gamma^{\nu} \gamma^{\rho} u_{s}(\vec{p}) \\
& =-2 i\left[p_{\nu}^{\prime} p_{\rho} \bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) \gamma^{\nu} \gamma^{\rho} u_{s}(\vec{p})-p_{\rho}^{\prime} p_{\nu} \bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right)\left(2 \eta^{\nu \rho}-\gamma^{\rho} \gamma^{\nu}\right) u_{s}(\vec{p})\right] \\
& =-4 i\left[m m^{\prime} \bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) u_{s}(\vec{p})-p \cdot p^{p^{\prime}} \bar{u}_{s^{\prime}}\left(\vec{p}^{\prime}\right) u_{s}(\vec{p})\right] \tag{3.172}
\end{align*}
$$

the contributions coming from $c_{3,2}, c_{3,3}$, and $c_{5,1}$ are of the same form as those coming from $c_{1,1}$, and the contributions from $c_{3,4}, c_{3,5}$, and $c_{5,2}$ are of the same form as those coming from $c_{1,2}$. This leaves three independent structures to parameterise the matrix elements. A similar argument can be made for an axial vector operator.

Instead of proceeding as above, we first simplify the argument for the vector operator. From here on we drop spin indices for simplicity. The matrix element on the left-hand side of Eq. (3.170) is a vector that must be built using only the vectors $P^{\mu}, q^{\mu}$, and the structures $\bar{u}\left(p^{\prime}\right) \Gamma^{A} u(p)$. The are then only five vector structures,

$$
\begin{equation*}
P^{\mu} \bar{u}\left(p^{\prime}\right) u(p), \quad q^{\mu} \bar{u}\left(p^{\prime}\right) u(p), \quad \bar{u}\left(p^{\prime}\right) \gamma^{\mu} u(p), \quad \bar{u}\left(p^{\prime}\right) \sigma^{\mu \nu} u(p) P_{\nu}, \quad \bar{u}\left(p^{\prime}\right) \sigma^{\mu \nu} u(p) q_{\nu} \tag{3.173}
\end{equation*}
$$

As we showed above, these can be reduced to three independent structures. The simplest way to do this is to use the generalised version of the Gordon identity. Since

$$
\begin{equation*}
\bar{u}\left(p^{\prime}\right) \gamma^{\alpha} \gamma^{\beta} u(p)=\bar{u}\left(p^{\prime}\right) \eta^{\alpha \beta} u(p)+\bar{u}\left(p^{\prime}\right) i \sigma^{\alpha \beta} u(p), \tag{3.174}
\end{equation*}
$$

contracting with $p$ and $p^{\prime}$ and using the Dirac equation we find

$$
\begin{align*}
\bar{u}\left(p^{\prime}\right) \gamma^{\alpha} \not p u(p) & =m \bar{u}\left(p^{\prime}\right) \gamma^{\alpha} u(p)=p^{\alpha} \bar{u}\left(p^{\prime}\right) u(p)+\bar{u}\left(p^{\prime}\right) i \sigma^{\alpha \beta} p_{\beta} u(p), \\
\bar{u}\left(p^{\prime}\right) \not p^{\prime} \gamma^{\beta} u(p) & =m^{\prime} \bar{u}\left(p^{\prime}\right) \gamma^{\beta} u(p)=p^{\prime \beta} \bar{u}\left(p^{\prime}\right) u(p)+\bar{u}\left(p^{\prime}\right) i \sigma^{\alpha \beta} p_{\alpha}^{\prime} u(p) . \tag{3.175}
\end{align*}
$$

After relabelling $\beta \rightarrow \alpha$ in the second equation, we find by adding and subtracting the two equations that

$$
\begin{align*}
& \left(m+m^{\prime}\right) \bar{u}\left(p^{\prime}\right) \gamma^{\alpha} u(p)=P^{\alpha} \bar{u}\left(p^{\prime}\right) u(p)+\bar{u}\left(p^{\prime}\right) i \sigma^{\alpha \beta} q_{\beta} u(p), \\
& \left(m-m^{\prime}\right) \bar{u}\left(p^{\prime}\right) \gamma^{\alpha} u(p)=q^{\alpha} \bar{u}\left(p^{\prime}\right) u(p)+\bar{u}\left(p^{\prime}\right) i \sigma^{\alpha \beta} P_{\beta} u(p), \tag{3.176}
\end{align*}
$$

which allow us to express the first and fourth terms in Eq. (3.173) as linear combinations of the other three. In passing, we get in the case $m=m^{\prime}$ the Gordon identity, which reads

$$
\begin{equation*}
\bar{u}\left(p^{\prime}\right) \gamma^{\alpha} u(p)=\frac{P^{\alpha}}{2 m} \bar{u}\left(p^{\prime}\right) u(p)+\bar{u}\left(p^{\prime}\right) i \sigma^{\alpha \beta} \frac{q_{\beta}}{2 m} u(p) . \tag{3.177}
\end{equation*}
$$

In the axial vector case we have other five structures,

$$
\begin{equation*}
P^{\mu} \bar{u}\left(p^{\prime}\right) \gamma^{5} u(p), \quad q^{\mu} \bar{u}\left(p^{\prime}\right) \gamma^{5} u(p), \quad \bar{u}\left(p^{\prime}\right) \gamma^{\mu} \gamma^{5} u(p), \quad \bar{u}\left(p^{\prime}\right) \sigma^{\mu \nu} \gamma^{5} u(p) P_{\nu}, \quad \bar{u}\left(p^{\prime}\right) \sigma^{\mu \nu} \gamma^{5} u(p) q_{\nu} \tag{3.178}
\end{equation*}
$$

but also in this case Gordon-type identities allow us to drop two of them. Indeed,

$$
\begin{equation*}
\bar{u}\left(p^{\prime}\right) \gamma^{\alpha} \gamma^{\beta} \gamma^{5} u(p)=\bar{u}\left(p^{\prime}\right) \eta^{\alpha \beta} \gamma^{5} u(p)+\bar{u}\left(p^{\prime}\right) i \sigma^{\alpha \beta} \gamma^{5} u(p), \tag{3.179}
\end{equation*}
$$

and proceeding as above we find

$$
\begin{align*}
\bar{u}\left(p^{\prime}\right) \gamma^{\alpha} \not p \gamma^{5} u(p) & =-m \bar{u}\left(p^{\prime}\right) \gamma^{\alpha} u(p)=p^{\alpha} \bar{u}\left(p^{\prime}\right) \gamma^{5} u(p)+\bar{u}\left(p^{\prime}\right) i \sigma^{\alpha \beta} \gamma^{5} p_{\beta} u(p) \\
\bar{u}\left(p^{\prime}\right) p^{\prime} \gamma^{\beta} \gamma^{5} u(p) & =m^{\prime} \bar{u}\left(p^{\prime}\right) \gamma^{\beta} u(p)=p^{\prime \beta} \bar{u}\left(p^{\prime}\right) \gamma^{5} u(p)+\bar{u}\left(p^{\prime}\right) i \sigma^{\alpha \beta} \gamma^{5} p_{\alpha}^{\prime} u(p) \tag{3.180}
\end{align*}
$$

Combining the two equations we find

$$
\begin{align*}
\left(m^{\prime}-m\right) \bar{u}\left(p^{\prime}\right) \gamma^{\alpha} u(p) & =P^{\alpha} \bar{u}\left(p^{\prime}\right) \gamma^{5} u(p)+\bar{u}\left(p^{\prime}\right) i \sigma^{\alpha \beta} \gamma^{5} q_{\beta} u(p) \\
\left(m^{\prime}+m^{\prime}\right) \bar{u}\left(p^{\prime}\right) \gamma^{\alpha} \gamma^{5} u(p) & =-q^{\alpha} \bar{u}\left(p^{\prime}\right) \gamma^{5} u(p)-\bar{u}\left(p^{\prime}\right) i \sigma^{\alpha \beta} \gamma^{5} P_{\beta} u(p), \tag{3.181}
\end{align*}
$$

which allows us to express, e.g., the first and fourth terms in Eq. (3.178) in terms of the others. One ends up with the three terms shown in Eq. (3.92).

Reality of the form factors in Eq. (3.92) follows from the $T$ invariance of the matrix elements (in the approximation in which the $t$ and $b$ quarks are neglected):

$$
\begin{align*}
& \langle T p| V_{+}^{\mu}|T n\rangle^{*}=\langle p| T^{\dagger} V_{+}^{\mu} T|n\rangle=\mathcal{P}^{\mu}{ }_{\nu}\langle p| V_{+}^{\nu}|n\rangle, \\
& \langle T p| A_{+}^{\mu}|T n\rangle^{*}=\langle p| T^{\dagger} A_{+}^{\mu} T|n\rangle=\mathcal{P}^{\mu}{ }_{\nu}\langle p| A_{+}^{\nu}|n\rangle . \tag{3.182}
\end{align*}
$$

Since $T\left|\vec{p}, s_{z}\right\rangle=\zeta(-1)^{s+s_{z}}\left|-\vec{p},-s_{z}\right\rangle$ for some phase $\zeta$, which is the same for proton and neutron, and using $i \gamma^{0} \gamma^{2} \gamma^{5} u_{s}(\vec{p})^{*}=(-1)^{\frac{1}{2}+s} u_{-s}(-\vec{p})$, we find

$$
\begin{align*}
\langle T p| V_{+}^{\mu}|T n\rangle^{*} & =\left\{(-1)^{\frac{1}{2}+s_{p}} \bar{u}_{p}\left(\mathcal{P} p_{p},-s_{p}\right) M^{\mu}(\mathcal{P} q) u_{n}\left(\mathcal{P} p_{n},-s_{n}\right)(-1)^{\frac{1}{2}+s_{n}}\right\}^{*} \\
& =-\bar{u}_{p}\left(p_{p}, s_{p}\right)\left[\gamma^{5} \gamma^{2} \gamma^{0} M^{\mu}(\mathcal{P} q) \gamma^{0} \gamma^{2} \gamma^{5}\right]^{*} u_{n}\left(p_{n}, s_{n}\right), \\
\langle T p| A_{+}^{\mu}|T n\rangle^{*} & =\left\{(-1)^{\frac{1}{2}+s_{p}} \bar{u}_{p}\left(\mathcal{P} p_{p},-s_{p}\right) M_{5}^{\mu}(\mathcal{P} q) u_{n}\left(\mathcal{P} p_{n},-s_{n}\right)(-1)^{\frac{1}{2}+s_{n}}\right\}^{*}  \tag{3.183}\\
& =-\bar{u}_{p}\left(p_{p}, s_{p}\right)\left[\gamma^{5} \gamma^{2} \gamma^{0} M_{5}^{\mu}(\mathcal{P} q) \gamma^{0} \gamma^{2} \gamma^{5}\right]^{*} u_{n}\left(p_{n}, s_{n}\right) .
\end{align*}
$$

It is now straightforward to show that

$$
\begin{align*}
\gamma^{5} \gamma^{2} \gamma^{0} \gamma^{0} \gamma^{2} \gamma^{5} & =-1, \\
\gamma^{5} \gamma^{2} \gamma^{0} \gamma^{\mu} \gamma^{0} \gamma^{2} \gamma^{5} & =-\gamma^{2} \gamma^{\mu \dagger} \gamma^{2}=-\mathcal{P}^{\mu}{ }_{\nu} \gamma^{\nu *}, \\
\gamma^{5} \gamma^{2} \gamma^{0} \sigma^{\mu \nu} \gamma^{0} \gamma^{2} \gamma^{5} & =\gamma^{2} \sigma^{\mu \nu \dagger} \gamma^{2}=\mathcal{P}_{\alpha}^{\mu} \mathcal{P}^{\nu}{ }_{\beta} \sigma^{\mu \nu *}, \\
\gamma^{5} \gamma^{2} \gamma^{0} \gamma^{5} \gamma^{0} \gamma^{2} \gamma^{5} & =-\gamma^{5},  \tag{3.184}\\
\gamma^{5} \gamma^{2} \gamma^{0} \gamma^{\mu} \gamma^{5} \gamma^{0} \gamma^{2} \gamma^{5} & =-\gamma^{2} \gamma^{\mu \dagger} \gamma^{2} \gamma^{5}=-\mathcal{P}^{\mu}{ }_{\nu} \gamma^{\nu *} \gamma^{5}, \\
\gamma^{5} \gamma^{2} \gamma^{0} \sigma^{\mu \nu} \gamma^{5} \gamma^{0} \gamma^{2} \gamma^{5} & =\gamma^{2} \sigma^{\mu \nu \dagger} \gamma^{2} \gamma^{5}=\mathcal{P}^{\mu}{ }_{\alpha}^{\nu}{ }_{\beta} \sigma^{\mu \nu *} \gamma^{5},
\end{align*}
$$

from which it follows that (recall $\gamma^{5}=\gamma^{5 *}$ )

$$
\begin{align*}
{\left[\gamma^{5} \gamma^{2} \gamma^{0} M^{\mu}(\mathcal{P} q) \gamma^{0} \gamma^{2} \gamma^{5}\right]^{*} } & =\mathcal{P}^{\mu}{ }_{\nu}\left(f_{1}\left(q^{2}\right)^{*} \gamma^{\nu}-f_{2}\left(q^{2}\right)^{*}(-i) \sigma^{\nu \alpha} \mathcal{P}_{\alpha}^{\rho} \mathcal{P}_{\rho}{ }^{\beta} \frac{q_{\beta}}{2 m}+f_{3}\left(q^{2}\right)^{*} \frac{q^{\nu}}{2 m}\right) \\
& =\mathcal{P}^{\mu}{ }_{\nu}\left(f_{1}\left(q^{2}\right)^{*} \gamma^{\nu}+f_{2}\left(q^{2}\right)^{*} \sigma^{\nu \rho} \frac{q_{\rho}}{2 m}+f_{3}\left(q^{2}\right)^{*} \frac{q^{\nu}}{2 m}\right), \\
{\left[\gamma^{5} \gamma^{2} \gamma^{0} M_{5}^{\mu}(\mathcal{P} q) \gamma^{0} \gamma^{2} \gamma^{5}\right]^{*} } & =\mathcal{P}^{\mu}{ }_{\nu}\left(g_{1}\left(q^{2}\right)^{*} \gamma^{\nu}-g_{2}\left(q^{2}\right)^{*}(-i) \sigma^{\nu \alpha} \mathcal{P}_{\alpha}^{\rho} \mathcal{P}_{\rho}{ }^{\beta} \frac{q_{\beta}}{2 m}+g_{3}\left(q^{2}\right)^{*} \frac{q^{\nu}}{2 m}\right) \gamma^{5} \\
& =\mathcal{P}^{\mu}{ }_{\nu}\left(g_{1}\left(q^{2}\right)^{*} \gamma^{\nu}+g_{2}\left(q^{2}\right)^{*} \sigma^{\nu \rho} \frac{q_{\rho}}{2 m}+g_{3}\left(q^{2}\right)^{*} \frac{q^{\nu}}{2 m}\right) \gamma^{5} . \tag{3.185}
\end{align*}
$$

These must equal $\mathcal{P}^{\mu}{ }_{\nu} M^{\nu}(q)$ and $\mathcal{P}^{\mu}{ }_{\nu} M_{5}^{\nu}(q)$, respectively, which is only possible if the $f_{i}$ and $g_{i}$ are real.

### 3.7 Appendix: G-parity

In the isospin limit it is convenient to make use of the following symmetry transformation called $G$-parity,

$$
\begin{equation*}
G=C e^{i \pi I_{2}} \tag{3.186}
\end{equation*}
$$

where $C$ is charge conjugation and $e^{i \pi I_{2}}$ an isospin rotation. It can be shown that

$$
\begin{equation*}
e^{i \pi I_{2}}\left|I i_{3}\right\rangle=(-1)^{I+i_{3}}\left|I-i_{3}\right\rangle . \tag{3.187}
\end{equation*}
$$

The way we are going to use this symmetry is however unrelated to the transformation properties of the states, but rather to those of the quantum fields associated to the particles. It is a general result (a particular case of the LSZ reduction formula) that matrix elements of operators, e.g., a current $J^{\mu}$, between a proton and a neutron state can be expressed as

$$
\begin{align*}
&\left\langle p\left(p^{\prime}, s^{\prime}\right)\right| J^{\mu}(z)|n(p, s)\rangle \\
&=(-i)^{2} \int d^{4} y \int d^{4} x e^{i\left(p^{\prime} \cdot y-p \cdot x\right)} \bar{u}\left(p^{\prime}, s^{\prime}\right)\left(i \not \ddot{y}_{y}-m\right) S^{\mu}(x, y)\left(i \overleftarrow{\not D}_{x}-m\right) u(p, s) \\
&=\bar{u}\left(p^{\prime}, s^{\prime}\right) \Gamma^{\mu}\left(p^{\prime}, p\right) u(p, s),  \tag{3.188}\\
& S^{\mu}(y, x)=\langle 0| T\left\{\psi_{p}(y) \bar{\psi}_{n}(x) J^{\mu}(z)\right\}|0\rangle, \\
& \Gamma^{\mu}\left(p^{\prime}, p\right)=\lim _{p, p^{\prime} \rightarrow \text { on-shell }}-\left(\not p^{\prime}-m\right)\left\{\int d^{4} y \int d^{4} x e^{i\left(p^{\prime} \cdot y-p \cdot x\right)} S^{\mu}(y, x)\right\}(\not p-m),
\end{align*}
$$

where the limit " $p, p^{\prime} \rightarrow$ on-shell" implies that we start from off-shell momenta $p, p^{\prime}$, compute the Fourier transform of $S^{\mu}(y, x)$ for off-shell momenta, then multiply by the inverse propagators and finally take $p, p^{\prime}$ on their mass shell. Here $\psi_{p}$ and $\psi_{n}$ are local, spin- $\frac{1}{2}$ fields which annihilate a proton and a neutron, respectively, normalised as

$$
\begin{equation*}
\langle 0| \psi_{p}(x)|p(p, s)\rangle=u(p, s) e^{-i p \cdot x}, \quad\langle 0| \psi_{n}(x)|n(p, s)\rangle=u(p, s) e^{-i p \cdot x} \tag{3.189}
\end{equation*}
$$

Of course, the form of the matrix element on the third line of Eq. (3.188) is dictated by Lorentz invariance, but in this way we have related $\Gamma^{\mu}$ directly to the vacuum expectation value of fields.

We can now discuss the consequences of $G$-parity invariance. Let us assume that $J^{\mu}$ has the simple transformation property $G^{\dagger} J^{\mu} G=\eta J^{\mu}$ with $\eta$ some phase. Under charge conjugation a spin- $\frac{1}{2}$ field transforms as

$$
\begin{equation*}
C^{\dagger} \psi_{\alpha}(x) C=\xi\left(\bar{\psi} i \gamma^{0} \gamma^{2}\right)_{\alpha}, \quad C^{\dagger} \bar{\psi}_{\alpha}(x) C=\xi^{*}\left(i \gamma^{0} \gamma^{2} \psi\right)_{\alpha} \tag{3.190}
\end{equation*}
$$

Concerning the isospin rotation, $p$ and $n$ are the $i_{3}=+\frac{1}{2}$ and $i_{3}=-\frac{1}{2}$ components of an isospin doublet, and so

$$
\begin{equation*}
e^{-i \pi I_{2}} \psi_{p}(x) e^{i \pi I_{2}}=-\psi_{n}(x), \quad e^{-i \pi I_{2}} \psi_{n}(x) e^{i \pi I_{2}}=\psi_{p}(x) \tag{3.191}
\end{equation*}
$$

Putting everything together we find

$$
\begin{align*}
S_{\alpha \beta}^{\mu}(y, x) & =\langle 0| T\left\{\psi_{p \alpha}(y) \bar{\psi}_{n \beta}(x) J^{\mu}(z)\right\}|0\rangle \\
& =-\eta\langle 0| T\left\{C^{\dagger} \psi_{n \alpha}(y) C C^{\dagger} \bar{\psi}_{p \beta}(x) C J^{\mu}(z)\right\}|0\rangle  \tag{3.192}\\
& =\eta\langle 0| T\left\{\left(i \gamma^{0} \gamma^{2} \psi_{p}\right)_{\beta}(x)\left(\bar{\psi}_{n} i \gamma^{0} \gamma^{2}\right)_{\alpha}(y) J^{\mu}(z)\right\}|0\rangle \\
& =\eta\left(i \gamma^{0} \gamma^{2}\langle 0| T\left\{\psi_{p}(x) \bar{\psi}_{n}(y) J^{\mu}(z)\right\}|0\rangle i \gamma^{0} \gamma^{2}\right)_{\beta \alpha},
\end{align*}
$$

or in a more manageable form

$$
\begin{equation*}
S^{\mu}(y, x)=\eta\left(i \gamma^{0} \gamma^{2} S^{\mu}(x, y) \eta i \gamma^{0} \gamma^{2}\right)^{T}=-\eta \gamma^{0} \gamma^{2}\left(S^{\mu}(x, y)\right)^{T} \gamma^{0} \gamma^{2} . \tag{3.193}
\end{equation*}
$$

Here and below we need the following identities,

$$
\begin{equation*}
\gamma^{\mu T}=-\gamma^{0} \gamma^{2} \gamma^{\mu} \gamma^{0} \gamma^{2}, \quad \sigma^{\mu \nu T}=-\gamma^{0} \gamma^{2} \sigma^{\mu \nu} \gamma^{0} \gamma^{2}, \quad\left(\gamma^{0} \gamma^{2}\right)^{2}=\mathbf{1} \tag{3.194}
\end{equation*}
$$

Plugging Eq. (3.193) into the expression for $\Gamma^{\mu}$ we find (the on-shell limit is understood)

$$
\begin{align*}
-\Gamma^{\mu}\left(p^{\prime}, p\right) & =-\eta\left(\not p^{\prime}-m\right) \gamma^{0} \gamma^{2}\left\{\int d^{4} y \int d^{4} x e^{i\left(p^{\prime} \cdot y-p \cdot x\right)} S^{\mu}(x, y)\right\}^{T} \gamma^{0} \gamma^{2}(\not p-m) \\
& =-\eta \gamma^{0} \gamma^{2}\left(-\not p^{\prime}-m\right)\left\{\int d^{4} y \int d^{4} x e^{i\left(-p \cdot y+p^{\prime} \cdot x\right)} S^{\mu}(y, x)\right\}^{T}\left(-\not p^{T}-m\right) \gamma^{0} \gamma^{2}  \tag{3.195}\\
& =-\eta \gamma^{0} \gamma^{2}\left\{(-\not p-m) \int d^{4} y \int d^{4} x e^{i\left(-p \cdot y+p^{\prime} \cdot x\right)} S^{\mu}(y, x)\left(-\not p^{\prime}-m\right)\right\}^{T} \gamma^{0} \gamma^{2} \\
& =\eta \gamma^{0} \gamma^{2} \Gamma^{\mu}\left(-p,-p^{\prime}\right)^{T} \gamma^{0} \gamma^{2},
\end{align*}
$$

i.e.,

$$
\begin{equation*}
\Gamma^{\mu}\left(p^{\prime}, p\right)=-\eta \gamma^{0} \gamma^{2} \Gamma^{\mu}\left(-p,-p^{\prime}\right)^{T} \gamma^{0} \gamma^{2} \tag{3.196}
\end{equation*}
$$

The currents we are interested in are the following vector and axial currents,

$$
\begin{equation*}
V^{\mu}=\bar{u} \gamma^{\mu} d, \quad A^{\mu}=\bar{u} \gamma^{\mu} \gamma^{5} d, \tag{3.197}
\end{equation*}
$$

which under $G$ transform as follows

$$
\begin{equation*}
G^{\dagger} V^{\mu} G=V^{\mu}, \quad G^{\dagger} A^{\mu} G=-A^{\mu} \tag{3.198}
\end{equation*}
$$

In fact (recall that $u$ and $d$ behave like $p$ and $n$ as far as isospin is concerned),

$$
\begin{align*}
G^{\dagger} V^{\mu} G & =C^{\dagger}(\bar{d}) \gamma^{\mu}(-u) C=-\left(i \gamma^{0} \gamma^{2} d\right)_{\alpha} \gamma_{\alpha \beta}^{\mu}\left(\bar{u} i \gamma^{0} \gamma^{1}\right)_{\beta}=\bar{u} i \gamma^{0} \gamma^{1} \gamma^{\mu T} i \gamma^{0} \gamma^{2} d \\
& =\bar{u} \gamma^{0} \gamma^{1} \gamma^{0} \gamma^{2} \gamma^{\mu} \gamma^{0} \gamma^{2} \gamma^{0} \gamma^{2} d=\bar{u} \gamma^{\mu} d=V^{\mu},  \tag{3.199}\\
G^{\dagger} A^{\mu} G & =C^{\dagger}(\bar{d}) \gamma^{\mu} \gamma^{5}(-u) C=-\left(i \gamma^{0} \gamma^{2} d\right)_{\alpha}\left(\gamma^{\mu} \gamma^{5}\right)_{\alpha \beta}\left(\bar{u} i \gamma^{0} \gamma^{1}\right)_{\beta}=\bar{u} i \gamma^{0} \gamma^{1}\left(\gamma^{\mu} \gamma^{5}\right)^{T} i \gamma^{0} \gamma^{2} d \\
& =\bar{u} \gamma^{0} \gamma^{1} \gamma^{5} \gamma^{0} \gamma^{2} \gamma^{\mu} \gamma^{0} \gamma^{2} \gamma^{0} \gamma^{2} d=-\bar{u} \gamma^{\mu} \gamma^{5} d=-A^{\mu} .
\end{align*}
$$

The associated matrices $\Gamma_{V}^{\mu}\left(p^{\prime}, p\right)$ and $\Gamma_{A}^{\mu}\left(p^{\prime}, p\right)$ must therefore satisfy

$$
\begin{equation*}
\Gamma_{V}^{\mu}\left(p^{\prime}, p\right)=-\gamma^{0} \gamma^{2} \Gamma_{V}^{\mu}\left(-p,-p^{\prime}\right)^{T} \gamma^{0} \gamma^{2}, \quad \Gamma_{A}^{\mu}\left(p^{\prime}, p\right)=\gamma^{0} \gamma^{2} \Gamma_{A}^{\mu}\left(-p,-p^{\prime}\right)^{T} \gamma^{0} \gamma^{2} . \tag{3.200}
\end{equation*}
$$

The general form of these matrices has been given above, and each consists of three linearly independent terms proportional to $\gamma^{\mu}, \sigma^{\mu \nu} q_{\nu}, \mathbf{1} q^{\mu}$, for the vector current, and $\gamma^{\mu} \gamma^{5}, \sigma^{\mu \nu} \gamma^{5} q_{\nu}, \gamma^{5} q^{\mu}$, for the axial current, where $q=p-p^{\prime}$. Since $q \rightarrow q$ under $p \leftrightarrow-p^{\prime}$ it now suffices to verify that

$$
\begin{align*}
& -\gamma^{0} \gamma^{2} \gamma^{\mu T} \gamma^{0} \gamma^{2}=\gamma^{\mu}, \quad-\gamma^{0} \gamma^{2} \sigma^{\mu \nu T} \gamma^{0} \gamma^{2}=\sigma^{\mu \nu}, \quad-\gamma^{0} \gamma^{2} \mathbf{1} \gamma^{0} \gamma^{2}=-\mathbf{1}, \\
& \gamma^{0} \gamma^{2}\left(\gamma^{\mu} \gamma^{5}\right)^{T} \gamma^{0} \gamma^{2}=\gamma^{\mu} \gamma^{5}, \quad \gamma^{0} \gamma^{2} \sigma^{\mu \nu T} \gamma^{0} \gamma^{2}=-\sigma^{\mu \nu} \gamma^{5}, \quad \gamma^{0} \gamma^{2} \gamma^{5} \gamma^{0} \gamma^{2}=\gamma^{5}, \tag{3.201}
\end{align*}
$$

to show that only $\gamma^{\mu}$ and $\sigma^{\mu \nu} q_{\nu}$ are admissible for the vector current, and $\gamma^{\mu} \gamma^{5}$ and $\gamma^{5} q^{\mu}$ for the axial current.

### 3.8 Appendix: pion pole in nucleon matrix elements

We show here a particular case of a general result: vacuum expectation values or matrix elements of products of (appropriate) fields have poles when combinations of external momenta approach the mass shell of a physical particles. Consider the correlation function

$$
\begin{align*}
\mathscr{G}_{a}\left(p^{\prime}, p, q\right) & =\int d^{4} x \int d^{4} y \int d^{4} z e^{i\left(p^{\prime} x \cdot x-p \cdot y-q \cdot z\right)}\langle 0| T\left\{\psi_{p}(x) \bar{\psi}_{n}(y) \phi_{a}(z)\right\}|0\rangle \\
& =(2 \pi)^{4} \delta^{(4)}\left(p^{\prime}-p-q\right) \int d^{4} x \int d^{4} y e^{i\left(p^{\prime} x \cdot x-p \cdot y\right)}\langle 0| T\left\{\psi_{p}(x) \bar{\psi}_{n}(y) \phi_{a}(0)\right\}|0\rangle  \tag{3.202}\\
& =(2 \pi)^{4} \delta^{(4)}\left(p^{\prime}-p-q\right) g_{a}\left(p^{\prime}, p, q\right),
\end{align*}
$$

where the proton and neutron fields $\psi_{p}$ and $\psi_{n}$ are normalised as in Eq. 3.189, while for the pion field $\phi_{a}(x)$

$$
\begin{equation*}
\langle 0| \phi_{a}(x)\left|\pi_{b}\right\rangle=\delta_{a b} e^{-i p \cdot x} \tag{3.203}
\end{equation*}
$$

The LSZ reduction formula relates $\mathscr{G}$ and the proton-neutron-pion vertex, e.g.,

$$
\begin{equation*}
{ }_{\text {out }}\left\langle p \mid n \pi^{+}\right\rangle_{\text {in }}=\bar{u}\left(p^{\prime}, s^{\prime}\right) \frac{\not p^{\prime}-m}{i} \frac{1}{\sqrt{2}} \mathscr{G}_{+}\left(p^{\prime}, p, q\right) \frac{\not p-m}{i} u(p, s) \frac{q^{2}-m_{\pi}^{2}}{i} \tag{3.204}
\end{equation*}
$$

It is understood here that momenta are off the mass-shell, so the left-hand side is actually the analytic continuation of the $S$-matrix element for $n \pi^{+} \rightarrow p$ (which is obviously zero for onshell momenta). With this proviso, we can extract the scattering amplitude for $n \pi^{+} \rightarrow p$ from Eq. (3.204) as

$$
\begin{equation*}
i \mathcal{M}_{n \pi^{+} \rightarrow p}=\bar{u}\left(p^{\prime}, s^{\prime}\right) \frac{\not p^{\prime}-m}{i} \frac{1}{\sqrt{2}} g_{+}\left(p^{\prime}, p, q\right) \frac{\not p-m}{i} u(p, s) \frac{q^{2}-m_{\pi}^{2}}{i} \tag{3.205}
\end{equation*}
$$

Similarly, using the last line of Eq. 3.202) one finds by means of LSZ reduction

$$
\begin{equation*}
{ }_{\text {out }}\langle p| \phi_{a}(0)|n\rangle_{\text {in }}=\bar{u}\left(p^{\prime}, s^{\prime}\right) \frac{\not p p^{\prime}-m}{i} g_{a}\left(p^{\prime}, p, q=p^{\prime}-p\right) \frac{\not p-m}{i} u(p, s) \tag{3.206}
\end{equation*}
$$

Let us now isolate, among the various time orderings in Eq. 3.202, those in which the pion field corresponds to the earliest time,

$$
\begin{align*}
\mathscr{G}_{a}\left(p^{\prime}, p, q\right)= & \int d^{4} x \int d^{4} y \int d^{4} z e^{i\left(p^{\prime} x \cdot x-p \cdot y-q \cdot z\right)}\langle 0| T\left\{\psi_{p}(x) \bar{\psi}_{n}(y)\right\} \phi_{a}(z)|0\rangle \theta\left(\min \left(x^{0}, y^{0}\right)-z^{0}\right) \\
& +\mathrm{OT} \tag{3.207}
\end{align*}
$$

where OT indicates "other terms". We now insert a complete set of states and focus on pion states, writing (no sum over $a$ )

$$
\begin{array}{r}
\mathscr{G}_{a}\left(p^{\prime}, p, q\right)=\int d^{4} x \int d^{4} y \int d^{4} z \int d \Omega_{k} e^{i\left(p^{\prime} x \cdot x-p \cdot y-q \cdot z\right)}\langle 0| T\left\{\psi_{p}(x) \bar{\psi}_{n}(y)\right\}\left|\pi_{a}(\vec{k})\right\rangle  \tag{3.208}\\
\times\left\langle\pi_{a}(\vec{k})\right| \phi_{a}(z)|0\rangle \theta\left(\min \left(x^{0}, y^{0}\right)-z^{0}\right)+\mathrm{OT}
\end{array}
$$

Here it is crucial that the fields $\phi_{a}$ have nonzero matrix elements between the vacuum and the pion states. Translation invariance implies

$$
\begin{align*}
\langle 0| T\left\{\psi_{p}(x) \bar{\psi}_{n}(y)\right\}\left|\pi_{a}(\vec{k})\right\rangle & =\langle 0| T\left\{\psi_{p}(0) \bar{\psi}_{n}(y-x)\right\}\left|\pi_{a}(\vec{k})\right\rangle e^{-i k \cdot x}, \\
\left\langle\pi_{a}(\vec{k})\right| \phi_{a}(z)|0\rangle & =e^{i k \cdot z}\left\langle\pi_{a}(\vec{k})\right| \phi_{a}(0)|0\rangle, \tag{3.209}
\end{align*}
$$

and of course $\min \left(x^{0}, y^{0}\right)=x^{0}+\min \left(0, y^{0}-x^{0}\right)$. Changing variables to $y^{\prime}=y-x$ and $z^{\prime}=x-z$ (and dropping the prime) we find

$$
\begin{align*}
\mathscr{G}_{a}\left(p^{\prime}, p, q\right)= & \int d^{4} x \int d^{4} y \int d^{4} z \int d \Omega_{k} e^{i\left[\left(p^{\prime}-p-q\right) \cdot x-p \cdot y+(q-k) \cdot z\right]} \\
& \times\langle 0| T\left\{\psi_{p}(0) \bar{\psi}_{n}(y)\right\}\left|\pi_{a}(\vec{k})\right\rangle\left\langle\pi_{a}(\vec{k})\right| \phi_{a}(0)|0\rangle \theta\left(z^{0}+\min \left(0, y^{0}\right)\right)+\mathrm{OT} \\
= & (2 \pi)^{4} \delta^{(4)}\left(p^{\prime}-p-q\right) \int d^{4} y \int d^{4} z \int d \Omega_{k} e^{i(-p \cdot y+(q-k) \cdot z)}  \tag{3.210}\\
& \times\langle 0| T\left\{\psi_{p}(0) \bar{\psi}_{n}(y)\right\}\left|\pi_{a}(\vec{k})\right\rangle \theta\left(z^{0}+\min \left(0, y^{0}\right)\right)+\mathrm{OT},
\end{align*}
$$

where in the second line we used Eq. (3.203). Dropping the momentum-conserving delta function, and making use of the Fourier transform representation of the theta function,

$$
\begin{equation*}
\theta\left(x^{0}\right)=-\int \frac{d \omega}{2 \pi i} e^{-i \omega x^{0}} \frac{1}{\omega+i \epsilon}, \tag{3.211}
\end{equation*}
$$

we obtain

$$
\begin{align*}
g_{a}\left(p^{\prime}, p, q\right)= & \int d^{4} y \int d^{4} z \int d \Omega_{k} \int \frac{d \omega}{2 \pi} e^{i(-p \cdot y+(q-k) \cdot z)} \\
& \times \frac{i}{\omega+i \epsilon} e^{-i \omega\left(z^{0}+\min \left(0, y^{0}\right)\right)}\langle 0| T\left\{\psi_{p}(0) \bar{\psi}_{n}(y)\right\}\left|\pi_{a}(\vec{k})\right\rangle+\mathrm{OT} \\
= & \int d^{4} y \int d \Omega_{k} \int \frac{d \omega}{2 \pi} e^{-i p \cdot y} \frac{i}{\omega+i \epsilon \epsilon} e^{-i \omega \min \left(0, y^{0}\right)}  \tag{3.212}\\
& \times(2 \pi)^{3} \delta^{(3)}(\vec{q}-\vec{k}) 2 \pi \delta\left(q^{0}-k^{0}-\omega\right)\langle 0| T\left\{\psi_{p}(0) \bar{\psi}_{n}(y)\right\}\left|\pi_{a}(\vec{k})\right\rangle+\mathrm{OT} \\
= & \int d^{4} y e^{-i p \cdot y} \frac{1}{2 k^{0}} \frac{i}{q^{0}-k^{0}+i \epsilon} e^{-i\left(q^{0}-k^{0}\right) \min \left(0, y^{0}\right)} \\
& \times\langle 0| T\left\{\psi_{p}(0) \bar{\psi}_{n}(y)\right\}\left|\pi_{a}(\vec{q})\right\rangle+\mathrm{OT} .
\end{align*}
$$

We now focus on the region $q^{0} \simeq k^{0}=\sqrt{\vec{q}^{2}+m_{\pi}^{2}}$, where the other terms can be dropped (since they do not have a pole there), and get

$$
\begin{align*}
g_{a}\left(p^{\prime}, p, q\right) & =d_{q^{0} \sim k^{0}}^{=} \int d^{4} y e^{-i p \cdot y} \frac{1}{2 k^{0}} \frac{i}{q^{0}-k^{0}+i \epsilon}\langle 0| T\left\{\psi_{p}(0) \bar{\psi}_{n}(y)\right\}\left|\pi_{a}(\vec{q})\right\rangle \\
& =\frac{1}{2 k^{0}} \frac{i\left(q^{0}+k^{0}\right)}{\left(q^{0}\right)^{2}-\vec{q}^{2}-m_{\pi}^{2}+i \epsilon} \int d^{4} y e^{-i p \cdot y}\langle 0| T\left\{\psi_{p}(0) \bar{\psi}_{n}(y)\right\}\left|\pi_{a}(\vec{q})\right\rangle  \tag{3.213}\\
& ==\frac{i}{q^{0} \sim k^{0}} \frac{2}{q^{2}-m_{\pi}^{2}+i \epsilon} \int d^{4} y e^{-i p \cdot y}\langle 0| T\left\{\psi_{p}(0) \bar{\psi}_{n}(y)\right\}\left|\pi_{a}(\vec{q})\right\rangle .
\end{align*}
$$

Plugging this result into Eq. (3.206) we find

$$
\begin{align*}
& \operatorname{out~}^{\langle p| \phi_{a}(0)|n\rangle_{\text {in }}} \\
& \quad=\frac{i}{{ }_{q^{0} \sim k^{0}}} \frac{i}{q^{2}-m_{\pi}^{2}+i \epsilon} \bar{u}\left(p^{\prime}, s^{\prime}\right)\left(\not p^{\prime}-m\right) \int d^{4} y e^{-i p \cdot y}\langle 0| T\left\{\psi_{p}(0) \bar{\psi}_{n}(y)\right\}\left|\pi_{a}(\vec{q})\right\rangle(\not p-m) u(p, s) . \tag{3.214}
\end{align*}
$$

We can then parameterise this matrix element, using Lorentz and isospin invariance and knowledge of the pion pole, as

$$
\begin{equation*}
{ }_{\mathrm{out}}\langle p| \phi_{a}(0)|n\rangle_{\mathrm{in}}=-i \frac{g_{\pi N N}\left(q^{2}\right)}{q^{2}-m_{\pi}^{2}} \bar{u}\left(p^{\prime}, s^{\prime}\right) \gamma^{5} \tau_{a} u(p, s), \tag{3.215}
\end{equation*}
$$

with $g_{\pi N N}$ regular at the pion mass squared. In particular,

$$
\begin{equation*}
{ }_{\mathrm{out}}\langle p| \phi_{+}(0)|n\rangle_{\mathrm{in}}=-i \frac{g_{\pi N N}\left(q^{2}\right)}{q^{2}-m_{\pi}^{2}} \bar{u}\left(p^{\prime}, s^{\prime}\right) \gamma^{5} \tau_{+} u(p, s)=2 i \frac{g_{\pi N N}\left(q^{2}\right)}{m_{\pi}^{2}-q^{2}} \bar{u}_{p}\left(p^{\prime}, s^{\prime}\right) \gamma^{5} u_{n}(p, s) . \tag{3.216}
\end{equation*}
$$

To understand the choice of prefactors, compute the left-hand side using an effective Lagrangian with the following nucleon-pion coupling,

$$
\begin{equation*}
\mathscr{L}^{\mathrm{eff}}=i g \bar{\psi} \tau_{a} \gamma^{5} \psi \phi_{a}=g\left(\bar{\psi}_{p} \tau_{3} \gamma^{5} \psi_{p}-\bar{\psi}_{n} \tau_{3} \gamma^{5} \psi_{n}\right) \phi_{3}+g \sqrt{2} \bar{\psi}_{p} \gamma^{5} \psi_{n} \frac{\phi_{-}}{\sqrt{2}}+g \sqrt{2} \bar{\psi}_{n} \gamma^{5} \psi_{p} \frac{\phi_{+}}{\sqrt{2}} . \tag{3.217}
\end{equation*}
$$

(The factor $i$ is required to make the Lagrangian $T$-invariant, since $\eta_{P} \eta_{C}=-1$ for pions. Actually: it makes the Lagrangian Hermitian.) It is straightforward to get to lowest order

$$
\begin{equation*}
{ }_{\text {out }}\langle p| \phi_{+}(0)|n\rangle_{\text {in }}=\sqrt{2} i(i g \sqrt{2}) \bar{u}_{p} \gamma^{5} u_{n} \frac{i}{q^{2}-m_{\pi}^{2}}=2 i g \bar{u}_{p} \gamma^{5} u_{n} \frac{1}{m_{\pi}^{2}-q^{2}}, \tag{3.218}
\end{equation*}
$$

so that we can identify $g=g_{\pi N N}\left(m_{\pi}^{2}\right)$ as the physical pion-nucleon-nucleon coupling.
The reason of this identification is that if one considers the $S$-matrix element for nucleonnucleon scattering, inserting a complete set of states and focussing on the unphysical region where $q^{2}=\left(p_{1}-p_{1}^{\prime}\right)^{2}$ is close to the pion mass squared, an argument similar to the one developed above shows that

$$
\begin{align*}
& \operatorname{out}\left\langle N\left(p_{1}^{\prime}\right) N\left(p_{2}^{\prime}\right) \mid N\left(p_{1}\right) N\left(p_{2}\right)\right\rangle_{\text {in }} \\
& \sim \text { const. } \times \frac{g_{\pi N N}\left(q^{2}\right)}{q^{2}-m_{\pi}^{2}} \bar{u}\left(p_{1}^{\prime}, s_{1}^{\prime}\right) \gamma^{5} \tau_{a} u\left(p_{1}, s_{1}\right) \bar{u}\left(p_{2}^{\prime}, s_{2}^{\prime}\right) \gamma^{5} \tau_{a} u\left(p_{2}, s_{2}\right)  \tag{3.219}\\
& \sim \text { const. } \times \frac{g_{\pi N N}\left(m_{\pi}^{2}\right)}{q^{2}-m_{\pi}^{2}} \bar{u}\left(p_{1}^{\prime}, s_{1}^{\prime}\right) \gamma^{5} \tau_{a} u\left(p_{1}, s_{1}\right) \bar{u}\left(p_{2}^{\prime}, s_{2}^{\prime}\right) \gamma^{5} \tau_{a} u\left(p_{2}, s_{2}\right) .
\end{align*}
$$

In the physical region $q^{2} \leq 0$, but since the pion mass is small the pole Eq. 3.219) is close to the physical region and can dominate the scattering amplitude. The resulting contribution is then precisely of the form corresponding to the tree-level exchange of a pseudoscalar isovector particle (i.e., the pion) between the colliding nucleons, with coupling constant $g_{\pi N N}\left(m_{\pi}^{2}\right)$.

## 4 Strangeness-changing leptonic decays of hadrons

Most leptonic decays of strange particles are strangeness-changing. The relevant interaction is the coupling of the hadronic currents $\bar{u} \mathcal{O}_{L}^{\alpha} s$ and $\bar{s} \mathcal{O}_{L}^{\alpha} u$ to the leptonic currents $\bar{\ell} \mathcal{O}_{L}^{\alpha} \nu_{\ell}$ and $\bar{\nu}_{\ell} \mathcal{O}_{L}^{\alpha} \ell$, respectively. The lightest hadrons undergoing strangeness-changing weak decays are strange octet mesons and baryons (as well as the decuplet $\Omega$ baryon). Kaon decays have the following special notation in the literature:

$$
\begin{array}{llll}
K_{\mu 2}^{+}: & K^{+} \rightarrow \mu^{+} \nu_{\mu}, & K_{\mu 4}^{+}: & K^{+} \rightarrow \mu^{+} \nu_{\mu} \pi^{+} \pi^{-}, \\
K_{\mu 3}^{+}: & K^{+} \rightarrow \mu^{+} \nu_{\mu} \pi^{0}, & K_{\mu 4}^{+1}: & K^{+} \rightarrow \mu^{+} \nu_{\mu} \pi^{0} \pi^{0},  \tag{4.1}\\
K_{\mu 3}^{0}: & K^{0} \rightarrow \mu^{+} \nu_{\mu} \pi^{-}, & K_{\mu 4}^{0}: & K^{0} \rightarrow \mu^{+} \nu_{\mu} \pi^{0} \pi^{-} .
\end{array}
$$

Similarly, $K_{\mu 2}^{-}, K_{\mu 3}^{-}$etc. are used to denote the corresponding decays of $K^{-}$into muons, and $K_{e 2}^{ \pm}, K_{e 3}^{ \pm}$etc. to denote processes with positrons or electrons in the final state. If the notation $K_{\ell 2}^{-}, K_{\ell 3}^{-}$is used, then the lepton type in the final state is summed over, i.e., one generically considers decays into whatever type of lepton (compatibly with mass contraints, of course). No special notation is used for hyperon decays, the most relevant of which are

$$
\begin{array}{ll}
\Lambda \rightarrow p \ell^{-} \bar{\nu}_{\ell}, & \Sigma^{-} \rightarrow n \ell^{-} \bar{\nu}_{\ell}, \\
\Xi^{-} \rightarrow \Lambda \ell^{-} \bar{\nu}_{\ell}, & \Xi^{-} \rightarrow \Sigma^{0} \ell^{-} \bar{\nu}_{\ell},  \tag{4.2}\\
\Xi^{0} \rightarrow \Sigma^{+} \ell^{-} \bar{\nu}_{\ell}, & \Omega^{-} \rightarrow \Xi^{0} \ell^{-} \bar{\nu}_{\ell} .
\end{array}
$$

Decay amplitude The general form of the decay amplitude is

$$
\begin{equation*}
\mathcal{M}_{\mathrm{fi}}=-\frac{G}{\sqrt{2}} \sin \theta_{C} H_{\alpha} L^{\alpha}, \tag{4.3}
\end{equation*}
$$

where the hadronic matrix element is

$$
\begin{equation*}
H^{\alpha}=\langle f| \bar{u} \mathcal{O}_{L}^{\alpha} s|i\rangle \quad \text { or } \quad H^{\alpha}=\langle f| \bar{s} \mathcal{O}_{L}^{\alpha} u|i\rangle \tag{4.4}
\end{equation*}
$$

and the leptonic matrix element is

$$
\begin{equation*}
L^{\alpha}=\bar{u}_{\ell} \mathcal{O}_{L}^{\alpha} v_{\nu_{\ell}}, \quad \text { or } \quad L^{\alpha}=\bar{u}_{\nu_{\ell}} \mathcal{O}_{L}^{\alpha} v_{\ell}, \tag{4.5}
\end{equation*}
$$

depending on the specific process.

Selection rules The currents $\bar{u} \mathcal{O}_{L}^{\alpha} s$ and $\bar{s} \mathcal{O}_{L}^{\alpha} u$ carry a change in strangeness $\Delta S= \pm 1$, so the selection rule $|\Delta S|=1$ follows. They also effect a change in electric charge equal to the change in strangeness, so the selection rule $\Delta Q=\Delta S$ follows. Processes that violate these selection rules are not strictly forbidden, since they could take place in higher orders of perturbation theory, but they are very strongly suppressed. Since $u$ has $I=\frac{1}{2}$ while $s$ has $I=0$, these currents belong to isodoublets, and so the selection rule $|\Delta I|=\frac{1}{2}$ follows (more on this below).

SU(3) flavour symmetry The $\mathrm{SU}(3)$ flavour symmetry allows one to relate strangeness-conserving and strangeness-changing processes. The currents $\bar{u} \mathcal{O}_{L}^{\alpha} d, \bar{d} \mathcal{O}_{L}^{\alpha} u, \bar{u} \mathcal{O}_{L}^{\alpha} s$ and $\bar{s} \mathcal{O}_{L}^{\alpha} u$ belong to the same octet of currents,

$$
\begin{equation*}
\left(j^{\alpha}\right)^{i}{ }_{j}=\bar{q}^{i} \mathcal{O}_{L}^{\alpha} q_{j}-\frac{1}{3} \sum_{m} \bar{q}^{m} \mathcal{O}_{L}^{\alpha} q_{m} \tag{4.6}
\end{equation*}
$$

where $q=(u, d, s)$. There are two independent flavour-diagonal currents, that can be taken as

$$
\begin{equation*}
\frac{1}{\sqrt{2}}\left(\bar{u} \mathcal{O}_{L}^{\alpha} u-\bar{d} \mathcal{O}_{L}^{\alpha} d\right), \quad \frac{1}{\sqrt{6}}\left(\bar{u} \mathcal{O}_{L}^{\alpha} u+\bar{d} \mathcal{O}_{L}^{\alpha} d-2 \bar{s} \mathcal{O}_{L}^{\alpha} s\right) \tag{4.7}
\end{equation*}
$$

Both are neutral currents, the first part of an isovector of strangeness-conserving currents with $\bar{u} \mathcal{O}_{L}^{\alpha} d, \bar{d} \mathcal{O}_{L}^{\alpha} u$, while the second is an isoscalar. The remaining two currents are $\bar{d} \mathcal{O}_{L}^{\alpha} s$ and $\bar{s} \mathcal{O}_{L}^{\alpha} d$, which complete the isospin doublets of the strangeness-changing currents. These are however flavour-changing neutral currents (FCNC) which do not appear in the weak Lagrangian. The different roles played by different parts of the octet should not be surprising since weak interactions are not $\mathrm{SU}(3)$ invariant. Since strong interactions approximately are, relations among decay amplitudes follow.

## 4.1 $\quad K_{\ell 2}$ decays

The processes

$$
\begin{equation*}
K^{+} \rightarrow \ell \nu_{\ell} \tag{4.8}
\end{equation*}
$$

are the analogues of the charged pion decays $\pi^{+} \rightarrow \ell \nu_{\ell}$, and would have exactly the same amplitude in the $\mathrm{SU}(3)$-symmetric limit. As in that case, the hadronic matrix elements receive contributions only from the axial current, and read

$$
\begin{equation*}
H^{\alpha}=\langle 0| \bar{s} \mathcal{O}_{L}^{\alpha} u\left|K^{+}\right\rangle=i \sqrt{2} f_{K} p^{\alpha} \tag{4.9}
\end{equation*}
$$

where $p^{\alpha}$ is the kaon momentum and $f_{K}$ is the kaon decay constant, a real quantity with dimensions of a mass, that would equal $f_{\pi}$ in the $\mathrm{SU}(3)$-symmetric limit (in which case also $m_{K}=m_{\pi}$ ). Using momentum conservation, $p=p_{\ell}+p_{\nu}$, we find

$$
\begin{align*}
\mathcal{M}_{\mathrm{fi}} & =-\frac{G}{\sqrt{2}} \sin \theta_{C} i \sqrt{2} f_{K} p^{\alpha} \bar{u}_{\nu_{\ell}} \mathcal{O}_{L}^{\alpha} v_{\ell}=-i G \sin \theta_{C} f_{K} \bar{u}_{\nu_{\ell}} \not p\left(1-\gamma^{5}\right) v_{\ell} \\
& =-i G \sin \theta_{C} f_{K} \bar{u}_{\nu_{\ell}}\left(\not p_{\nu}+\not p_{\ell}\right)\left(1-\gamma^{5}\right) v_{\ell}=-i G \sin \theta_{C} f_{K} \bar{u}_{\nu_{\ell}}\left(1+\gamma^{5}\right) \not p_{\ell} v_{\ell}  \tag{4.10}\\
& =i G \sin \theta_{C} f_{K} m_{\ell} \bar{u}_{\nu_{\ell}}\left(1+\gamma^{5}\right) v_{\ell}
\end{align*}
$$

This is entirely analogous to charged pion decay, and following the same steps one obtains for the total decay width

$$
\begin{equation*}
\Gamma=\frac{G^{2} \sin ^{2} \theta_{C} f_{K}^{2}}{8 \pi} m_{K} m_{\ell}^{2}\left(1-\frac{m_{\ell}^{2}}{m_{K}^{2}}\right)^{2} \tag{4.11}
\end{equation*}
$$

Assuming knowledge of $\sin \theta_{C} \simeq 0.21$, of the decay times and branching ratios of pion and kaon decays,

$$
\begin{align*}
\Gamma\left(\pi^{+} \rightarrow \mu^{+} \nu_{\mu}\right) & \simeq \Gamma\left(\pi^{+}\right)=\left(2.6 \cdot 10^{-8} s\right)^{-1}  \tag{4.12}\\
\Gamma\left(K^{+} \rightarrow \mu^{+} \nu_{\mu}\right) & \simeq 0.63 \Gamma\left(K^{+}\right)=0.63\left(1.2 \cdot 10^{-8} s\right)^{-1}
\end{align*}
$$

and of the particle masses $m_{\pi}=140 \mathrm{MeV}, m_{K}=497 \mathrm{MeV}$, and $m_{\mu}=106 \mathrm{MeV}$, one obtains $f_{\pi} / f_{K}=1.3$. Besides a $28 \%$ of hadronic decays, significant contributions to the charged kaon decay come from the $K_{e 3}$ and $K_{\mu 3}$ processes. Since kaons are more than 3 times as massive as pions, the ratio between $\Gamma\left(K^{+} \rightarrow e^{+} \nu_{\mu}\right)$ and $\Gamma\left(K^{+} \rightarrow \mu^{+} \nu_{\mu}\right)$ is closer to the asymptotic limit $\left(m_{e} / m_{\mu}\right)^{2}=2 \cdot 10^{-5}$ for large mass of the decaying particle.

## $4.2 \quad K_{\ell 3}$ decays

The analysis of $K^{+} \rightarrow \ell^{+} \nu_{\ell} \pi^{0}$ and $K^{0} \rightarrow \ell^{+} \nu_{\ell} \pi^{-}$decays is similar to that of the $\beta$ decay of the pion. Writing $p_{K}=p_{\pi}+p_{\ell}+p_{\nu}$ for momentum conservation, and denoting $p=p_{K}+p_{\pi}$ and $q=p_{K}-p_{\pi}$, we have for the relevant hadronic matrix elements

$$
\begin{align*}
H_{\alpha}^{(+)} & =\left\langle\pi^{0}\right| \bar{s} \mathcal{O}_{L \alpha} u\left|K^{+}\right\rangle=f_{+}^{(+)}\left(q^{2}\right) p_{\alpha}+f_{-}^{(+)}\left(q^{2}\right) q_{\alpha} \\
H_{\alpha}^{(0)} & =\left\langle\pi^{-}\right| \bar{s} \mathcal{O}_{L \alpha} u\left|K^{0}\right\rangle=f_{+}^{(0)}\left(q^{2}\right) p_{\alpha}+f_{-}^{(0)}\left(q^{2}\right) q_{\alpha} \tag{4.13}
\end{align*}
$$

which receive contributions only from the vector current. In the $C P$-symmetric case $f_{ \pm}^{(+, 0)}$ are real quantities. The decay amplitude reads

$$
\begin{equation*}
\mathcal{M}_{\mathrm{fi}}^{(+, 0)}=-\frac{G}{\sqrt{2}} \sin \theta_{c}\left[f_{+}^{(+, 0)}\left(q^{2}\right) p_{\alpha}+f_{-}^{(+, 0)}\left(q^{2}\right) q_{\alpha}\right] \bar{u}_{\nu_{\ell}} \gamma^{\alpha}\left(1-\gamma^{5}\right) v_{\ell} \tag{4.14}
\end{equation*}
$$

The $f_{-}^{(+, 0)}$ term is easily seen to be proportional to $m_{\ell}$ (just plug $q=p_{\ell}+p_{\nu}$ into the leptonic matrix element), and thus it is negligible in $K_{e 3}$ decays.

The approximation $q^{2} \simeq 0$ is here less accurate than in the $\beta$ decay of the pion, but it is still reasonable. In the $\mathrm{SU}(3)$-symmetric limit we can determine $f_{+}^{(+, 0)}(0)$ using only symmetry considerations. In fact, in this limit the relevant weak current is related to one of the generators of the $\mathrm{SU}(3)$ symmetry,

$$
\begin{equation*}
\int d^{3} x\left(\bar{s} \mathcal{O}_{L}^{0} u\right)(0, \vec{x})=V_{-} \tag{4.15}
\end{equation*}
$$

Recall from the algebra of $\mathrm{SU}(3)$ that $\left[I_{3}, V_{-}\right]=-\frac{1}{2} V_{-}$and $\left[Y, V_{-}\right]=-V_{-}$, and furthermore that $\left[I_{-}, V_{-}\right]=0$. Taking matrix elements and using translation invariance,

$$
\begin{equation*}
\left.(2 \pi)^{3} \delta^{(3)}(\vec{q})\langle f|\left(\bar{s} \mathcal{O}_{L}^{0} u\right)(0)|i\rangle=\langle f| V_{-}|i\rangle=2 p^{0}(2 \pi)^{3} \delta^{(3)}(\vec{q})\left\langle\langle f| V_{-} \mid i\right\rangle\right\rangle, \tag{4.16}
\end{equation*}
$$

where double angular brackets indicate that only the flavour part of the wave functions is involved. Using Eq. (4.13) we find

$$
\begin{equation*}
\left.\left.f_{+}^{(+)}(0)=\left\langle\left\langle\pi^{0}\right| V_{-} \mid K^{+}\right\rangle\right\rangle, \quad f_{+}^{(0)}(0)=\left\langle\left\langle\pi^{-}\right| V_{-} \mid K^{0}\right\rangle\right\rangle . \tag{4.17}
\end{equation*}
$$

A look at the meson octet shows that $\pi^{-}$and $K^{0}$ are the $V_{3}=-\frac{1}{2}$ and the $V_{3}=\frac{1}{2}$ members of a $V$-spin doublet $V=\frac{1}{2}$, and $\mathrm{sq}^{33}$

$$
\begin{equation*}
f_{+}^{(0)}(0)=\left.\sqrt{V(V+1)-V_{3}\left(V_{3}-1\right)}\right|_{V=V_{3}=\frac{1}{2}}=1 \tag{4.18}
\end{equation*}
$$

[^20]Recalling that $I_{+}\left|\pi^{-}\right\rangle=\sqrt{2}\left|\pi^{0}\right\rangle$, we can show that

$$
\begin{equation*}
\left.\left.\left.\left.\left\langle\left\langle\pi^{0}\right| V_{-} \mid K^{+}\right\rangle\right\rangle=\frac{1}{\sqrt{2}}\left\langle\left\langle\pi^{-}\right| I_{-} V_{-} \mid K^{+}\right\rangle\right\rangle=\frac{1}{\sqrt{2}}\left\langle\left\langle\pi^{-}\right| V_{-} I_{-} \mid K^{+}\right\rangle\right\rangle=\frac{1}{\sqrt{2}}\left\langle\left\langle\pi^{-}\right| V_{-} \mid K^{0}\right\rangle\right\rangle, \tag{4.19}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
f_{+}^{(+)}(0) \underset{\text { isospin limit }}{ } \frac{1}{\sqrt{2}} f_{+}^{(0)}(0) \underset{\mathrm{SU}(3)}{\overline{=} \text { limit }} \frac{1}{\sqrt{2}} . \tag{4.20}
\end{equation*}
$$

The first relation is exact in case of isospin invariance, and should therefore have the same degree of accuracy. A result known as the Ademollo-Gatto theorem guarantees that corrections to $f_{+}^{(0)}(0)$ due to $\mathrm{SU}(3)$ symmetry breaking are only quadratic in the symmetry breaking parameter $\delta_{\mathrm{SU}(3)}$ (which from the modern point of view is the strange-light quark mass difference $m_{s}$ $\frac{m_{u}+m_{d}}{2}$ ), and so $f_{+}^{(0)}(0)=1$ should be a reasonable approximation.

The proof of the Ademollo-Gatto theorem is similar to the discussion of isospin breaking effects in Eqs. (3.57)-(3.61). In fact, the proof is the same, using $V$-spin instead of $I$-spin. Starting from $\left[V_{+}, V_{-}\right]=2 V_{3}=I_{3}+\frac{3}{2} Y$, one finds

$$
\begin{equation*}
\left.\left.2\left\langle\left\langle V V_{3}\right| V_{3} \mid V V_{3}\right\rangle\right\rangle=2 V_{3}=\left\langle\left\langle V V_{3}\right|\left[V_{+}, V_{-}\right] \mid V V_{3}\right\rangle\right\rangle=\left.\sum_{n}\left|\left\langle\langle n| V_{-} \mid V V_{3}\right\rangle\right\rangle\right|^{2}-\left.\left|\left\langle\langle n| V_{+} \mid V V_{3}\right\rangle\right\rangle\right|^{2}, \tag{4.21}
\end{equation*}
$$

where the generators are the unperturbed ones in the $\mathrm{SU}(3)$ symmetric case, while the states are the physical, perturbed states including the effects of $\mathrm{SU}(3)$ breaking. Using $\left.\left.\left|K^{0}\right\rangle\right\rangle=|V, V\rangle\right\rangle$ with $V=V_{3}=\frac{1}{2}$, and separating out from the sum the $\left.\left|\pi^{-}\right\rangle\right\rangle$state, which is the only state in the octet having a nonzero $V_{-}$matrix element with $K^{0}$, we find

$$
\begin{equation*}
\left.\left.1=\left.\left|\left\langle\left\langle\pi^{-}\right| V_{-} \mid K^{0}\right\rangle\right\rangle\right|^{2}+\left.\sum_{n}^{\prime}\left|\left\langle\langle n| V_{-} \mid K^{0}\right\rangle\right\rangle\right|^{2}-\left|\langle n| V_{+}\right| K^{0}\right\rangle\right\rangle\left.\right|^{2}, \tag{4.22}
\end{equation*}
$$

and since the matrix element in the sum are already of order $\delta_{\mathrm{SU}(3)}$, and $\left.\left|\left\langle\left\langle\pi^{-}\right| V_{-} \mid K^{0}\right\rangle\right\rangle\right|^{2} \mid \mathrm{SU}(3)$ limit $=$ 1 , we find

$$
\begin{equation*}
\left.\left.\left|1+\delta\left\langle\left\langle\pi^{-}\right| V_{-} \mid K^{0}\right\rangle\right\rangle\right|^{2}-1=\operatorname{Re}\left(\delta\left\langle\left\langle\pi^{-}\right| V_{-} \mid K^{0}\right\rangle\right\rangle\right)+\mathcal{O}\left(\delta_{\mathrm{SU}(3)}^{2}\right)=\mathcal{O}\left(\delta_{\mathrm{SU}(3)}^{2}\right), \tag{4.23}
\end{equation*}
$$

where $\left.\delta\left\langle\left\langle\pi^{-}\right| V_{-} \mid K^{0}\right\rangle\right\rangle$ is the deviation from the $\operatorname{SU}(3)$-symmetric limit. Since $\left.f_{+}^{(0)}(0)=\left\langle\left\langle\pi^{-}\right| V_{-} \mid K^{0}\right\rangle\right\rangle$ is real, it follows $f_{+}^{(0)}(0)=1+\mathcal{O}\left(\delta_{\mathrm{SU}(3)}^{2}\right)$.
In the $\operatorname{SU}(3)$-symmetric limit $f_{-}^{(+, 0)}\left(q^{2}\right)=0$ due to current conservation (which extends to the whole octet of currents), but no theorem prevents corrections due to $\mathrm{SU}(3)$ breaking from being large, and it is well possible that $f_{-}^{(+, 0)}(0) / f_{+}^{(+, 0)}(0)$ be of order 1 . Using the PCAC hypothesis it is possible to derive the Callan-Treiman relation, which relates $K_{\ell 2}$ and $K_{\ell 3}$ decays:

$$
\begin{equation*}
f_{+}\left(m_{K}^{2}\right)+f_{-}\left(m_{K}^{2}\right)=f_{K} / f_{\pi} . \tag{4.24}
\end{equation*}
$$

Like the Goldberger-Treiman relation, this one also involves form factors at unphysical values of their arguments (here $m_{\ell}^{2} \leq q^{2} \leq\left(m_{K}-m_{\pi}\right)^{2}$ ), which have to be reconstructed through extrapolation from experimental data. The formula is in good agreement with the experimental results.

In the case of $K_{e 3}^{+}$it is possible to obtain a reasonably accurate theoretical prediction, since the $f_{-}^{(+)}$term can be neglected. The form factor $f_{+}^{(+)}\left(q^{2}\right)$ is usually fitted to experimental results with expressions of the form

$$
\begin{equation*}
f_{+}^{(+)}\left(q^{2}\right)=f_{+}^{(+)}(0)\left(1+\lambda_{+}^{(+)} \frac{q^{2}}{m_{\pi^{+}}^{2}}\right), \tag{4.25}
\end{equation*}
$$

having included a quadratic term. The factor $f_{+}^{(+)}(0)$ is approximately known from $\mathrm{SU}(3)$ symmetry; fit to experimental results give $\lambda_{+}^{(+)} \simeq 0.03$. Due to its lightness, the electron is ultrarelativistic and its mass can be neglected. Calculation of the total decay width gives ${ }^{34}$

$$
\begin{align*}
\Gamma & =\frac{G^{2} \sin \theta_{C}^{2}\left(f_{+}^{(+)}(0)\right)^{2}}{12 \pi^{3}} m_{K}^{5}\left(\frac{m_{\pi^{0}}}{m_{K}}\right)^{4}\left\{1.62+\lambda_{+}^{(+)} 5.988\right\}  \tag{4.26}\\
& =\frac{G^{2} \sin \theta_{C}^{2}\left(f_{+}^{(+)}(0)\right)^{2}}{768 \pi^{3}} m_{K}^{5}\left\{0.58+\lambda_{+}^{(+)} 2.1\right\}
\end{align*}
$$

Comparison with experiments allows to determine $\sin \theta_{C}$.

## $4.3 \quad K_{e 4}$ decay

We conclude this subsection with a brief remark about the four-body decay process $K^{+} \rightarrow$ $\pi^{+} \pi^{-} e^{+} \nu_{e}$. Four-momentum conservation reads $p=p_{1}+p_{2}+k_{1}+k_{2}$. In this case both terms in the current contribute. In fact, since $K^{+}$is a pseudoscalar, $\left\langle\pi^{+} \pi^{-}\right| V_{+}^{\mu}\left|K^{+}\right\rangle$is a pseudovector and $\left\langle\pi^{+} \pi^{-}\right| A_{+}^{\mu}\left|K^{+}\right\rangle$is a vector ${ }^{35}$ Since there are now three independent momenta, it is possible to build a pseudovector using the Levi-Civita tensor. One has

$$
\begin{align*}
\left\langle\pi^{+} \pi^{-}\right| A_{+}^{\mu}\left|K^{+}\right\rangle & =f_{1}\left(p_{1}+p_{2}\right)^{\mu}+f_{2}\left(p_{1}-p_{2}\right)^{\mu}+f_{3}\left(p-p_{1}-p_{2}\right)^{\mu} \\
\left\langle\pi^{+} \pi^{-}\right| V_{+}^{\mu}\left|K^{+}\right\rangle & =\frac{f_{4}}{m_{K}^{2}} \varepsilon_{\nu \rho \sigma}^{\mu} p^{\nu} p_{1}^{\rho} p_{2}^{\sigma} \tag{4.27}
\end{align*}
$$

with $f_{j}$ real functions of mass dimension -1 of the scalar quantities $p \cdot p_{1}, p \cdot p_{2}$ and $p_{1} \cdot p_{2}$. The $f_{3}$ term contribution is small: in fact, $p-p_{1}-p_{2}=k_{1}+k_{2}$, and a familiar calculation shows that this contribution is proportional to the lepton mass (while the $f_{1}$ term is of order $m_{K}$ ). The $f_{4}$ term contribution is also suppressed: in fact, it is proportional to $m_{K}\left(\vec{p}_{1} / m_{K}\right) \wedge\left(\vec{p}_{2} / m_{K}\right)$, and so it is suppressed by two powers of momentum. An estimate using PCAC yields $f_{1,2} \approx 1 / f_{\pi}$.

### 4.4 Leptonic decays of hyperons

Octet baryons undergo weak decay processes analogue to the $\beta$ decay of the neutron, of the general form $h \rightarrow h^{\prime} \ell \nu_{\ell}$, where $\ell$ and $\nu_{\ell}$ are the appropriate combination of a charged lepton and a neutrino or antineutrino of the same family, respecting charge and lepton family conservation, but possibly not strangeness.

The hadronic matrix elements relevant to such leptonic decays of octet baryons, both the strangeness-conserving and the strangeness-changing ones, are among the matrix elements of the octet of currents $\left(j^{\alpha}\right)^{i}{ }_{j}$ of Eq. 4.6). Their octet nature is made explicit by writing them as

$$
\begin{equation*}
j_{a}^{\alpha}=\bar{q} \mathcal{O}_{L}^{\alpha} t_{a} q=\bar{q} \gamma^{\alpha}\left(1-\gamma^{5}\right) t_{a} q, \tag{4.28}
\end{equation*}
$$

[^21]where $t_{a}$ are the $\mathrm{SU}(3)$ generators in the fundamental representation, obeying the usual commutation relations $\left[t_{a}, t_{b}\right]=i f_{a b c} t_{c}$ and normalisation condition $2 \operatorname{tr} t_{a} t_{b}=\delta_{a b}$. The original currents can be recovered from the explicit expressions for the $t_{a}$. The hadronic matrix elements we are interested in are of the general form
\[

$$
\begin{equation*}
\mathcal{A}_{a}^{\mu}\left(B \rightarrow B^{\prime}\right)=\left\langle B^{\prime}\right| j_{a}^{\mu}(0)|B\rangle, \tag{4.29}
\end{equation*}
$$

\]

where $|B\rangle$ and $\left|B^{\prime}\right\rangle$ are generic octet baryon states, $|B\rangle=B^{a}|a\rangle$ and similarly $\left|B^{\prime}\right\rangle=B^{\prime} a|a\rangle$, where $|a\rangle, a=1, \ldots, 8$ are a basis for octet baryon states - not necessarily the physical one. We are allowing here states that are linear combinations of octet baryons, even though these are forbidden in the real world by superselection rules, because they would be allowed in the ideal, $\mathrm{SU}(3)$-symmetric world, and because they will be useful in order to find relations between the matrix elements of interest. Any dependence on the particle spins and momenta are enconded in the vectors $|a\rangle$, while $B^{a}$ and $B^{\prime a}$ are independent of them.

To make the abstract Hilbert space spanned by the states $|a\rangle$ concrete, we use the linear space of traceless hermitian $3 \times 3$ matrices as representation space for the flavour part of the wave function, thus writing $B=B^{a} t_{a}$ instead of $|B\rangle$. With this choice the basis states $|a\rangle$ are associated with the basis vectors $t^{a}$, i.e., the group generators; physical states are obtained as suitable linear combinations of them, e.g., the proton corresponds to $\frac{1}{\sqrt{2}} t_{4+i 5}=\frac{1}{\sqrt{2}}\left(t_{4}+i t_{5}\right)$. To make this linear space a Hilbert space we need to define a positive-definite Hermitian product: we take this to be

$$
\begin{equation*}
\left(B^{\prime}, B\right) \equiv 2 \operatorname{tr} B^{\prime \dagger} B \tag{4.30}
\end{equation*}
$$

This is easily seen to satisfy the requirements of a positive-definite Hermitian bilinear form. Using linearity and the normalisation of the generators we can write

$$
\begin{equation*}
\left(B^{\prime}, B\right)=\bar{B}^{\prime a} B^{b} 2 \operatorname{tr} t_{a} t_{b}=\bar{B}^{\prime a} B^{b} \delta_{a b}, \tag{4.31}
\end{equation*}
$$

where an overbar denotes complex conjugation $\left(\bar{B}^{\prime a}=B^{\prime a *}\right)$. It is now easy to find out the general structure of the matrix elements Eq. 4.29). Under a $\mathrm{SU}(3)$ transformation, $|B\rangle \rightarrow$ $\left|B_{U}\right\rangle=\mathbf{U}|B\rangle$ (boldface type denotes here the abstract transformation operator), one has for the representative vectors/matrices $B$ that $B \rightarrow U B U^{\dagger}=B^{a} D^{(8)}(U)_{b a} t_{b}$, since they transform according to the adjoint (octet) representation. On the other hand, $\mathbf{U}^{\dagger} j_{a}^{\alpha} \mathbf{U}=D^{(8)}(U)_{a b} j_{b}^{\alpha}$ due to the octet nature of the currents, so

$$
\begin{align*}
\mathcal{A}_{a}^{\mu}\left(B_{U} \rightarrow B_{U}^{\prime}\right) & =\left\langle B_{U}^{\prime}\right| j_{a}^{\mu}(0)\left|B_{U}\right\rangle=\left\langle B^{\prime}\right| \mathbf{U}^{\dagger} j_{a}^{\mu}(0) \mathbf{U}|B\rangle=D^{(\boldsymbol{8})}(U)_{a b}\left\langle B^{\prime}\right| j_{b}^{\mu}(0)|B\rangle  \tag{4.32}\\
& =D^{(\boldsymbol{8})}(U)_{a b} \mathcal{A}_{b}^{\mu}\left(B \rightarrow B^{\prime}\right)
\end{align*}
$$

The matrix elements $\mathcal{A}_{a}^{\mu}\left(B \rightarrow B^{\prime}\right)$ are then linear (antilinear) functionals of the matrix $B$ $\left(B^{\prime}\right)$, that under $B \rightarrow U B U^{\dagger}$ and $B^{\prime} \rightarrow U B^{\prime} U^{\dagger}$ transform as an octet. In representationtheoretic terms, we have to look for octet representations in the decomposition of the matrix $\mathcal{A}_{a}^{\mu}\left(B \rightarrow B^{\prime}\right)$ in irreducible components. This matrix is by construction transforming in the $\mathbf{8} \otimes \mathbf{8}=\overline{\mathbf{8}} \otimes \mathbf{8}$ representation (notice that $\mathbf{8}=\overline{\mathbf{8}}$ is self-conjugate). A general result is that in the decomposition of the representation $\overline{\mathbf{R}} \otimes \mathbf{R}$ of $\mathrm{SU}(3)$, the octet representation can appear at most twice. This means that there are at most two independent structures with the desired transformation properties. These have to be bilinear in $B$ and $B^{\prime}$, and linear in the current index. It is easy to write down two such structures,

$$
\begin{equation*}
2 \operatorname{tr} B^{\prime \dagger} t_{a} B, \quad 2 \operatorname{tr} B t_{a} B^{\prime \dagger} \tag{4.33}
\end{equation*}
$$

where the factor of 2 is introduced for normalisation purposes; the general result mentioned above guarantees that there are no more independent structures. Instead of the objects in Eq. 4.33) it is convenient to use their symmetric and antisymmetric combinations, and write

$$
\begin{align*}
\mathcal{A}_{a}^{\mu}\left(B \rightarrow B^{\prime}\right) & =\left(D^{\mu}+F^{\mu}\right) 2 \operatorname{tr} B^{\prime \dagger} t_{a} B+\left(D^{\mu}-F^{\mu}\right) 2 \operatorname{tr} B t_{a} B^{\prime \dagger} \\
& =D^{\mu} 2 \operatorname{tr}{B^{\prime \dagger}\left\{t_{a}, B\right\}+F^{\mu} 2 \operatorname{tr} B^{\prime \dagger}\left[t_{a}, B\right]=D^{\mu} d_{a b c} B^{b} \bar{B}^{\prime c}+F^{\mu} i f_{a b c} B^{b} \bar{B}^{\prime c}}=D^{\mu} d_{a b c} \bar{B}^{\prime b} B^{c}+F^{\mu}\left(-i f_{a b c}\right) \bar{B}^{\prime b} B^{c}=D^{\mu} \bar{B}^{\prime b}\left(\tilde{T}_{a}\right)_{b c} B^{c}+F^{\mu} \bar{B}^{\prime b}\left(T_{a}\right)_{b c} B^{c} . \tag{4.34}
\end{align*}
$$

Here $\left(T_{a}\right)_{b c}=-i f_{a b c}$ are the generators in the adjoint representation, and $\left(\tilde{T}_{a}\right)_{b c}=d_{a b c}$ are a second set of eight matrices. Equation (4.34) determines entirely the flavour structure of the matrix elements up to two unknown objects $D^{\mu}$ and $F^{\mu}$, which contain both a vector and an axial vector part. These objects depend on the spin and momenta of the initial and final baryons, but not on their type (in the exact $\mathrm{SU}(3)$ approximation considered here).

Lorentz invariance further imposes that $D^{\mu}$ and $F^{\mu}$ have the general structure

$$
\begin{align*}
& D^{\mu}=D_{V}^{\mu}-D_{A}^{\mu}, \\
& D_{V}^{\mu}=\bar{u}\left(p^{\prime}, s^{\prime}\right)\left\{f_{D 1}\left(q^{2}\right) \gamma^{\mu}+i f_{D 2}\left(q^{2}\right) \frac{\sigma^{\mu \nu} q_{\nu}}{2 m}+f_{D 3}\left(q^{2}\right) \frac{q^{\mu}}{2 m}\right\} u(p, s), \\
& D_{A}^{\mu}=\bar{u}\left(p^{\prime}, s^{\prime}\right)\left\{g_{D 1}\left(q^{2}\right) \gamma^{\mu}+i g_{D 2}\left(q^{2}\right) \frac{\sigma^{\mu \nu} q_{\nu}}{2 m}+g_{D 3}\left(q^{2}\right) \frac{q^{\mu}}{2 m}\right\} \gamma^{5} u(p, s),  \tag{4.35}\\
& F^{\mu}=F_{V}^{\mu}-F_{A}^{\mu}, \\
& F_{V}^{\mu}=\bar{u}\left(p^{\prime}, s^{\prime}\right)\left\{f_{F 1}\left(q^{2}\right) \gamma^{\mu}+i f_{F 2}\left(q^{2}\right) \frac{\sigma^{\mu \nu} q_{\nu}}{2 m}+f_{F 3}\left(q^{2}\right) \frac{q^{\mu}}{2 m}\right\} u(p, s), \\
& F_{A}^{\mu}=\bar{u}\left(p^{\prime}, s^{\prime}\right)\left\{g_{F 1}\left(q^{2}\right) \gamma^{\mu}+i g_{F 2}\left(q^{2}\right) \frac{\sigma^{\mu \nu} q_{\nu}}{2 m}+g_{F 3}\left(q^{2}\right) \frac{q^{\mu}}{2 m}\right\} \gamma^{5} u(p, s),
\end{align*}
$$

where $p, s$ and $p^{\prime}, s^{\prime}$ are the momenta and spin compent of the initial and final baryon states, and $q=p-p^{\prime}$. The decays of octet baryons involve small momentum transfers due to the relatively small mass difference, which makes it possible to work in the static approximation $q=0$ for the evaluation of matrix elements. Equation (4.35) then reduces to

$$
\begin{align*}
D^{\mu} & =\bar{u}(p) \gamma^{\mu}\left(D_{V}-D_{A} \gamma^{5}\right) u(p),  \tag{4.36}\\
F^{\mu} & =\bar{u}(p) \gamma^{\mu}\left(F_{V}-F_{A} \gamma^{5}\right) u(p)
\end{align*}
$$

with $D_{V}=f_{D 1}(0), D_{A}=g_{D 1}(0), F_{V}=f_{F 1}(0)$, and $F_{A}=g_{F 1}(0)$.
We can further exploit $\mathrm{SU}(3)$ symmetry to determine $D_{V}$ and $F_{V}$ in the case when this symmetry is exact. In fact, the integral over space of the vector part of $j_{a}^{\mu}(t, \vec{x})$ is just the ( $t$-independent since conserved) abstract generator $\mathbf{T}_{a}$ of $\operatorname{SU}(3)$, and so

$$
\begin{equation*}
\int d^{3} x\left\langle B^{\prime}\right| j_{V a}^{\mu}(0, \vec{x})|B\rangle=\left\langle B^{\prime}\right| \mathbf{T}_{a}|B\rangle=\bar{B}^{\prime b} B^{c}\langle b| \mathbf{T}_{a}|c\rangle=(2 \pi)^{3} \delta(\vec{q}) 2 p^{0} \delta_{s^{\prime} s} \bar{B}^{\prime b} B^{c}\left(T_{a}\right)_{b c} \tag{4.37}
\end{equation*}
$$

having used the fact that the baryon octet transforms precisely in the adjoint representation, and the relativistic normalisation of particle states. Using translation invariance on the left-hand side we obtain

$$
\begin{align*}
(2 \pi)^{3} \delta(\vec{q})\left\langle B^{\prime}\right| j_{V a}^{\mu}(0)|B\rangle & =(2 \pi)^{3} \delta(\vec{q}) 2 p^{0} \bar{B}^{\prime b} B^{c}\left(T_{a}\right)_{b c},  \tag{4.38}\\
\left.\left\langle B^{\prime}\right| j_{V a}^{\mu}(0)|B\rangle\right|_{\vec{q}=0, s=s^{\prime}} ^{\mu} & =2 p^{0} \bar{B}^{\prime b} B^{c}\left(T_{a}\right)_{b c},
\end{align*}
$$

having matched the coefficients of the Dirac delta. Using the general structures Eqs. (4.34) and 4.35) we find

$$
\begin{align*}
\bar{B}^{\prime b} B^{c}\left[D_{V}\left(\tilde{T}_{a}\right)_{b c}+F_{V}\left(T_{a}\right)_{b c}\right] \bar{u}(p, s) \gamma^{0} u(p, s) & =\bar{B}^{\prime b} B^{c}\left[D_{V}\left(\tilde{T}_{a}\right)_{b c}+F_{V}\left(T_{a}\right)_{b c}\right] 2 p^{0} \\
& =2 p^{0} \bar{B}^{\prime b} B^{c}\left(T_{a}\right)_{b c},  \tag{4.39}\\
D_{V}\left(\tilde{T}_{a}\right)_{b c}+F_{V}\left(T_{a}\right)_{b c} & =\left(T_{a}\right)_{b c} .
\end{align*}
$$

Since $\tilde{T}_{a}$ is a symmetric while $T_{a}$ is an antisymmetric matrix, they are linearly independent and so it follows $D_{V}=0$ and $F_{V}=1$. In the exact $\operatorname{SU}(3)$, static approximation we then have

$$
\begin{equation*}
\mathcal{A}_{a}^{\mu}\left(B \rightarrow B^{\prime}\right)=\bar{u}\left(p, s^{\prime}\right) \gamma^{\mu}\left\{\left(T_{a}\right)_{b c}-\left[F_{A}\left(T_{a}\right)_{b c}+D_{A}\left(\tilde{T}_{a}\right)_{b c}\right] \gamma^{5}\right\} u(p, s) \bar{B}^{\prime b} B^{c} . \tag{4.40}
\end{equation*}
$$

In order to determine the specific form of the amplitude for the various decays of octet baryons, it is convenient to return to the original set of structures, writing Eq. 4.40) as

$$
\begin{align*}
\mathcal{A}_{a}^{\mu}\left(B \rightarrow B^{\prime}\right)=\bar{u}\left(p, s^{\prime}\right) \gamma^{\mu} & \left\{\left[\left(D_{V}+F_{V}\right) 2 \operatorname{tr} \bar{B}^{\prime} t_{a} B+\left(D_{V}-F_{V}\right) 2 \operatorname{tr} B t_{a} \bar{B}^{\prime}\right]\right. \\
& \left.-\left[\left(D_{A}+F_{A}\right) 2 \operatorname{tr} \bar{B}^{\prime} t_{a} B+\left(D_{A}-F_{A}\right) 2 \operatorname{tr} B t_{a} \bar{B}^{\prime}\right] \gamma^{5}\right\} u(p, s), \tag{4.41}
\end{align*}
$$

and compute the quantities $\operatorname{tr} \bar{B}^{\prime} t_{a} B$ and $\operatorname{tr} B t_{a} \bar{B}^{\prime}$ explicitly for the relevant currents. Here we put back $D_{V}$ and $F_{V}$ for notational symmetry, and for allowing deviations from the $\mathrm{SU}(3)$-exact case. To this end, notice that the wave function $B=B^{a} t_{a}$ of octet baryons can be written as the following matrix ${ }^{36}$

$$
B=\frac{1}{\sqrt{2}}\left(\begin{array}{ccc}
\frac{1}{\sqrt{2}} \Sigma^{0}+\frac{1}{\sqrt{6}} \Lambda & \Sigma^{+} & p  \tag{4.42}\\
\Sigma^{-} & -\frac{1}{\sqrt{2}} \Sigma^{0}+\frac{1}{\sqrt{6}} \Lambda & n \\
\Xi^{-} & -\Xi^{0} & -\sqrt{\frac{2}{3}} \Lambda
\end{array}\right)
$$

Here we have replaced the wave function components $B^{a}$ with the appropriate linear combinations $p, n, \Sigma^{0}$, to make explicit which component corresponds to which baryon. Eventually, one of them will be set to 1 and the other to 0 to get the desired decay amplitudes. The form Eq.(4.42) can be obtained by direct calculation using the explicit form of the generators, or more simply by recalling that the transformation law $B \rightarrow U B U^{\dagger}$ identifies the traceless part of the upper left $2 \times 2$ block as an isospin triplet, with hypercharge 0 ; the first two elements of the rightmost column as an isodoublet with hypercharge 1 ; the first two elements of the bottom row as a (complex-conjugate) isodoublet with hypercharge -1 ; and the trace part of the top left block and the bottom right element as an isosinglet with hypercharge 0 . The association between matrix entries and baryons then follows naturally. Suitable normalisation factors are introduced to normalise each baryonic component to $\frac{1}{2}$. The currents of interest are the strangeness-conserving current

$$
\begin{equation*}
\bar{u} \mathcal{O}_{L}^{\mu} d=j_{1}^{\mu}+i j_{2}^{\mu}=j_{1+i 2}^{\mu} \tag{4.43}
\end{equation*}
$$

and its Hermitian conjugate, and the strangeness-changing current

$$
\begin{equation*}
\bar{u} \mathcal{O}_{L}^{\mu} s=j_{4}^{\mu}+i j_{5}^{\mu}=j_{4+i 5}^{\mu} \tag{4.44}
\end{equation*}
$$

[^22]and its Hermitian conjugate. Since
\[

t_{1+i 2}=t_{1}+i t_{2}=\left($$
\begin{array}{ccc}
0 & 1 & 0  \tag{4.45}\\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
$$\right), \quad t_{4+i 5}=t_{4}+i t_{5}=\left($$
\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
$$\right),
\]

one readily finds

$$
\begin{align*}
2 \operatorname{tr} \bar{B}^{\prime} t_{1+i 2} B & =2 \sum_{n} \bar{B}_{n 1}^{\prime} B_{2 n} \\
& =\left(\frac{1}{\sqrt{2}} \bar{\Sigma}^{0}+\frac{1}{\sqrt{6}} \bar{\Lambda}\right) \Sigma^{-}+\bar{\Sigma}^{+}\left(-\frac{1}{\sqrt{2}} \Sigma^{0}+\frac{1}{\sqrt{6}} \Lambda\right)+\bar{p} n  \tag{4.46}\\
2 \operatorname{tr} B t_{1+i 2} \bar{B}^{\prime} & =2 \sum_{n} \bar{B}_{2 n}^{\prime} B_{n 1} \\
& =\bar{\Sigma}^{+}\left(\frac{1}{\sqrt{2}} \Sigma^{0}+\frac{1}{\sqrt{6}} \Lambda\right)+\left(-\frac{1}{\sqrt{2}} \bar{\Sigma}^{0}+\frac{1}{\sqrt{6}} \bar{\Lambda}\right) \Sigma^{-}-\bar{\Xi}^{0} \Xi^{-} .
\end{align*}
$$

Including also the cosine of the Cabibbo angle, the coefficients multiplying $\bar{u}\left(p, s^{\prime}\right) \gamma^{\mu} u(p, s)$ and $\bar{u}\left(p, s^{\prime}\right) \gamma^{\mu} \gamma^{5} u(p, s)$ in the hadronic matrix elements relevant to the various strangeness-conserving decays of octet baryons (here and below the upper sign is for $V$, the lower sign for $A$ ),

$$
\begin{align*}
& \pm \cos \theta_{C}\left[\left(D_{V, A}+F_{V, A}\right) 2 \operatorname{tr} \bar{B}^{\prime} t_{1+i 2} B+\left(D_{V, A}-F_{V, A}\right) 2 \operatorname{tr} B t_{1+i 2} \bar{B}^{\prime}\right]  \tag{4.47}\\
& = \pm \cos \theta_{C}\left[D_{V, A}\left(2 \operatorname{tr} \bar{B}^{\prime} t_{1+i 2} B+2 \operatorname{tr} B t_{1+i 2} \bar{B}^{\prime}\right)+F_{V, A}\left(2 \operatorname{tr} \bar{B}^{\prime} t_{1+i 2} B-2 \operatorname{tr} B t_{1+i 2} \bar{B}^{\prime}\right)\right]
\end{align*}
$$

read explicitly

$$
\begin{align*}
n \rightarrow p: & \pm \cos \theta_{C}\left(D_{V, A}+F_{V, A}\right) \\
\Xi^{-} \rightarrow \Xi^{0}: & \mp \cos \theta_{C}\left(D_{V, A}-F_{V, A}\right) \\
\Lambda \rightarrow \Sigma^{+}: & \pm \cos \theta_{C} \sqrt{\frac{2}{3}} D_{V, A} \\
\Sigma^{-} \rightarrow \Lambda: & \pm \cos \theta_{C} \sqrt{\frac{2}{3}} D_{V, A}  \tag{4.48}\\
\Sigma^{0} \rightarrow \Sigma^{+}: & \mp \cos \theta_{C} \sqrt{2} F_{V, A} \\
\Sigma^{-} \rightarrow \Sigma^{0}: & \pm \cos \theta_{C} \sqrt{2} F_{V, A}
\end{align*}
$$

The same calculation for the strangeness-changing current gives

$$
\begin{align*}
2 \operatorname{tr} \bar{B}^{\prime} t_{4+i 5} B & =2 \sum_{n} \bar{B}_{n 1}^{\prime} B_{3 n} \\
& =\left(\frac{1}{\sqrt{2}} \bar{\Sigma}^{0}+\frac{1}{\sqrt{6}} \bar{\Lambda}\right) \Xi^{-}-\bar{\Sigma}^{+} \Xi^{0}+\bar{p}\left(-\sqrt{\frac{2}{3}} \Lambda\right),  \tag{4.49}\\
2 \operatorname{tr} B t_{4+i 5} \bar{B}^{\prime} & =2 \sum_{n} \bar{B}_{3 n}^{\prime} B_{n 1} \\
& =\bar{p}\left(\frac{1}{\sqrt{2}} \Sigma^{0}+\frac{1}{\sqrt{6}} \Lambda\right)+\bar{n} \Sigma^{-}+\left(-\sqrt{\frac{2}{3}} \bar{\Lambda}\right) \Xi^{-}
\end{align*}
$$

and including also the sine of the Cabibbo angle one finds for the coefficients of $\bar{u}(p) \gamma^{\mu} u(p)$ and $\bar{u}(p) \gamma^{\mu} \gamma^{5} u(p)$ in Eq. (4.41),

$$
\begin{align*}
& \pm \sin \theta_{C}\left[\left(D_{V, A}+F_{V, A}\right) 2 \operatorname{tr} \bar{B}^{\prime} t_{4+i 5} B+\left(D_{V, A}-F_{V, A}\right) 2 \operatorname{tr} B t_{4+i 5} \bar{B}^{\prime}\right]  \tag{4.50}\\
& = \pm \sin \theta_{C}\left[D_{V, A}\left(2 \operatorname{tr} B t_{4+i 5} \bar{B}^{\prime}+2 \operatorname{tr} \bar{B}^{\prime} t_{4+i 5} B\right)+F_{V, A}\left(2 \operatorname{tr} \bar{B}^{\prime} t_{4+i 5} B-2 \operatorname{tr} B t_{4+i 5} \bar{B}^{\prime}\right)\right]
\end{align*}
$$

the explicit expressions

$$
\begin{array}{rr}
\Lambda \rightarrow p: & \mp \sin \theta_{C} \frac{1}{\sqrt{6}}\left(D_{V, A}+3 F_{V, A}\right), \\
\Xi^{-} \rightarrow \Lambda: & \mp \sin \theta_{C} \frac{1}{\sqrt{6}}\left(D_{V, A}-3 F_{V, A}\right), \\
\Sigma^{0} \rightarrow p: & \pm \sin \theta_{C} \frac{1}{\sqrt{2}}\left(D_{V, A}-F_{V, A}\right),  \tag{4.51}\\
\Xi^{-} \rightarrow \Sigma^{0}: & \pm \sin \theta_{C} \frac{1}{\sqrt{2}}\left(D_{V, A}+F_{V, A}\right), \\
\Sigma^{-} \rightarrow n: & \pm \sin \theta_{C}\left(D_{V, A}-F_{V, A}\right), \\
\Xi^{0} \rightarrow \Sigma^{+}: & \mp \sin \theta_{C}\left(D_{V, A}+F_{V, A}\right) .
\end{array}
$$

Not all these matrix elements are relevant for actual physical processes in the real, $\mathrm{SU}(3)$ nonsymmetric world: for example, the decay $\Lambda \rightarrow \Sigma$ is forbidden since the Sigmas are heavier than the Lambda. Including the relevant gamma matrices and our knowledge of the vector form factors, we can write the following for the important hadronic matrix elements (to be sandwiched between initial and final hadron bispinors):

$$
\begin{align*}
n \rightarrow p e^{-} \bar{\nu}_{e}: & \cos \theta_{C}\left[\gamma^{\mu}-\left(D_{A}+F_{A}\right) \gamma^{\mu} \gamma^{5}\right], \\
\Sigma^{ \pm} \rightarrow \Lambda e^{ \pm} \nu_{e}\left(\bar{\nu}_{e}\right): & -\cos \theta_{C} D_{A} \gamma^{\mu} \gamma^{5}, \\
\Lambda \rightarrow p e^{-} \bar{\nu}_{e}: & -\sqrt{\frac{3}{2}} \sin \theta_{C}\left[\gamma^{\mu}-\left(F_{A}+\frac{1}{3} D_{A}\right) \gamma^{\mu} \gamma^{5}\right], \\
\Sigma^{-} \rightarrow n e^{-} \bar{\nu}_{e}: & -\sin \theta_{C}\left[\gamma^{\mu}-\left(F_{A}-D_{A}\right) \gamma^{\mu} \gamma^{5}\right],  \tag{4.52}\\
\Xi^{-} \rightarrow \Lambda e^{-} \bar{\nu}_{e}: & \sqrt{\frac{3}{2}} \sin \theta_{C}\left[\gamma^{\mu}-\left(F_{A}-\frac{1}{3} D_{A}\right) \gamma^{\mu} \gamma^{5}\right], \\
\Xi^{-} \rightarrow \Sigma^{0} e^{-} \bar{\nu}_{e}: & \frac{1}{\sqrt{2}} \sin \theta_{C}\left[\gamma^{\mu}-\left(F_{A}+D_{A}\right) \gamma^{\mu} \gamma^{5}\right], \\
\Xi^{0} \rightarrow \Sigma^{+} e^{-} \bar{\nu}_{e}: & -\sin \theta_{C}\left[\gamma^{\mu}-\left(F_{A}+D_{A}\right) \gamma^{\mu} \gamma^{5}\right] .
\end{align*}
$$

The decay rates of all these processes are parameterised by the three quantities $D_{A}, F_{A}$ and $\theta_{C}$, which can be determined by fitting the experimental data. Measurements of the neutron $\beta$ decay fix $D_{A}+F_{A}=g_{A}=1.25$. A best fit to experimental results yields $D_{A}=0.80, F_{A}=0.45$ and $\sin \theta_{C}=0.23$ (to be compared to $\sin \theta_{C}=0.21$ obtained from kaon decays).


Figure 7: Strangeness-changing non-leptonic decay processes: $\Sigma^{+} \rightarrow n \pi^{+}$(top) and $K^{+} \rightarrow$ $\pi^{+} \pi^{0}$ (bottom).

## 5 Strangeness-changing non-leptonic interactions

Non-leptonic interactions are responsible for process like $K \rightarrow 2 \pi, 3 \pi$ or $\Sigma \rightarrow N \pi$ (see Fig. 77). They involve the product of two hadronic currents (see Fig. 8). Restricting to the lightest quarks, and recalling $d^{\prime}=\cos \theta_{C} d+\sin \theta_{C} s$, the relevant part of the weak Lagrangian is

$$
\begin{align*}
-\frac{G}{\sqrt{2}} \bar{d}^{\prime} \mathcal{O}_{L}^{\alpha} u \bar{u} \mathcal{O}_{L \alpha} d^{\prime}=-\frac{G}{\sqrt{2}} & {\left[\cos ^{2} \theta_{C} \bar{d} \mathcal{O}_{L}^{\alpha} u \bar{u} \mathcal{O}_{L \alpha} d+\sin ^{2} \theta_{C} \bar{s} \mathcal{O}_{L}^{\alpha} u \bar{u} \mathcal{O}_{L \alpha} s\right.}  \tag{5.1}\\
& \left.+\sin \theta_{C} \cos \theta_{C}\left(\bar{s} \mathcal{O}_{L}^{\alpha} u \bar{u} \mathcal{O}_{L \alpha} d+\bar{d} \mathcal{O}_{L}^{\alpha} u \bar{u} \mathcal{O}_{L \alpha} s\right)\right] .
\end{align*}
$$

The first two terms are neutral interactions that do not change the flavour content of the system, while the term in brackets is the strangeness-changing non-leptonic interaction we are after. Recalling that $\mathcal{O}_{L}^{\alpha}=2 \gamma^{\alpha} P_{L}$ with $P_{L}$ the left-handed chiral projector, we can write the interesting part as

$$
\begin{equation*}
-2 \sqrt{2} G \sin \theta_{C} \cos \theta_{C}\left(\bar{s}_{L} \gamma^{\alpha} u_{L} \bar{u}_{L} \gamma_{\alpha} d_{L}+\bar{d}_{L} \gamma^{\alpha} u_{L} \bar{u}_{L} \gamma_{\alpha} s_{L}\right) . \tag{5.2}
\end{equation*}
$$

This term realises effectively a transition $s \rightarrow d$, but not directly through a flavour-changing neutral current. Since strangeness is either decreased or increased by 1 by this interaction, the selection rule $|\Delta S|=1$ follows. Processes with higher $|\Delta S|$ are not strictly forbidden but strongly suppressed, since they take place in higher orders of perturbation theory. Concerning isospin, the terms $\bar{s}_{L} \gamma^{\alpha} u_{L}$ and $\bar{u}_{L} \gamma_{\alpha} s_{L}$ are clearly part of isodoublets, while $\bar{u}_{L} \gamma_{\alpha} d_{L}$ and its Hermitian conjugate contain the representations $\frac{1}{2} \otimes \frac{1}{2}=0 \oplus 1$. Since $0 \otimes \frac{1}{2}=\frac{1}{2}$ and $\frac{1}{2} \otimes 1=\frac{1}{2} \oplus \frac{3}{2}$, the selection rule $|\Delta I|=\frac{1}{2}, \frac{3}{2}$ on total isospin follows. Moreover, since the field $d$ carries $I_{3}=-\frac{1}{2}$ (and so $d$ carries $I_{3}=\frac{1}{2}$ ), for $I_{3}$ one has the selection rule $\Delta I_{3}= \pm \frac{1}{2}$, the sign being plus if the first term in Eq. $\sqrt{5.2}$ ) is involved, and minus if it is the second term instead.


Figure 8: Four-fermion interaction vertices for strangeness changing non-leptonic interactions.

### 5.1 Suppression of $\Delta I=\frac{3}{2}$ transitions

Let us elaborate further on the issue of isospin. Take the second term in Eq. (5.2) (an entirely analogous argument works for the first one) and write it as

$$
\begin{align*}
2 \bar{d}_{L} \gamma^{\alpha} u_{L} \bar{u}_{L} \gamma_{\alpha} s_{L}= & \left(\bar{d}_{L} \gamma^{\alpha} u_{L} \bar{u}_{L} \gamma_{\alpha} s_{L}-\bar{u}_{L} \gamma^{\alpha} u_{L} \bar{d}_{L} \gamma_{\alpha} s_{L}\right) \\
& +\left(\bar{d}_{L} \gamma^{\alpha} u_{L} \bar{u}_{L} \gamma_{\alpha} s_{L}+\bar{u}_{L} \gamma^{\alpha} u_{L} \bar{d}_{L} \gamma_{\alpha} s_{L}\right)=M_{-}+M_{+} . \tag{5.3}
\end{align*}
$$

The combinations $M_{ \pm}$can be written as

$$
\begin{equation*}
M_{ \pm}=\left(\bar{d}_{L} \otimes \bar{u}_{L} \pm \bar{u}_{L} \otimes \bar{d}_{L}\right)_{a i, b j}\left(\gamma^{\alpha} u_{L} \otimes \gamma_{\alpha} s_{L}\right)_{a i, b j} \tag{5.4}
\end{equation*}
$$

where the Dirac indices $a, b=1, \ldots, 4$ and colour indices $i, j=1,2,3$ have been written explicitly. The first factor is a symmetric or antisymmetric product of two isospin doublets and fundamental (3) colour representations, and therefore an isotriplet $I=1$ transforming in the colour 6 representation, when taking the plus sign, or an isosinglet $I=0$ transforming in the colour $\overline{3}$ representation, when taking the minus sign. The second factor has clearly $I=\frac{1}{2}$, so that overall the antisymmetric combination mediates $|\Delta I|=\frac{1}{2}$ transitions, while the symmetric combination mediates both $|\Delta I|=\frac{1}{2}$ and $|\Delta I|=\frac{3}{2}$ transitions. Setting $\mathcal{I}_{3}=M_{-}+$h.c. and $\mathcal{I}_{6}=M_{+}+$h.c., we can write for the relevant part of the Lagrangian

$$
\begin{equation*}
\mathscr{L}=-\sqrt{2} G \sin \theta_{C} \cos \theta_{C}\left(\mathcal{I}_{3}+\mathcal{I}_{6}\right) \tag{5.5}
\end{equation*}
$$

Although there is no particular difference between the $|\Delta I|=\frac{1}{2}$ and $|\Delta I|=\frac{3}{2}$ parts of the Lagrangian, it is an experimental fact that $|\Delta I|=\frac{1}{2}$ transitions are around one order of magnitude enhanced (in amplitude) with respect to the $|\Delta I|=\frac{3}{2}$ transitions. The reason for this is therefore dynamical in nature, and enhancement and suppression result from the interplay of weak and other interactions, mainly the strong ones.

To get a handle on how strong interactions contribute to the relevant matrix elements of $\mathcal{I}_{3,6}$ one would like to use perturbation theory, but doing so is not straightforward. Perturbatively, strong interactions are described by the exchange of gluons between the quarks participating in the process (as well as by more complicated virtual processes involving gluon self-interactions and quark-antiquark pair creation). To do this accurately, one should use for each interaction vertex the so-called running coupling constant, $g_{s}(\mu)$, where $\mu$ is an energy scale, and the dependence of the running coupling on $\mu$ is determined by the theory. One should then choose $\mu$ judiciously, depending on the process under consideration. For gluon exchange, this scale is set by the amount of energy transferred by the gluon. In QCD, due to a phenomenon known
as asymptotic freedom, the running coupling constant becomes small at high energy, while it is large at low energies. For this reason, high-energy processes can be dealt with perturbatively, while low-energy ones cannot. One should then separate the exchange of "soft gluons", i.e., with low transferred momentum, and the exchange of "hard gluons", i.e., with high transferred momentum: while the latter can be treated perturbatively, the former cannot. Adopting a description of hadrons as bound states of quarks and antiquarks (and gluons), the effects of soft-gluon exchange should be included in the wave functions that describe the various hadronic states, while hard-gluon exchange between the constituents can be treated perturbatively. The separation scale between soft and hard gluons should then be taken around the inverse of the confinement scale $\mu \sim 1 / l_{\text {conf }}$, as gluons of longer wavelength than $l_{\text {conf }}$ do not resolve the inside of the hadron, determining instead their long distance properties as encoded in the wave function. The relevant scale is then of the order of $\mu \sim 1 / l_{\text {conf }} \sim 100 \mathrm{MeV} \div 1 \mathrm{GeV}$. Given gluons carrying four-momentum $q$, for $q^{2}<\mu^{2}$ we speak of soft gluons, while for $q^{2}>\mu^{2}$ we speak of hard gluons.

In the context of weak hadronic interactions, the effects of soft-gluon exchange is then included in the (anyway unknown) wave functions of the hadrons, while hard-gluon exchange can be studied explicitly in perturbation theory. The Lagrangian $\mathscr{L}$ in Eq. (5.5) becomes then the "bare" Lagrangian, $\mathscr{L}_{\text {bare }}$, describing only weak-interaction effects at the level of quark and antiquarks, that should be "dressed" by including the effects of further hard-gluon exchanges on top of $W$-boson exchanges. Notice that since $\mu \ll m_{W}$, we can still treat the exchange of a $W$ boson effectively as a four-fermion interaction when including the effects of hard-gluon exchange. The resulting Lagrangian can then be used to describe weak-interaction effects at the constituent level; eventually, one should fold the resulting amplitudes with suitable wave functions.

Let us see how the perturbative approach works in practice at a qualitative level. One type of diagrams are those in which we add gluons to $W$-boson exchanges between different quark lines. In the low-energy limit where the interaction becomes a local four-fermion interaction they look like those in Fig. 9, where one should add any possible number of gluons. Effectively these diagrams reduce to the original vertex except that the $\mathcal{I}_{3}$ and $\mathcal{I}_{6}$ parts receive different contributions and have therefore different prefactors. After "dressing" with gluons one has then to replace [see Eq. (5.5)]

$$
\begin{equation*}
\mathscr{L}_{\text {bare }}=-\sqrt{2} G \sin \theta_{C} \cos \theta_{C}\left(\mathcal{I}_{3}+\mathcal{I}_{6}\right) \rightarrow-\sqrt{2} G \sin \theta_{C} \cos \theta_{C}\left(a_{3} \mathcal{I}_{3}+a_{6} \mathcal{I}_{6}\right) . \tag{5.6}
\end{equation*}
$$

A second type of diagrams is obtained by including gluon exchanges to the emission and subsequent reabsorption of a $W$-boson in the same quark line, see Fig. 10. After emitting the $W$, the $s$ quark can turn into a $u$ or a $c$ quark, but the sign of the coupling is opposite in the two cases. The two diagrams would therefore cancel exactly if $m_{u}=m_{c}$, but since $m_{u} \ll m_{c}$ they do not. The effective four-fermion interaction obtained from these diagrams and the more complicated ones obtained by further adding gluons exchanged between the quark lines is of a new type, and is described by the operator

$$
\begin{equation*}
\mathcal{I}_{R}=-\left(\bar{d}_{L} \gamma^{\alpha} \lambda^{a} s_{L}\right)\left(\bar{u}_{R} \gamma_{\alpha} \lambda^{a} u_{R}+\bar{d}_{R} \gamma_{\alpha} \lambda^{a} d_{R}\right), \tag{5.7}
\end{equation*}
$$

where $\lambda^{a}$ are the Gell-Mann matrices, and summation over $a, a=1, \ldots, 8$, is understood. Since the first factor is $I=\frac{1}{2}$ while the second is an isosinglet, $I=0$, this term carries $\Delta I=\frac{1}{2}$. This is referred to as the "gluonic monopole vertex". The main differences with the other contributions is that this one involves colour, and most importantly also the right-handed component of the


Figure 9: Dressing the four-fermion interaction with hard-gluon exchanges.


Figure 10: Lowest-order contributions to gluonic monopole vertex.
quark fields ${ }^{37}$ All in all, the effective, "dressed" Lagrangian reads

$$
\begin{equation*}
\mathscr{L}_{\text {dressed }}=-\sqrt{2} G \sin \theta_{C} \cos \theta_{C}\left(a_{3} \mathcal{I}_{3}+a_{6} \mathcal{I}_{6}+a_{R} \mathcal{I}_{R}\right) \tag{5.8}
\end{equation*}
$$

A perturbative calculation yields $a_{3} \simeq 3, a_{6} \simeq 0.6, a_{R} \simeq 0.12$ (see Ref. [6] for details about the estimate), and since $\Delta I=\frac{3}{2}$ transitions are mediated only by $\mathcal{I}_{6}$, one sees that these are suppressed compared to the $\Delta I=\frac{1}{2}$ transitions. However, the degree of suppression obtained with this perturbative estimate is not enough to agree with experiments.

Of course, in order to take fully into account the effects of strong interactions one should rather employ a non-perturbative technique. Good agreement between the experimentally observed degree of suppression and the prediction of QCD has been recently obtained using lattice techniques [8].

### 5.2 Non-leptonic decays of kaons

The most relevant non-leptonic decays of kaons are into two or three pions,

$$
\begin{align*}
K^{0}, \bar{K}^{0} & \rightarrow \pi^{+} \pi^{-}, \pi^{0} \pi^{0}, & K^{0}, \bar{K}^{0} & \rightarrow \pi^{+} \pi^{-} \pi^{0}, \pi^{0} \pi^{0} \pi^{0}, \\
K^{ \pm} & \rightarrow \pi^{ \pm} \pi^{0}, & K^{ \pm} & \rightarrow \pi^{ \pm} \pi^{+} \pi^{-}, \pi^{ \pm} \pi^{0} \pi^{0} . \tag{5.9}
\end{align*}
$$

[^23]
### 5.2.1 Neutral kaons

An important aspect of neutral kaons concerning their weak decays is that the eigenstates of strangeness produced in strong interaction processes, $K^{0}(d \bar{s})$ and $\bar{K}^{0}(-s \bar{d})$, are not eigenstates of $C P$ (nor of $C$ ). At the same time, weak interactions are $C P$ invariant as long as we ignore the third generation of fermions, while they do not conserve strangeness. For this reason it is more convenient in this context to use the linear combinations of neutral kaons that are $C P$ eigenstates rather than strangeness eigenstates, as these will have definite decay properties. Using linear combinations of neutral kaons is physically allowed since $K^{0}$ and $\bar{K}^{0}$ differ only in strangeness, for which there is no superselection rule, and are mixed by a second-order weak interaction to form the physical eigenstates with definite decay properties (we will come back to this below).

Since kaons are pseudoscalars, $P\left|K^{0}\right\rangle=-\left|K^{0}\right\rangle$ and $P\left|\bar{K}^{0}\right\rangle=-\left|\bar{K}^{0}\right\rangle$, and choosing phases such that $C\left|K^{0}\right\rangle=\left|\bar{K}^{0}\right\rangle$, and $C\left|\bar{K}^{0}\right\rangle=\left|K^{0}\right\rangle$, the $C P$ eigenstates are

$$
\begin{align*}
\left|K_{1}^{0}\right\rangle=\frac{K^{0}-\bar{K}^{0}}{\sqrt{2}}, & C P\left|K_{1}^{0}\right\rangle=\left|K_{1}^{0}\right\rangle \\
\left|K_{2}^{0}\right\rangle=\frac{K^{0}+\bar{K}^{0}}{\sqrt{2}}, & C P\left|K_{2}^{0}\right\rangle=-\left|K_{2}^{0}\right\rangle \tag{5.10}
\end{align*}
$$

Two- and three-pion states with definite orbital angular momenta are also eigenstates of $C P$. Working in the centre-of-mass frame and writing the definite $\ell$ states $\left|\pi^{a} \pi^{b} ; \ell\right\rangle, a, b=0, \pm$, in terms of definite momentum states $\left|\pi^{a}(\vec{p}) \pi^{b}(-\vec{p})\right\rangle$ and a suitable wave function $f_{\ell}(\vec{p})$, with $f_{\ell}(-\vec{p})=(-1)^{\ell} f_{\ell}(\vec{p})$, one has

$$
\begin{equation*}
\left|\pi^{a} \pi^{b} ; \ell\right\rangle=\int d \Omega_{p}\left|\pi^{a}(\vec{p}) \pi^{b}(-\vec{p})\right\rangle f_{\ell}(\vec{p}) \tag{5.11}
\end{equation*}
$$

A $\pi^{+} \pi^{-}$system has $C P=1$ independently of its orbital angular momentum $\ell$ :

$$
\begin{align*}
C P\left|\pi^{+} \pi^{-} ; \ell\right\rangle & =\int d \Omega_{p} C P\left|\pi^{+}(\vec{p}) \pi^{-}(-\vec{p})\right\rangle f_{\ell}(\vec{p})=(-1)^{2} \int d \Omega_{p}\left|\pi^{-}(-\vec{p}) \pi^{+}(\vec{p})\right\rangle f_{\ell}(\vec{p}) \\
& =\int d \Omega_{p}\left|\pi^{+}(\vec{p}) \pi^{-}(-\vec{p})\right\rangle f_{\ell}(\vec{p})=\left|\pi^{+} \pi^{-} ; \ell\right\rangle \tag{5.12}
\end{align*}
$$

having used in the last passage the symmetry of the state vector, due to the bosonic nature of pions. For a $\pi^{0} \pi^{0}$ system one has instead

$$
\begin{align*}
C P\left|\pi^{0} \pi^{0} ; \ell\right\rangle & =\int d \Omega_{p} C P\left|\pi^{0}(\vec{p}) \pi^{0}(-\vec{p})\right\rangle f_{\ell}(\vec{p})=(-1)^{2} \int d \Omega_{p}\left|\pi^{0}(-\vec{p}) \pi^{0}(\vec{p})\right\rangle f_{\ell}(\vec{p}) \\
& =\int d \Omega_{p}\left|\pi^{0}(\vec{p}) \pi^{0}(-\vec{p})\right\rangle f_{\ell}(-\vec{p})=(-1)^{\ell}\left|\pi^{0} \pi^{0} ; \ell\right\rangle  \tag{5.13}\\
& =\int d \Omega_{p}\left|\pi^{0}(\vec{p}) \pi^{0}(-\vec{p})\right\rangle f_{\ell}(\vec{p})=\left|\pi^{0} \pi^{0} ; \ell\right\rangle,
\end{align*}
$$

having used the change of variable $\vec{p} \rightarrow-\vec{p}$ to get the second line from the first, and again the symmetry of the state vector to get the third one from the first. We then find that a $\pi^{0} \pi^{0}$ system has $C P=(-1)^{\ell}$, but must also have $C P=1$, so it cannot exist in an odd $\ell$ state. A neutral two-pion state with definite $\ell$ therefore always has $C P=1$.

For the three-pion system one can consider states $\left|\pi^{a} \pi^{b} \pi^{c} ; \ell, L\right\rangle$ with definite values of two orbital angular momenta, one corresponding to the $\pi^{+} \pi^{-}$pair or one pair of $\pi^{0}$ s, that we will denote again with $\ell$, and a second one corresponding to the motion of the third pion with respect to the centre of mass of the other two, that we will denote with $L$. Since the kaon has spin zero, one must have $\ell=L$. For the state $\pi^{+} \pi^{-} \pi^{0}$ we have shown above that the charged pair is $C P$ invariant, while for the neutral pion the only effect is an intrinsic parity factor and a change $\vec{p} \rightarrow-\vec{p}$ in the corresponding wave function $f_{L}$ (see the second line of Eq. (5.13)), and so

$$
\begin{equation*}
C P\left|\pi^{+} \pi^{-} \pi^{0} ; \ell, \ell\right\rangle=-(-1)^{\ell}\left|\pi^{+} \pi^{-} \pi^{0} ; \ell, \ell\right\rangle=(-1)^{\ell+1}\left|\pi^{+} \pi^{-} \pi^{0} ; \ell, \ell\right\rangle \tag{5.14}
\end{equation*}
$$

For the $\pi^{0} \pi^{0} \pi^{0}$ case we get a factor $(-1)^{\ell}$ from the first pair if it has orbital angular momentum $\ell$, and also a factor $(-1)^{L}=(-1)^{\ell}$ from the third pion and so

$$
\begin{equation*}
C P\left|\pi^{0} \pi^{0} \pi^{0} ; \ell, \ell\right\rangle=-(-1)^{2 \ell}\left|\pi^{0} \pi^{0} \pi^{0} ; \ell, \ell\right\rangle=-\left|\pi^{0} \pi^{0} \pi^{0} ; \ell, \ell\right\rangle \tag{5.15}
\end{equation*}
$$

In summary, one has $C P=1$ for two-pion final states, and for three-pion $\pi^{+} \pi^{-} \pi^{0}$ if $\ell$ of the charged pair is odd, although this process is relatively suppressed for phase-space reasons; and $C P=-1$ for three-pion $\pi^{0} \pi^{0} \pi^{0}$ states and for three-pion $\pi^{+} \pi^{-} \pi^{0}$ states if $\ell$ of the charged pair is even.

In the approximation of conserved $C P, K_{1}^{0}$ and $K_{2}^{0}$ correspond exactly to the "short" and "long" kaons $K_{S, L}^{0}$ that come from the diagonalisation of the effective Hamiltonian that describes the temporal evolution of the neutral kaon state on the kaon subspace (see below), and have therefore a definite mass and decay width. "Short" and "long" clearly refer to their lifetime. Imposing $C P$ conservation one has predominantly $K_{1}^{0} \rightarrow 2 \pi$ and exclusively $K_{2}^{0} \rightarrow 3 \pi$. The first process violates parity, while the second one conserves it, and since $P$-preserving and $P$-violating interactions have the same strength at the level of the Lagrangian, the difference between the widths of the two types of processes comes from the difference in the available phase space. Since there is less kinetic energy available for three pions than for two, the available phase space is smaller in $K_{2}^{0}$ decays, so the width of the $K_{2}^{0}$ is smaller than that of the $K_{1}^{0}$ and its lifetime longer. In other words, $K_{1}^{0}=K_{S}^{0}$ and $K_{2}^{0}=K_{L}^{0}$. Indeed, the decay rate of $K_{1}^{0} \rightarrow 2 \pi$ is about three orders of magnitude larger than that of $K_{2}^{0} \rightarrow 3 \pi$, with $\tau_{1}=1 / \Gamma_{1} \simeq 10^{-10} s$ and $\tau_{2}=1 / \Gamma_{2} \simeq 5 \cdot 10^{-8} s$. The two states $K_{1,2}^{0}$ (identified with $K_{S, L}^{0}$ ) also differ slightly in mass, with $\Delta m=m_{2}-m_{1}=3.5 \cdot 10^{-12} \mathrm{MeV}$. We will see below how this mass difference can be measured experimentally

### 5.2.2 Isospin wave functions in two-pion decays and ratios of decay widths

Isospin conservation can be used to predict the ratios of kaon decay widths based on symmetry considerations alone. To this end, we write a generic two-pion state as follows,

$$
\begin{equation*}
|\pi \pi\rangle=A^{a} B^{b}\left|\pi_{a} \pi_{b}\right\rangle, \quad a, b=1,2,3 \tag{5.16}
\end{equation*}
$$

where

$$
\begin{equation*}
\pi_{1}=\frac{\pi^{+}+\pi^{-}}{\sqrt{2}}, \quad \pi_{2}=\frac{\pi^{+}-\pi^{-}}{\sqrt{2} i}, \quad \pi_{3}=\pi^{0} \tag{5.17}
\end{equation*}
$$

and $\vec{A}$ and $\vec{B}$ are three-component complex vectors. Although the electric charge superselection rule forbids the physical realisation of superpositions of states with different charge, we are still
allowed to build these combinations in the Hilbert space. In this language, the transformation properties of the pions under isospin rotations are passed on to $A^{a}$ and $B^{b}$, which are therefore $I=1$ isotriplets. Since $A^{a} B^{b}$ is the composition of two isotriplets, it can be decomposed in the usual way into $I=0, I=1$ and $I=2$ parts that transform irreducibly under isospin rotations:

$$
\begin{equation*}
A^{a} B^{b}=\underbrace{\frac{1}{3} \vec{A} \cdot \vec{B} \delta^{a b}}_{I=0}+\underbrace{\frac{1}{2} \varepsilon^{a b c}(\vec{A} \wedge \vec{B})^{c}}_{I=1}+\underbrace{\frac{1}{2}\left(A^{a} B^{b}+A^{b} B^{a}-\frac{2}{3} \vec{A} \cdot \vec{B} \delta^{a b}\right)}_{I=2} . \tag{5.18}
\end{equation*}
$$

Bose-Einstein symmetry requires that the total pion wave function be symmetric, so for a $\ell=0$ state like those involved in kaon decays the flavour wave function must be symmetric, and so the $I=1$ component must be absent. Furthermore, in $K_{1}^{0} \rightarrow \pi^{+} \pi^{-}, \pi^{0} \pi^{0}$ one has $I_{3}=0$ in the final state, so both $I=0,2$ can be present, while in $K^{+} \rightarrow \pi^{+} \pi^{0}$ one has $I_{3}=1$ and so only $I=2$ is present.

If $|\Delta I|=\frac{3}{2}$ transitions were completely suppressed and the $|\Delta I|=\frac{1}{2}$ enhancement turned into an exact selection rule, then since kaons have $I=\frac{1}{2}$ one could only get final states with $\frac{1}{2} \otimes \frac{1}{2}=0 \oplus 1$, and therefore the process $K^{+} \rightarrow \pi^{+} \pi^{0}$ would be forbidden, and only the $I=0$ component would be present in the $K_{1}^{0}$ decay. In this case one would have for the amplitude the simple form $\mathcal{M} \propto \vec{A} \cdot \vec{B}$, and for the width

$$
\begin{equation*}
\Gamma\left(K_{1}^{0} \rightarrow 2 \pi\right) \propto \frac{|\vec{A} \cdot \vec{B}|^{2}}{|\vec{A}|^{2}|\vec{B}|^{2}+|\vec{A} \cdot \vec{B}|^{2}}, \tag{5.19}
\end{equation*}
$$

where the numerator takes care of the normalisation of the state. In the exact isospin limit considered here, the proportionality factor is the same for the two two-pion decay processes, as it involves the same $(I=0)$ decay amplitude, and a phase-space factor that is identical for charged and neutral pion pairs. For $K_{1}^{0} \rightarrow \pi^{+} \pi^{-}$, we can take

$$
\begin{equation*}
\vec{A}=\frac{1}{\sqrt{2}}(1, i, 0), \quad \vec{B}=\frac{1}{\sqrt{2}}(1,-i, 0), \quad \vec{A} \cdot \vec{B}=1, \quad|\vec{A}|^{2}=|\vec{B}|^{2}=1, \quad \vec{A}^{*} \cdot \vec{B}=0 . \tag{5.20}
\end{equation*}
$$

For $K_{1}^{0} \rightarrow \pi^{0} \pi^{0}$ we take instead

$$
\begin{equation*}
\vec{A}=(0,0,1), \quad \vec{B}=(0,0,1), \quad \vec{A} \cdot \vec{B}=1, \quad|\vec{A}|^{2}=|\vec{B}|^{2}=1, \quad \overrightarrow{A^{*}} \cdot \vec{B}=1 \tag{5.21}
\end{equation*}
$$

We then find

$$
\begin{equation*}
\frac{\Gamma\left(K_{1}^{0} \rightarrow \pi^{+} \pi^{-}\right)}{\Gamma\left(K_{1}^{0} \rightarrow \pi^{0} \pi^{0}\right)}=\frac{1 / 1}{1 / 2}=2 . \tag{5.22}
\end{equation*}
$$

The experimental value is $2.255(5)$, estimated identifying $K_{1}^{0}$ with the short kaon $K_{S}^{0}$, and using current values for the $K_{S}^{0}$ branching fractions [9].

As matter of fact the process $K^{+} \rightarrow \pi^{+} \pi^{0}$ is not strictly forbidden, but it takes place mediated by $\mathcal{I}_{6}$ (see Eqs. 5.5) and (5.8), which carries also a $I=\frac{3}{2}$ component. This means also that the $I=2$ component of two-pion state wave function can contribute to the decay amplitude. Let us then include both the $I=\frac{1}{2}$ and $I=\frac{3}{2}$ contributions and refine the analysis above. To this end, notice that the relevant part $\mathscr{L}$ of the interaction Lagrangian can be written as

$$
\begin{equation*}
\mathscr{L}=\bar{s} \mathcal{O}_{L}^{\alpha} u \bar{u} \mathcal{O}_{L \alpha} d+\bar{d} \mathcal{O}_{L}^{\alpha} u \bar{u} \mathcal{O}_{L \alpha} s=O_{\frac{3}{2} \frac{1}{2}}+O_{\frac{3}{2}-\frac{1}{2}}+O_{\frac{1}{2} \frac{1}{2}}+O_{\frac{1}{2}-\frac{1}{2}} \tag{5.23}
\end{equation*}
$$

i.e., the sum of four terms $O_{I I_{3}}$ carrying total isospin $I=\frac{3}{2}, \frac{1}{2}$ and third component $I_{3}=$ $\pm \frac{1}{2}$. Notice that $O_{I \frac{1}{2}}$ and $O_{I-\frac{1}{2}}$ are transformed into each other by Hermitean conjugation, as
well as by a $C P$ transformation, $(C P)^{\dagger} O_{I I_{3}}(C P)=O_{I-I_{3}}$. Using the known Clebsch-Gordan coefficients we can decompose the states $O_{I I_{3}}\left|K^{+}\right\rangle=O_{I I_{3}}\left|\frac{1}{2}+\frac{1}{2}\right\rangle$ and $O_{I I_{3}}\left|K^{0}\right\rangle=O_{I I_{3}}\left|\frac{1}{2}-\frac{1}{2}\right\rangle$ into total isospin eigenstates:

$$
\begin{align*}
& O_{\frac{3}{2} \frac{1}{2}}\left|K^{0}\right\rangle=O_{\frac{3}{2} \frac{1}{2}}\left|\frac{1}{2}-\frac{1}{2}\right\rangle=\frac{1}{\sqrt{2}}\left(|20\rangle_{K}-|10\rangle_{K}\right), \\
& O_{\frac{1}{2} \frac{1}{2}}\left|K^{0}\right\rangle=O_{\frac{1}{2} \frac{1}{2}}\left|\frac{1}{2}-\frac{1}{2}\right\rangle=\frac{1}{\sqrt{2}}\left(|10\rangle_{K}-|00\rangle_{K}\right),  \tag{5.24}\\
& O_{\frac{3}{2} \frac{1}{2}}\left|K^{+}\right\rangle=O_{\frac{3}{2} \frac{1}{2}}\left|\frac{1}{2}+\frac{1}{2}\right\rangle=\frac{\sqrt{3}}{2}\left(|21\rangle_{K}+|11\rangle_{K}\right) .
\end{align*}
$$

One can decompose the two-pion states in a similar manner:

$$
\begin{align*}
\left|\pi^{+} \pi^{-}\right\rangle & =\frac{1}{\sqrt{3}}|20\rangle_{\pi}+\sqrt{\frac{2}{3}}|00\rangle_{\pi} \\
\left|\pi^{0} \pi^{0}\right\rangle & =\sqrt{\frac{2}{3}}|20\rangle_{\pi}-\frac{1}{\sqrt{3}}|00\rangle_{\pi},  \tag{5.25}\\
\left|\pi^{+} \pi^{0}\right\rangle & =|21\rangle_{\pi}
\end{align*}
$$

The subscripts $K$ and $\pi$ are used to distinguish the two sets of isospin eigenstates pertaining to the kaon and the two-pion systems. When considering $K_{1}^{0}$ decays, we can use the fact that the two-pion states of interest have $I_{3}=0$ and $C P=1$ to show that

$$
\begin{align*}
\langle\pi \pi| O_{I \frac{1}{2}}+O_{I-\frac{1}{2}}\left|K_{1}^{0}\right\rangle & =\frac{1}{\sqrt{2}}\left[\langle\pi \pi| O_{I \frac{1}{2}}\left|K^{0}\right\rangle-\langle\pi \pi| O_{I-\frac{1}{2}}\left|\bar{K}^{0}\right\rangle\right] \\
& =\frac{1}{\sqrt{2}}\left[\langle\pi \pi| O_{I \frac{1}{2}}\left|K^{0}\right\rangle-\langle\pi \pi|(C P)^{\dagger} O_{I \frac{1}{2}}(C P)\left|\bar{K}^{0}\right\rangle\right]  \tag{5.26}\\
& =\sqrt{2}\langle\pi \pi| O_{I \frac{1}{2}}\left|K^{0}\right\rangle
\end{align*}
$$

All in all, we can parameterise the three relevant amplitudes in terms of two definite-isospin amplitudes,

$$
\begin{align*}
\left\langle\pi^{+} \pi^{-}\right| \mathscr{L}\left|K_{1}^{0}\right\rangle & =\frac{1}{\sqrt{3}} \pi\langle 20 \mid 20\rangle_{K}-\sqrt{\frac{2}{3}} \pi\langle 00 \mid 00\rangle_{K}=\frac{1}{\sqrt{3}} \mathcal{A}_{2}-\sqrt{\frac{2}{3}} \mathcal{A}_{0} \\
\left\langle\pi^{0} \pi^{0}\right| \mathscr{L}\left|K_{1}^{0}\right\rangle & =\sqrt{\frac{2}{3}} \pi\langle 20 \mid 20\rangle_{K}+\frac{1}{\sqrt{3}} \pi\langle 00 \mid 00\rangle_{K}=\sqrt{\frac{2}{3}} \mathcal{A}_{2}+\frac{1}{\sqrt{3}} \mathcal{A}_{0}  \tag{5.27}\\
\left\langle\pi^{+} \pi^{0}\right| \mathscr{L}\left|K^{+}\right\rangle & =\frac{\sqrt{3}}{2} \pi\langle 21 \mid 21\rangle_{K}=\frac{\sqrt{3}}{2} \pi\langle 20 \mid 20\rangle_{K}=\frac{\sqrt{3}}{2} \mathcal{A}_{2}
\end{align*}
$$

In the case discussed above of an exact $|\Delta I|=\frac{1}{2}$ selection rule, only $\mathcal{A}_{0}$ entered the decay amplitudes. In the real world $|\Delta I|=\frac{3}{2}$ transitions are suppressed but not perfectly, so we expect $\left|\mathcal{A}_{2}\right| \ll 1$ but nonzero. We then get for the ratio of widths

$$
\begin{align*}
& \frac{\Gamma\left(K_{1}^{0} \rightarrow \pi^{+} \pi^{-}\right)}{\Gamma\left(K_{1}^{0} \rightarrow \pi^{0} \pi^{0}\right)}=\frac{p_{ \pm}}{p_{0}} \frac{\left|\frac{1}{\sqrt{3}} \mathcal{A}_{2}-\sqrt{\frac{2}{3}} \mathcal{A}_{0}\right|^{2}}{\left|\sqrt{\frac{2}{3}} \mathcal{A}_{2}+\frac{1}{\sqrt{3}} \mathcal{A}_{0}\right|^{2}}=\frac{p_{ \pm}}{p_{0}} \frac{2\left|\mathcal{A}_{0}\right|^{2}+\left|\mathcal{A}_{2}\right|^{2}-2 \sqrt{2} \operatorname{Re} \mathcal{A}_{2}^{*} \mathcal{A}_{0}}{\left|\mathcal{A}_{0}\right|^{2}+2\left|\mathcal{A}_{2}\right|^{2}+2 \sqrt{2} \operatorname{Re} \mathcal{A}_{2}^{*} \mathcal{A}_{0}} \\
& \simeq\left(2-3 \sqrt{2} \frac{\operatorname{Re} \mathcal{A}_{2}^{*} \mathcal{A}_{0}}{\left|\mathcal{A}_{0}\right|^{2}}\right) \frac{p_{ \pm}}{p_{0}}  \tag{5.28}\\
& \frac{\Gamma\left(K^{+} \rightarrow \pi^{+} \pi^{0}\right)}{\Gamma\left(K_{1}^{0} \rightarrow 2 \pi\right)} \simeq \frac{\frac{3}{4}\left|\mathcal{A}_{2}\right|^{2}}{\left|\mathcal{A}_{0}\right|^{2}+\left|\mathcal{A}_{2}\right|^{2}} \simeq \frac{3}{4} \frac{\left|\mathcal{A}_{2}\right|^{2}}{\left|\mathcal{A}_{0}\right|^{2}}
\end{align*}
$$

where $\Gamma\left(K_{1}^{0} \rightarrow 2 \pi\right)=\Gamma\left(K_{1}^{0} \rightarrow \pi^{+} \pi^{-}\right)+\Gamma\left(K_{1}^{0} \rightarrow \pi^{0} \pi^{0}\right)$. We also included a factor that takes into account the slight difference in the phase space of the two final states of $K_{1}^{0}$ decays, i.e., the ratio $p_{ \pm} / p_{0} \simeq 0.99$ of final-state momenta in the centre-of-mass frame for the charged pion and the neutral pion pair. From experiments one gets a suppression factor for $K^{+}$against $K_{1}^{0}$ decay of about 670 [9], corresponding to $\left|\mathcal{A}_{2}\right| /\left|\mathcal{A}_{0}\right| \simeq 0.045$, which a posteriori justifies neglecting $\left|\mathcal{A}_{2}\right|^{2} /\left|\mathcal{A}_{0}\right|^{2}$ terms in Eq. (5.28).

### 5.2.3 Neutral kaon oscillations

As already remarked above, the neutral kaons produced by strong interactions are eigenstates of strangeness: for example, one can produce a beam of $K^{0}$ directing a beam of negatively charged pions on ordinary matter via the process $\pi^{-} p \rightarrow K^{0} \Lambda$, but one cannot similarly produce a beam of $\bar{K}^{0}$. On the other hand, decays of neutral kaons are governed by weak interactions which are (almost) $C P$-conserving but not strangeness-conserving, and decays into two or three pions proceed respectively through the $C P$-even and $C P$-odd components of the neutral kaons. A consequence of this is the possibility for $K^{0}$ and $\bar{K}^{0}$ to oscillate into each other: since both can decay into two pions, they can also oscillate into each other through a pion loop, i.e., via a decay of the (say) $K^{0}$ into two (virtual) pions and their subsequent annihilation to form a $\bar{K}^{0}$. Due to this oscillation, if one prepares a beam of $K^{0}$ at time $t=0$, then at some later time $t$ the beam (thinner, due to real decay processes taking place) will contain a linear superposition of $K^{0}$ and $\bar{K}^{0}$ states. The oscillation of $K^{0}$ and $\bar{K}^{0}$ into each other is possible since the charge distinguishing them (strangeness) is not exactly conserved in Nature, differently from, e.g., neutron and antineutron (differing in baryon number). We now show that in the approximation of exact $C P$ symmetry the frequency of this oscillation is given by the mass difference of the $K_{1,2}^{0}$ particles.

The exact, unitary temporal evolution of a neutral kaon state at rest that at $t=0$ equals $\left|K^{0}\right\rangle$ reads

$$
\begin{equation*}
e^{-i H t}\left|K^{0}\right\rangle=c_{1}(t)\left|K^{0}\right\rangle+c_{2}(t)\left|\bar{K}^{0}\right\rangle+|R(t)\rangle, \tag{5.29}
\end{equation*}
$$

where $|R(t)\rangle$ accounts for the non-kaon states in which the neutral kaon can decay. The projection $\left|K^{0}(t)\right\rangle=\left(\left|K^{0}\right\rangle\left\langle K^{0}\right|+\left|\bar{K}^{0}\right\rangle\left\langle\bar{K}^{0}\right|\right) e^{-i H t}\left|K^{0}\right\rangle$ of the neutral kaon state at time $t$ on the kaon subspace (thus projecting out the states into which it can decay) reads then

$$
\begin{equation*}
\left|K^{0}(t)\right\rangle=c_{1}(t)\left|K^{0}\right\rangle+c_{2}(t)\left|\bar{K}^{0}\right\rangle . \tag{5.30}
\end{equation*}
$$

Due to the projection, the evolution of $\left|K^{0}(t)\right\rangle$ is not unitary. Nonetheless, under certain approximations (Weisskopf-Wigner approximation) it can be described in terms of a non-Hermitian effective Hamiltonian $H_{\text {eff }}$ as $\left|K^{0}(t)\right\rangle=e^{-i H_{\text {eff }} t}\left|K^{0}\right\rangle$ (see Ref. [2], Appendix I, for details). The corresponding effective Schrödinger equation reads

$$
\begin{equation*}
i \frac{\partial}{\partial t}\left|K^{0}(t)\right\rangle=H_{\mathrm{eff}}\left|K^{0}(t)\right\rangle \tag{5.31}
\end{equation*}
$$

and can be solved in the usual way by diagonalising $H_{\text {eff }}$. Since this is not a Hermitean operator, its eigenvalues are generally complex. This procedure yields the "short" and "long" neutral kaons, $K_{S, L}^{0}$,

$$
\begin{equation*}
H_{\mathrm{eff}}\left|K_{S, L}^{0}\right\rangle=\left(m_{S, L}-\frac{i}{2} \Gamma_{S, L}\right)\left|K_{S, L}^{0}\right\rangle, \tag{5.32}
\end{equation*}
$$

where by definition $\Gamma_{S}>\Gamma_{L}$ (assumed to be non-degenerate), and one can show that $\Gamma_{S, L}>0[2]$. The real part of the eigenvalue is naturally identified with the particle mass. The imaginary part of the eigenvalue governs instead the exponential decay with time of the corresponding component of the wave function, thus providing the decay width of the two eigenstates. Since the eigenvalues are different, in the limit of exact $C P$ symmetry the eigenstates of $H_{\text {eff }}$ must also be eigenstates of $C P$, and since $K_{1}^{0}$ is the short lived one (it has a larger decay rate since it decays in two rather than three pions) we have $K_{1}^{0}=K_{S}^{0}$ and $K_{2}^{0}=K_{L}^{0}$, so $m_{S, L}=m_{1,2}$ and $\Gamma_{S, L}=\Gamma_{1,2}$.

For a state coinciding with $K^{0}$ at $t=0$,

$$
\begin{equation*}
\left|K^{0}(0)\right\rangle=\left|K^{0}\right\rangle=\frac{\left|K_{1}^{0}\right\rangle+\left|K_{2}^{0}\right\rangle}{\sqrt{2}} \tag{5.33}
\end{equation*}
$$

we have at time $t$,

$$
\begin{equation*}
\left|K^{0}(t)\right\rangle=\frac{1}{\sqrt{2}}\left(e^{-i\left(m_{1}-i \frac{\Gamma_{1}}{2}\right) t}\left|K_{1}^{0}\right\rangle+e^{-i\left(m_{2}-i \frac{\Gamma_{2}}{2}\right) t}\left|K_{2}^{0}\right\rangle\right) \tag{5.34}
\end{equation*}
$$

and the probability of observing the neutral kaon as a $K^{0}$ or a $\bar{K}^{0}$ at time $t$ is the absolute value square of the amplitudes

$$
\begin{align*}
\left\langle K^{0} \mid K^{0}(t)\right\rangle & =\frac{1}{2}\left(e^{-i\left(m_{1}-i \frac{\Gamma_{1}}{2}\right) t}+e^{-i\left(m_{2}-i \frac{\Gamma_{2}}{2}\right) t}\right) \\
\left\langle\bar{K}^{0} \mid K^{0}(t)\right\rangle & =-\frac{1}{2}\left(e^{-i\left(m_{1}-i \frac{\Gamma_{1}}{2}\right) t}-e^{-i\left(m_{2}-i \frac{\Gamma_{2}}{2}\right) t}\right) \tag{5.35}
\end{align*}
$$

The number of $K^{0}$ and $\bar{K}^{0}$ observed in the neutral kaon beam at time $t$ is thus equal to

$$
\begin{align*}
& N_{K^{0}}(t)=\frac{1}{4}\left(e^{-\Gamma_{1} t}+e^{-\Gamma_{2} t}+2 \cos \left(\left(m_{2}-m_{1}\right) t\right) e^{-\frac{\Gamma_{1}+\Gamma_{2}}{2} t}\right)  \tag{5.36}\\
& N_{\bar{K}^{0}}(t)=\frac{1}{4}\left(e^{-\Gamma_{1} t}+e^{-\Gamma_{2} t}-2 \cos \left(\left(m_{2}-m_{1}\right) t\right) e^{-\frac{\Gamma_{1}+\Gamma_{2}}{2} t}\right)
\end{align*}
$$

Besides the overall exponential decay of these quantities due to the various decay processes of the neutral kaons, these expressions show that $K^{0}$ and $\bar{K}^{0}$ oscillate into each other: for example, while $N_{K^{0}}(0)-N_{\bar{K}^{0}}(0)=1$ at $t_{*}=\frac{\pi}{2}\left(m_{2}-m_{1}\right)$ one finds $N_{K^{0}}\left(t_{*}\right)-N_{\bar{K}^{0}}\left(t_{*}\right)=0$.

The number of $K^{0}$ and $\bar{K}^{0}$ in the beam can be inferred by measuring the positrons and the electrons produced in their semileptonic decays. Due to the $\Delta Q=\Delta S$ selection rule,

$$
\begin{equation*}
K^{0} \rightarrow e^{+} \nu_{e} \pi^{-}, \quad \bar{K}^{0} \rightarrow e^{-} \bar{\nu}_{e} \pi^{+} \tag{5.37}
\end{equation*}
$$

while $K^{0} \rightarrow e^{-}$and $\bar{K}^{0} \rightarrow e^{+}$are forbidden. The number of positron and electron detected as decay products from the beam at a certain time are then proportional to $N_{K^{0}}(t)$ and $N_{\bar{K}^{0}}(t)$ (with the same proportionality factor since the two processes are related by a $C P$ transformation). Another possibility is to direct the beam against a fixed target of ordinary (non-strange) matter and observe hyperon production: while $\bar{K}^{0}$ can be absorbed through the process $\bar{K}^{0} p \rightarrow \Lambda \pi^{+}$, strangeness conservation forbids the creation of hyperons from a $K^{0}$. These processes depend on the strangeness content of the kaon beam, and thus see the $K^{0}, \bar{K}^{0}$ amplitudes. On the other hand, non-leptonic decays into pions see the $C P$ content of the kaon beam, and thus see the $K_{1}^{0}$ and $K_{2}^{0}$ amplitudes. These are different, and non-compatible aspects of the quantum-mechanical state of the kaon, much like two different components of the spin of an electron.


Figure 11: Effective $\Delta S=2$ vertex.

### 5.2.4 Neutral kaon mass difference and the GIM mechanism

A measurement of the oscillation frequency of neutral kaons gives the mass difference $\Delta m=$ $m_{2}-m_{1}$ between the two $C P$ eigenstates. This can be predicted from the theory. To this end, it is convenient to express it in terms of the mixing matrix element between strangeness eigenstates as follows: since (for states normalised to 1 )

$$
\begin{align*}
& m_{j}=\operatorname{Re}\left\langle K_{j}^{0}\right| H_{\mathrm{eff}}\left|K_{j}^{0}\right\rangle, \\
& m_{1}=\frac{1}{2} \operatorname{Re}\left(\left\langle K^{0}\right|-\left\langle\bar{K}^{0}\right|\right) H_{\text {eff }}\left(\left|K^{0}\right\rangle-\left|\bar{K}^{0}\right\rangle\right),  \tag{5.38}\\
& m_{2}=\frac{1}{2} \operatorname{Re}\left(\left\langle K^{0}\right|+\left\langle\bar{K}^{0}\right|\right) H_{\text {eff }}\left(\left|K^{0}\right\rangle+\left|\bar{K}^{0}\right\rangle\right),
\end{align*}
$$

we have

$$
\begin{equation*}
m_{2}-m_{1}=\operatorname{Re}\left(\left\langle\bar{K}^{0}\right| H_{\text {eff }}\left|K^{0}\right\rangle+\left\langle K^{0}\right| H_{\text {eff }}\left|\bar{K}^{0}\right\rangle\right) . \tag{5.39}
\end{equation*}
$$

Since

$$
\begin{equation*}
\left\langle\bar{K}^{0}\right| H_{\mathrm{eff}}\left|K^{0}\right\rangle^{*}=\left\langle K^{0}\right| H_{\mathrm{eff}}^{\dagger}\left|\bar{K}^{0}\right\rangle, \tag{5.40}
\end{equation*}
$$

we have

$$
\begin{align*}
m_{2}-m_{1} & =\frac{1}{2}\left(\left\langle\bar{K}^{0}\right| H_{\mathrm{eff}}\left|K^{0}\right\rangle+\left\langle\bar{K}^{0}\right| H_{\mathrm{eff}}\left|K^{0}\right\rangle^{*}+\left\langle K^{0}\right| H_{\mathrm{eff}}\left|\bar{K}^{0}\right\rangle+\left\langle K^{0}\right| H_{\mathrm{eff}}\left|\bar{K}^{0}\right\rangle^{*}\right) \\
& =\frac{1}{2}\left(\left\langle\bar{K}^{0}\right| H_{\mathrm{eff}}\left|K^{0}\right\rangle+\left\langle K^{0}\right| H_{\mathrm{eff}}^{\dagger}\left|\bar{K}^{0}\right\rangle+\left\langle K^{0}\right| H_{\mathrm{eff}}\left|\bar{K}^{0}\right\rangle+\left\langle\bar{K}^{0}\right| H_{\mathrm{eff}}^{\dagger}\left|K^{0}\right\rangle\right)  \tag{5.41}\\
& =2 \operatorname{Re}\left\langle\bar{K}^{0}\right| H_{\mathrm{eff} H}\left|K^{0}\right\rangle,
\end{align*}
$$

with $H_{\text {eff } H}=\frac{1}{2}\left(H_{\text {eff }}+H_{\text {eff }}^{\dagger}\right)$ the Hermitean part of the effective Hamiltonian. This can be obtained as minus the spatial integral of the Hermitean part of a suitable effective interaction Lagrangian density that describes the oscillation process.

Since $S\left(K^{0}\right)=1$ and $S\left(\bar{K}^{0}\right)=-1$, the neutral kaon oscillation is a $\Delta S=2$ process which requires a second-order weak interaction to take place. If there were a $\Delta S=2$ vertex with some coupling $G_{2}$ (see Fig. 11),

$$
\begin{equation*}
\mathscr{L}^{\Delta S=2}=-G_{2}\left(\bar{d} \mathcal{O}_{L}^{\alpha} s \bar{d} \mathcal{O}_{L \alpha} s+\bar{s} \mathcal{O}_{L}^{\alpha} d \bar{s} \mathcal{O}_{L \alpha} d\right), \tag{5.42}
\end{equation*}
$$

one would easily find $(x=(0, \vec{x}))$ fro relativistically normalised states

$$
\begin{align*}
\Delta m & =2 G_{2} \int d^{3} x \frac{\operatorname{Re}\left\langle\bar{K}^{0}\right| \bar{s} \mathcal{O}_{L}^{\alpha} d(x) \bar{s} \mathcal{O}_{L \alpha} d(x)\left|K^{0}\right\rangle}{\left(\left\langle\bar{K}^{0} \mid \bar{K}^{0}\right\rangle\left\langle K^{0} \mid K^{0}\right\rangle\right)^{\frac{1}{2}}}=\frac{G_{2}}{m_{K}} \operatorname{Re}\left\langle\bar{K}^{0}\right| \bar{s} \mathcal{O}_{L}^{\alpha} d(0) \bar{s} \mathcal{O}_{L \alpha} d(0)\left|K^{0}\right\rangle \\
& =\frac{G_{2}}{m_{K}} \operatorname{Re} \sum_{n}\left\langle\bar{K}^{0}\right| \bar{s} \mathcal{O}_{L}^{\alpha} d(0)|n\rangle\langle n| \bar{s} \mathcal{O}_{L \alpha} d(0)\left|K^{0}\right\rangle  \tag{5.43}\\
& \sim \frac{G_{2}}{m_{K}}\left\langle\bar{K}^{0}\right| \bar{s} \mathcal{O}_{L}^{\alpha} d(0)|0\rangle\langle 0| \bar{s} \mathcal{O}_{L \alpha} d(0)\left|K^{0}\right\rangle=G_{2} f_{K}^{2} m_{K},
\end{align*}
$$



Figure 12: GIM mechanism: four fermion loops with $f_{1}=u, c$ and $f_{2}=u, c$ have to be summed up.
where in the first passage we have braved the laws of mathematics to cancel out a $\delta^{(3)}(0)$ in the numerator, coming from the spatial integral, and a $\delta^{(3)}(0)$ in the denominator, coming from the normalisation of the states, that also leave a factor $\frac{1}{2 m_{k}}$; and in the third passage we have approximated the sum over states with the vacuum contribution, ignoring some numerical factors that will be clarified later. The quantity $f_{K}$ is the same kaon decay constant appearing in $K^{+} \rightarrow \ell^{+} \nu_{\ell}$ decays: in fact,

$$
\begin{equation*}
\langle 0| \bar{s} \mathcal{O}_{L \alpha} d\left|K^{0}\right\rangle=\frac{1}{\sqrt{2}}\langle 0| \bar{s} \mathcal{O}_{L \alpha} u\left|K^{+}\right\rangle=-i p_{\alpha} f_{K} \tag{5.44}
\end{equation*}
$$

due to isospin symmetry. Comparing Eq. (5.43) with the experimental value of $\Delta m=3.5$. $10^{-12} \mathrm{MeV}$ obtained studying kaon oscillations, one finds that one would need $G_{2} \sim 10^{-7} G$.

Although a $\Delta S=2$ vertex like Eq. 5.42 is not present in the $V-A$ theory, it can be obtained as an effective vertex in second order perturbation theory. Since this effective vertex involves a loop integral which diverges quadratically, one needs to impose a cut-off $\Lambda$, which is naturally chosen to be of the order of the $W$-boson mass, where the whole four-fermion interaction picture ceases to be adequate. One then obtains for the effective coupling constant the estimate $G_{2}=G^{2} \Lambda^{2}=G^{2} m_{W}^{2} \simeq 10^{-1} G$. This is too big a coupling to explain the small mass difference $\Delta m$. As shown below, a possible way out of this problem is to assume the existence of a fourth type of quark, the charm $c$, with the same charge as the $u$ quark, and coupled in the same way as the $u$ to weak interactions. This quark would form a second family together with the strange quark, and would be coupled to a combination $s^{\prime}$ of $d$ and $s$, similarly to the $u$ being coupled to the combination $d^{\prime}$ of $d$ and $s$.

Including the $c$ quark, the charged weak hadronic current reads $\bar{u} \mathcal{O}_{L}^{\alpha} d^{\prime}+\bar{c} \mathcal{O}_{L}^{\alpha} s^{\prime}$, and removing all unphysical, unobservable phases, the most general form of $d$-s mixing reads

$$
\binom{d^{\prime}}{s^{\prime}}=\left(\begin{array}{cc}
\cos \theta_{C} & \sin \theta_{C}  \tag{5.45}\\
-\sin \theta_{C} & \cos \theta_{C}
\end{array}\right)\binom{d}{s}=V_{C}\binom{d}{s}=\left(\begin{array}{ll}
\left(V_{C}\right)_{u d} & \left(V_{C}\right)_{u s} \\
\left(V_{C}\right)_{c d} & \left(V_{C}\right)_{c s}
\end{array}\right)\binom{d}{s}
$$

Repeating the calculation outlined above, one finds know not one but four loop diagrams, either with identical upper and lower line corresponding to $u u$ or $c c$, or with different upper and lower line corresponding to $u c c u$, see Fig. 12. The couplings corresponding to the two cases are $\sin ^{2} \theta_{C} \cos ^{2} \theta_{C}$ and $-\sin ^{2} \theta_{C} \cos ^{2} \theta_{C}$, and so if $u$ and $c$ had the same mass these four loop diagrams would cancel each other out exactly. For sure, large-momentum contributions to the loop integrals do cancel out exactly, since at very large loop momenta all quarks are effectively
massless, and so the sum of the four loop diagrams is finite and no UV cutoff is needed. A detailed calculation shows that the effective coupling of the $\Delta S=2$ effective vertex is

$$
\begin{equation*}
G_{2}=\frac{G^{2}}{(4 \pi)^{2}} \sin ^{2} \theta_{C} \cos ^{2} \theta_{C}\left(m_{c}-m_{u}\right)^{2} \tag{5.46}
\end{equation*}
$$

The mechanism discussed above, suppressing the unwanted large mass difference by having a fourth quark running through loops, is known as the GIM mechanism after Glashow, Iliopoulos and Maiani who proposed it.

Going now back to the second line of Eq. (5.44), treating the neutral kaons as pure $q \bar{q}$ states, $K^{0}=d \bar{s}$ and $\bar{K}^{0}=s \bar{d}$, and properly taking into account the colour degree of freedom of quarks we obtain the estimate

$$
\begin{align*}
\Delta m & =\frac{G_{2}}{m_{K}} \operatorname{Re}\left\langle\bar{K}^{0}\right| \bar{s} \mathcal{O}_{L}^{\alpha} d(0) \bar{s} \mathcal{O}_{L \alpha} d(0)\left|K^{0}\right\rangle \simeq \frac{8}{3} \frac{G_{2}}{m_{K}} \operatorname{Re}\left\langle\bar{K}^{0}\right| \bar{s} \mathcal{O}_{L}^{\alpha} d|0\rangle\langle 0| \bar{s} \mathcal{O}_{L \alpha} d\left|K^{0}\right\rangle \\
& =\frac{8}{3 m_{K}} \frac{G^{2}}{(4 \pi)^{2}} \sin ^{2} \theta_{C} \cos ^{2} \theta_{C}\left(m_{c}-m_{u}\right)^{2} f_{K}^{2} m_{K}^{2} \simeq \frac{G^{2}}{6 \pi} \sin ^{2} \theta_{C} \cos ^{2} \theta_{C} m_{c}^{2} f_{K}^{2} m_{K} \tag{5.47}
\end{align*}
$$

The factor $\frac{8}{3}$ arises as follows. The first matrix element in Eq. (5.47) can be computed pairing the $d$ and $\bar{s}$ fields with the $d, \bar{d}$ and $s, \bar{s}$ quarks in the initial and final states. Colour indices are contracted within each operator, i.e., $\sum_{i} \bar{s}_{i} \mathcal{O}_{L}^{\alpha} d_{i}$, and within each meson, i.e., $\sum_{i}\left|d_{i} \bar{s}_{i}\right\rangle$ and $\sum_{i}\left\langle s_{i} \bar{d}_{i}\right|$. There are two types of operator-state pairing: one where each $\bar{s} \mathcal{O}_{L}^{\alpha} d$ pairs with a single state, and one where each $\bar{s} \mathcal{O}_{L}^{\alpha} d$ pairs with one quark from both states. Each of these pairings can be done in two equivalent ways. For the first type of pairing, one has to compute in practice the product of the matrix elements $\langle 0| \bar{s}^{\prime} \mathcal{O}_{L \alpha} d\left|K^{0}\right\rangle=\sum_{i}\langle 0| \bar{s}_{i} \mathcal{O}_{L \alpha} d_{i}\left|d_{i} \bar{s}_{i}\right\rangle=N_{c}\langle 0| \bar{s}_{1} \mathcal{O}_{L \alpha} d_{1}\left|d_{1} \bar{s}_{1}\right\rangle$ and $\left\langle\bar{K}^{0}\right| \bar{s} \mathcal{O}_{L}^{\alpha} d|0\rangle=\sum_{i}\left\langle s_{i} \bar{d}_{i}\right| \bar{s}_{i} \mathcal{O}_{L}^{\alpha} d_{i}|0\rangle=N_{c}\left\langle s_{1} \bar{d}_{1}\right| \bar{s}_{1} \mathcal{O}_{L}^{\alpha} d_{1}|0\rangle$, having taken into account that each colour gives the same contribution. For the second type of pairing one has instead

$$
\begin{align*}
& \sum_{i j k l} \sum_{a b c d}\left\langle s_{i} \bar{d}_{i}\right| \bar{s}_{a k} d_{d l}|0\rangle\langle 0| \bar{s}_{c l} d_{b k}\left|d_{j} \bar{s}_{j}\right\rangle\left(\mathcal{O}_{L}^{\alpha} \otimes \mathcal{O}_{L \alpha}\right)_{a b, c d} \\
& =\sum_{i j k l} \sum_{a b c d} \delta_{i k} \delta_{i l} \delta_{j l} \delta_{j k}\left\langle s_{1} \bar{d}_{1}\right| \bar{s}_{a 1} d_{d 1}|0\rangle\langle 0| \bar{s}_{c 1} d_{b 1}\left|d_{1} \bar{s}_{1}\right\rangle\left(\mathcal{O}_{L}^{\alpha} \otimes \mathcal{O}_{L \alpha}\right)_{a b, c d}  \tag{5.48}\\
& =N_{c}\left\langle s_{1} \bar{d}_{1}\right| \bar{s}_{1} \mathcal{O}_{L}^{\alpha} d_{1}|0\rangle\langle 0| \bar{s}_{1} \mathcal{O}_{L \alpha} d_{1}\left|d_{1} \bar{s}_{1}\right\rangle
\end{align*}
$$

where $a, b, c, d$ are Dirac indices. The full contribution is then

$$
\begin{equation*}
2\left(N_{c}^{2}+N_{c}\right)\left\langle s_{1} \bar{d}_{1}\right| \bar{s}_{1} \mathcal{O}_{L}^{\alpha} d_{1}|0\rangle\langle 0| \bar{s}_{1} \mathcal{O}_{L \alpha} d_{1}\left|d_{1} \bar{s}_{1}\right\rangle=2 \frac{N_{c}^{2}+N_{c}}{N_{c}^{2}}\left\langle\bar{K}^{0}\right| \bar{s} \mathcal{O}_{L}^{\alpha} d|0\rangle\langle 0| \bar{s} \mathcal{O}_{L \alpha} d\left|K^{0}\right\rangle, \tag{5.49}
\end{equation*}
$$

and setting $N_{c}=3$ one finds $2\left(1+\frac{1}{3}\right)=\frac{8}{3}$. Comparing with the $K^{+} \rightarrow \mu^{+} \nu_{\mu}$ decay width one finds

$$
\begin{equation*}
\Delta m \simeq \frac{4 \cos ^{2} \theta_{C} m_{c}^{2}}{3 \pi m_{\mu}^{2}} \Gamma\left(K^{+} \rightarrow \mu^{+} \nu_{\mu}\right) \tag{5.50}
\end{equation*}
$$

From the known values of $\Gamma\left(K^{+} \rightarrow \mu^{+} \nu_{\mu}\right), \cos \theta_{C}$, and $m_{\mu}$, one can predict the charm mass.

## $6 \quad \tau$ decays

The $\tau$ lepton was discovered in 1975 at SLAC in $e^{+} e^{-}$collisions. The corresponding neutrino, $\nu_{\tau}$, was immediately theorised but observed only much later by the DONUT experiment at Fermilab in 2000. The $\tau$ has spin $\frac{1}{2}$, mass $m_{\tau}=1.78 \mathrm{GeV}$, and lifetime $\tau_{\tau}=3.4 \cdot 10^{-13} \mathrm{~s}$. Under the assumption of leptonic universality, the current $\bar{\nu}_{\tau} \mathcal{O}_{L}^{\alpha} \tau$ has to be added to the charged weak leptonic current with the same coupling as the electronic and muonic ones, i.e.,

$$
\begin{equation*}
J_{l}^{\alpha}=\bar{\nu}_{e} \mathcal{O}_{L}^{\alpha} e+\bar{\nu}_{\mu} \mathcal{O}_{L}^{\alpha} \mu+\bar{\nu}_{\tau} \mathcal{O}_{L}^{\alpha} \tau \tag{6.1}
\end{equation*}
$$

A new feature is that besides the leptonic decays $\tau \rightarrow \ell \bar{\nu}_{\ell} \nu_{\tau}, \ell=e, \mu$, the heavy mass of the tau lepton allows also semi-hadronic decays $\tau \rightarrow \nu_{\tau}+$ hadrons. Since the tau is lighter than the lightest charmed particle, $m_{\tau}<m_{D^{0}}=1.864 \mathrm{GeV}$ (see below), only decays involving $u, d, s$ quarks are allowed.

Leptonic decays The decay rate for leptonic decays can be obtained exactly as in the case of muon decay $\mu \rightarrow e \bar{\nu}_{e} \nu_{\mu}$. Since $m_{e} \ll m_{\mu} \ll m_{\tau}$, for both decays of the $\tau$ one can treat the final lepton as massless. In this approximation, one thus has only to replace $m_{\mu} \rightarrow m_{\tau}$ in Eq. (2.32), and get ( $m_{\mu}=106 \mathrm{MeV}, \tau_{\mu}=2.2 \cdot 10^{-6} s$ )

$$
\begin{equation*}
\Gamma\left(\tau \rightarrow e \bar{\nu}_{e} \nu_{\tau}\right)=\Gamma\left(\tau \rightarrow \mu \bar{\nu}_{\mu} \nu_{\tau}\right)=\frac{G^{2} m_{\tau}^{5}}{192 \pi^{3}}=\left(\frac{m_{\tau}}{m_{\mu}}\right)^{5} \Gamma\left(\mu \rightarrow e \bar{\nu}_{e} \nu_{\mu}\right) \simeq 6.1 \cdot 10^{11} \mathrm{~s}^{-1} \tag{6.2}
\end{equation*}
$$

Semi-hadronic decays For decay or scattering processes governed by electromagnetic or weak interactions, but producing hadrons in the final state, one can ideally separate the process into two parts. At first, electromagnetic or weak interactions produce quarks: for example, an $e^{+} e^{-}$pair annihilates into a photon which subsequently turns into a quark-antiquark pair $q_{i} \bar{q}_{i}$ pair for some flavour $q$ and colour $i$ of quarks. After this, the hadronisation process takes place, during which strong interactions build up hadrons from the quarks. If one is interested in inclusive processes, it suffices to know what is the total cross section or decay width for quark production starting from the given initial state: after this, the resulting quarks will become hadrons with probability one, and even though we do not know well how hadronisation works we can rest assured that it will always take place. The total hadron production rate coincides then with the total quark-antiquark production rate.

In computing the total cross section or decay width of interest it suffices to consider the production of a single quark-antiquark pair, since electromagnetic and weak interactions have small coupling constants and producing more pairs is suppressed. On the other hand, this stage of the process may receive large corrections from additional strong interaction effects in the intermediate quark-antiquark state. However, thanks to the asymptotic freedom property of QCD, at high energy it is possible to use perturbation theory to estimate these corrections. For example, for hadron production in $e^{+} e^{-}$collisions at total centre of mass energy squared $s$ one has

$$
\begin{equation*}
R(s) \equiv \frac{\sigma\left(e^{+} e^{-} \rightarrow \text { hadrons }\right)}{\sigma\left(e^{+} e^{-} \rightarrow \mu^{+} \mu^{-}\right)}=\frac{\sum_{f} \sigma\left(e^{+} e^{-} \rightarrow \bar{f} f\right)}{\sigma\left(e^{+} e^{-} \rightarrow \mu^{+} \mu^{-}\right)}=3 \sum_{f} Q_{f}^{2}\left(1+\frac{\alpha_{s}(s)}{\pi}+\ldots\right), \tag{6.3}
\end{equation*}
$$

where $\alpha_{s}(s) \equiv \frac{g_{s}(s)^{2}}{4 \pi}$ with $g_{s}\left(\mu^{2}\right)$ the running coupling constant of QCD, evaluated here at the energy scale squared $s$. Moreover, $Q_{f}$ is the electric charge of quark $f$, and the sum runs over
those flavours $f$ such that $4 m_{f}^{2} \leq s$. The factor 3 corresponds to the number of colours. Ignoring next-to-leading order effects, for $s<4 m_{c}^{2}$ one has $R \simeq 3\left(\frac{4}{9}+\frac{1}{9}+\frac{1}{9}\right)=2$. As the threshold for $c \bar{c}$ production is crossed, this should increase to $R \simeq 2+3 \frac{4}{9}=\frac{10}{3}$, and further to $R \simeq \frac{10}{3}+3 \frac{1}{9}=\frac{11}{3}$ as soon as also $b \bar{b}$ production becomes possible.

Using the same idea, one can estimate $\Gamma\left(\tau \rightarrow \nu_{\tau}+\right.$ hadrons $)$. The three processes $\tau \rightarrow \nu_{\tau} \mu \bar{\nu}_{\mu}$, $\tau \rightarrow \nu_{\tau} u \bar{d}$ and $\tau \rightarrow \nu_{\tau} u \bar{s}$ are described by the same diagram except for the value of the coupling constant, which is $G, G \cos \theta_{C}$ and $G \sin \theta_{C}$ in the three cases, respectively. The quark and the antiquark produced in the process must have the same colour, but this can be any of the $N_{c}=3$ possible colours; all the possibile outcomes must be included in the total decay rate. Ignoring corrections to the tree-level contributions, since they are of order $\mathcal{O}\left(\alpha_{s}\left(m_{\tau}^{2}\right)\right)$, we find

$$
\begin{equation*}
\frac{\Gamma\left(\tau \rightarrow \nu_{\tau}+\text { hadrons }\right)}{\Gamma\left(\tau \rightarrow \nu_{\tau} \mu \bar{\nu}_{\mu}\right)}=\frac{\Gamma\left(\tau \rightarrow \nu_{\tau} u \bar{d}\right)+\Gamma\left(\tau \rightarrow \nu_{\tau} u \bar{s}\right)}{\Gamma\left(\tau \rightarrow \nu_{\tau} \mu \bar{\nu}_{\mu}\right)}=\left(\cos ^{2} \theta_{C}+\sin ^{2} \theta_{C}\right) N_{c}=3 \tag{6.4}
\end{equation*}
$$

so that $\frac{\Gamma\left(\tau \rightarrow \nu_{\tau}+\text { hadrons }\right)}{\sum_{\ell} \Gamma\left(\tau \rightarrow \ell \bar{\nu}_{\ell} \nu_{\tau}\right)}=\frac{3}{2}$.
Total width and lifetime Using Eqs. (6.2) and (6.4) we can estimate the total width of the $\tau$ lepton:

$$
\begin{equation*}
\Gamma(\tau)=\Gamma\left(\tau \rightarrow \nu_{\tau}+\text { hadrons }\right)+\Gamma\left(\tau \rightarrow \nu_{\tau} e \bar{\nu}_{e}\right)+\Gamma\left(\tau \rightarrow \nu_{\tau} \mu \bar{\nu}_{\mu}\right) \simeq 5 \Gamma\left(\tau \rightarrow \nu_{\tau} \mu \bar{\nu}_{\mu}\right) \tag{6.5}
\end{equation*}
$$

from which it follows

$$
\begin{equation*}
\tau_{\tau} \simeq \frac{1}{5 \Gamma\left(\tau \rightarrow \nu_{\tau} \mu \bar{\nu}_{\mu}\right)}=3.3 \cdot 10^{-13} \mathrm{~s} \tag{6.6}
\end{equation*}
$$

which compares well with the experimental value $\tau_{\tau}=3.4 \cdot 10^{-13} \mathrm{~s}$.

### 6.1 Semi-hadronic decay modes

We discuss now two specific decay modes of the $\tau$ involving hadrons in the final state.

### 6.1.1 $\quad \tau^{-} \rightarrow \pi^{-} \nu_{\tau}$ and $\tau^{-} \rightarrow K^{-} \nu_{\tau}$

The decay amplitude reads

$$
\begin{align*}
\mathcal{M}_{\mathrm{fi}} & =-\frac{G \cos \theta_{C}}{\sqrt{2}} \bar{u}_{\nu}\left(p_{\nu}\right) \gamma^{\alpha}\left(1-\gamma^{5}\right) u_{\tau}\left(p_{\tau}\right)\left\langle\pi^{-}\right|\left(\bar{d} \gamma_{\alpha}\left(1-\gamma^{5}\right) u\right)(0)|0\rangle \\
& =i \frac{G \cos \theta_{C}}{\sqrt{2}} f_{\pi} \sqrt{2} \bar{u}_{\nu}\left(p_{\nu}\right) \phi_{\pi}\left(1-\gamma^{5}\right) u_{\tau}\left(p_{\tau}\right) \tag{6.7}
\end{align*}
$$

where we used the already known matrix element [see Eq. (3.32)]

$$
\begin{align*}
\left\langle\pi^{-}\right|\left(\bar{d} \gamma_{\alpha}\left(1-\gamma^{5}\right) u\right)(0)|0\rangle^{*} & =\langle 0|\left(\bar{u} \gamma_{\alpha}\left(1-\gamma^{5}\right) d\right)(0)\left|\pi^{-}\right\rangle=\langle 0|\left(\bar{d} \gamma_{\alpha}\left(1-\gamma^{5}\right) u\right)(0)\left|\pi^{+}\right\rangle \\
& =-\langle 0|\left(\bar{d} \gamma_{\alpha} \gamma^{5} u\right)(0)\left|\pi^{+}\right\rangle=-i \sqrt{2} f_{\pi}\left(p_{\pi}\right)_{\alpha} \tag{6.8}
\end{align*}
$$

The pion decay constant $f_{\pi}=92 \mathrm{MeV}$ is exactly the same appearing in the amplitude for pion decay. Using now momentum conservation, $p_{\pi}=p_{\tau}-p_{\nu}$, we find

$$
\begin{equation*}
\bar{u}_{\nu}\left(p_{\nu}\right) p_{\pi}\left(1-\gamma^{5}\right) u_{\tau}\left(p_{\tau}\right)=\bar{u}_{\nu}\left(p_{\nu}\right) p_{\tau}\left(1-\gamma^{5}\right) u_{\tau}\left(p_{\tau}\right)=m_{\tau} \bar{u}_{\nu}\left(p_{\nu}\right)\left(1+\gamma^{5}\right) u_{\tau}\left(p_{\tau}\right) . \tag{6.9}
\end{equation*}
$$

Squaring the amplitude, averaging over the spins of the $\tau$ and integrating over the phase space of the final state, we obtain for the total decay width after including the appropriate factors

$$
\begin{align*}
\Gamma\left(\tau^{-} \rightarrow \pi^{-} \nu_{\tau}\right) & =\frac{\Phi^{(2)} G^{2} \cos ^{2} \theta_{C} m_{\tau}^{2} f_{\pi}^{2}}{2 m_{\tau}} \frac{1}{2} \sum_{s_{\tau}} \bar{u}_{\nu}\left(p_{\nu}\right)\left(1+\gamma^{5}\right) u_{\tau}\left(p_{\tau}\right) \bar{u}_{\tau}\left(p_{\tau}\right)\left(1-\gamma^{5}\right) u_{\nu}\left(p_{\nu}\right) \\
& =\frac{G^{2} \cos ^{2} \theta_{C}}{4} m_{\tau} f_{\pi}^{2} \Phi^{(2)} \operatorname{tr}\left(\not p_{\tau}+m_{\tau}\right)\left(1-\gamma^{5}\right) \not p_{\nu}\left(1+\gamma^{5}\right)  \tag{6.10}\\
& =2 G^{2} \cos ^{2} \theta_{C} m_{\tau} f_{\pi}^{2} \Phi^{(2)} p_{\tau} \cdot p_{\nu} .
\end{align*}
$$

Notice that we have integrated over phase space before evaluating the spin-summed matrix element: this is allowed since this matrix element can only depend on Lorentz-invariant combinations of the final momenta, and since the process has a two-body final state where such invariants are fixed by four-momentum conservation. Recalling that

$$
\begin{equation*}
\Phi^{(2)}=\frac{p_{\mathrm{CM}}}{4 \pi E_{\mathrm{CM}}}=\frac{E_{\nu}}{4 \pi m_{\tau}}, \tag{6.11}
\end{equation*}
$$

and that

$$
\begin{equation*}
p_{\tau} \cdot p_{\nu}=m_{\tau} E_{\nu}, \quad E_{\nu}=\frac{m_{\tau}^{2}-m_{\pi}^{2}}{2 m_{\tau}}=\frac{m_{\tau}}{2}\left(1-\frac{m_{\pi}^{2}}{m_{\tau}^{2}}\right) \tag{6.12}
\end{equation*}
$$

we obtain

$$
\begin{align*}
\Gamma\left(\tau^{-} \rightarrow \pi^{-} \nu_{\tau}\right) & =\frac{G^{2} \cos ^{2} \theta_{C} f_{\pi}^{2}}{2 \pi} m_{\tau} E_{\nu}^{2}=\frac{G^{2} \cos ^{2} \theta_{C} f_{\pi}^{2}}{8 \pi} m_{\tau}^{3}\left(1-\frac{m_{\pi}^{2}}{m_{\tau}^{2}}\right)^{2} \\
& =\Gamma\left(\tau \rightarrow \mu \bar{\nu}_{\mu} \nu_{\tau}\right) \frac{24 \pi^{2}}{m_{\tau}^{2}} \cos ^{2} \theta_{C} f_{\pi}^{2}\left(1-\frac{m_{\pi}^{2}}{m_{\tau}^{2}}\right)^{2} \simeq 0.6 \Gamma\left(\tau \rightarrow \mu \bar{\nu}_{\mu} \nu_{\tau}\right) \tag{6.13}
\end{align*}
$$

The decay width for the process $\tau^{-} \rightarrow K^{-} \nu_{\tau}$ is obtained from Eq. (6.13) simply by replacing $m_{\pi} \rightarrow m_{K}, f_{\pi} \rightarrow f_{K}$ and $\cos \theta_{C} \rightarrow \sin \theta_{C}$. Using the experimental values $m_{K}=495 \mathrm{MeV}$, $f_{K}=1.2 f_{\pi}, \sin \theta_{C}=0.22$ and $\cos \theta_{C}=0.97$, we find

$$
\begin{equation*}
\Gamma\left(\tau^{-} \rightarrow K^{-} \nu_{\tau}\right)=\Gamma\left(\tau^{-} \rightarrow \pi^{-} \nu_{\tau}\right)\left(\frac{f_{K}}{f_{\pi}}\right)^{2}\left(\frac{\sin \theta_{C}}{\cos \theta_{C}}\right)^{2}\left(\frac{1-\frac{m_{K}^{2}}{m_{\tau}^{2}}}{1-\frac{m_{\pi}^{2}}{m_{\tau}^{2}}}\right)^{2} \simeq 0.06 \tag{6.14}
\end{equation*}
$$

in reasonable agreement with the experimental value 0.07 .

### 6.1.2 $\tau^{-} \rightarrow \rho^{-} \nu_{\tau}$

The rho mesons are vector particles $(J=1, P=-1)$ forming an isotriplet $(I=1)$, with masses $m_{\rho}=770 \mathrm{MeV}$. Their quark content is the same as that of the pions ( $\rho^{+}=-u \bar{d}$, $\left.\rho^{0}=(u \bar{u}-d \bar{d}) / \sqrt{2}, \rho^{-}=d \bar{u}\right)$, but the quark spin state is $S=1$ instead of $S=0$. The analysis of this decay process follows the same lines of the decay into a pion discussed above, except that now the hadronic current matrix element has a different parameterisation:

$$
\begin{equation*}
H_{\alpha}=\left\langle\rho^{-}\right| \bar{d} \gamma_{\alpha}\left(1-\gamma^{5}\right) u|0\rangle=\left\langle\rho^{-}\right| \bar{d} \gamma_{\alpha} u|0\rangle=g_{\rho} \varepsilon_{\alpha}+f_{\rho} p_{\alpha} . \tag{6.15}
\end{equation*}
$$

In fact, only the vector current contributes, and there are two possible vectors, the fourmomentum $p$ of the rho meson and its polarisation vector $\varepsilon_{\alpha}$ (which is a dimensionless quantity). Being a vector particle, there are three independent polarisation vectors, corresponding
to $J_{z}=0, \pm 1$, which satisfy $p \cdot \varepsilon=0$. This is because the polarisation vector has only spatial components in the rest frame of the rho, and so must be orthogonal (in the Minkowskian sense) to $p{ }^{38}$ For future utility, we notice that the sum over three independent and orthogonal polarisations yields

$$
\begin{equation*}
\sum_{s} \varepsilon_{\alpha}^{(s)} \varepsilon_{\beta}^{(s) *}=-\eta_{\alpha \beta}+\frac{p_{\alpha} p_{\beta}}{m_{\rho}^{2}} \tag{6.16}
\end{equation*}
$$

where we allow $\varepsilon$ to be complex in order to describe circular polarisations. Imposing now conservation of the vector current (we work in the isospin limit) we get

$$
\begin{equation*}
0=p \cdot V=f_{\rho} m_{\rho}^{2} \Rightarrow f_{\rho}=0 \tag{6.17}
\end{equation*}
$$

We then have $H_{\alpha}=V_{\alpha}=g_{\rho} \varepsilon_{\alpha}$, with $g_{\rho}$ a constant of mass dimension $m^{2}$.
We can now write the decay amplitude,

$$
\begin{equation*}
\mathcal{M}_{\mathrm{fi}}=-\frac{G}{\sqrt{2}} \cos \theta_{C} g_{\rho} \varepsilon_{\alpha} \bar{u}_{\nu}\left(p_{\nu}\right) \gamma^{\alpha}\left(1-\gamma^{5}\right) u_{\tau}\left(p_{\tau}\right) \tag{6.18}
\end{equation*}
$$

and the spin-summed amplitude square,

$$
\begin{align*}
\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle & =\frac{G^{2}}{2} \cos ^{2} \theta_{C} g_{\rho}^{2}\left(-\eta_{\alpha \beta}+\frac{p_{\alpha} p_{\beta}}{m_{\rho}^{2}}\right) \operatorname{tr} \gamma^{\alpha}\left(1-\gamma^{5}\right)\left(\not p_{\tau}+m_{\tau}\right) \gamma^{\beta}\left(1-\gamma^{5}\right) \not p_{\nu} \\
& =G^{2} \cos ^{2} \theta_{C} g_{\rho}^{2}\left(-\eta_{\alpha \beta}+\frac{p_{\alpha} p_{\beta}}{m_{\rho}^{2}}\right) \operatorname{tr} \gamma^{\alpha} \not \phi_{\tau} \gamma^{\beta} \not p_{\nu} \\
& =G^{2} \cos ^{2} \theta_{C} g_{\rho}^{2}\left(-\operatorname{tr} \gamma^{\alpha} \not p_{\tau} \gamma_{\alpha} \not \phi_{\nu}+\frac{1}{m_{\rho}^{2}} \operatorname{tr} \not \not \not p_{\tau} \not \not p p_{\nu}\right)  \tag{6.19}\\
& =G^{2} \cos ^{2} \theta_{C} g_{\rho}^{2}\left(2 \operatorname{tr} \not p_{\tau} \not \phi_{\nu}+\frac{1}{m_{\rho}^{2}} \operatorname{tr} \not p \not \phi_{\tau} \not p \not p_{\nu}\right) \\
& =4 G^{2} \cos ^{2} \theta_{C} g_{\rho}^{2}\left(2 p_{\tau} \cdot p_{\nu}+\frac{1}{m_{\rho}^{2}}\left(2 p \cdot p_{\tau} p \cdot p_{\nu}-m_{\rho}^{2} p_{\tau} \cdot p_{\nu}\right)\right) \\
& =4 G^{2} \cos ^{2} \theta_{C} \frac{g_{\rho}^{2}}{m_{\rho}^{2}}\left(2 p \cdot p_{\tau} p \cdot p_{\nu}+p_{\tau} \cdot p_{\nu} m_{\rho}^{2}\right) .
\end{align*}
$$

Using now $p \cdot p_{\nu}=\left(p_{\tau}-p_{\nu}\right) \cdot p_{\nu}=p_{\tau} \cdot p_{\nu}$, and $0=p_{\nu}^{2}=m_{\tau}^{2}+m_{\rho}^{2}-2 p \cdot p_{\tau}$, we obtain

$$
\begin{align*}
\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{f}}\right|^{2}\right\rangle\right\rangle & =4 G^{2} \cos ^{2} \theta_{C} \frac{g_{\rho}^{2}}{m_{\rho}^{2}} p_{\tau} \cdot p_{\nu}\left(2 p \cdot p_{\tau}+m_{\rho}^{2}\right) \\
& =4 G^{2} \cos ^{2} \theta_{C} \frac{g_{\rho}^{2}}{m_{\rho}^{2}} p_{\tau} \cdot p_{\nu}\left(2 m_{\rho}^{2}+m_{\tau}^{2}\right) . \tag{6.20}
\end{align*}
$$

Including all the appropriate factors we obtain for the decay width

$$
\begin{equation*}
\Gamma=\frac{1}{2} \frac{1}{2 m_{\tau}} \Phi^{(2)} 4 G^{2} \cos ^{2} \theta_{C} \frac{g_{\rho}^{2}}{m_{\rho}^{2}} p_{\tau} \cdot p_{\nu} m_{\tau}^{2}\left(1+\frac{2 m_{\rho}^{2}}{m_{\tau}^{2}}\right) \tag{6.21}
\end{equation*}
$$

[^24]The phase space is again

$$
\begin{equation*}
\Phi^{(2)}=\frac{p_{\mathrm{CM}}}{4 \pi E_{\mathrm{CM}}}=\frac{E_{\nu}}{4 \pi m_{\tau}} \tag{6.22}
\end{equation*}
$$

and since $m_{\rho}^{2}=m_{\tau}^{2}-2 p_{\tau} \cdot p_{\nu}=m_{\tau}^{2}-2 m_{\tau} E_{\nu}$ we find

$$
\begin{align*}
\Gamma & =\frac{1}{4 m_{\tau}} \frac{E_{\nu}}{4 \pi m_{\tau}} 4 G^{2} \cos ^{2} \theta_{C} \frac{g_{\rho}^{2}}{m_{\rho}^{2}} m_{\tau}^{3} E_{\nu}\left(1+\frac{2 m_{\rho}^{2}}{m_{\tau}^{2}}\right) \\
& =\frac{G^{2} \cos ^{2} \theta_{C}}{4 \pi}\left(\frac{g_{\rho}}{m_{\rho}}\right)^{2} m_{\tau} E_{\nu}^{2}\left(1+\frac{2 m_{\rho}^{2}}{m_{\tau}^{2}}\right)  \tag{6.23}\\
& =\frac{G^{2} \cos ^{2} \theta_{C}}{16 \pi}\left(\frac{g_{\rho}}{m_{\rho}}\right)^{2} m_{\tau}^{3}\left(1-\frac{m_{\rho}^{2}}{m_{\tau}^{2}}\right)^{2}\left(1+\frac{2 m_{\rho}^{2}}{m_{\tau}^{2}}\right)
\end{align*}
$$

The unknown constant $g_{\rho}$ can be related to the electromagnetic decay process $\rho^{0} \rightarrow e^{+} e^{-}$using isospin invariance. The amplitude for this process reads

$$
\begin{equation*}
\mathcal{M}_{\mathrm{fi}}=\frac{4 \pi \alpha_{\mathrm{em}}}{q^{2}} \bar{u}_{e} \gamma^{\mu} v_{e}\langle 0| J_{\mathrm{em} \mu}\left|\rho^{0}\right\rangle \tag{6.24}
\end{equation*}
$$

where $J_{\mathrm{em} \mu}$ is the electromagnetic current, and $q=p_{e^{+}}+p_{e^{-}}$is the rho meson momentum. Using conservation of the current, the hadronic matrix element can be written as

$$
\begin{equation*}
\langle 0| J_{\mathrm{em} \mu}\left|\rho^{0}\right\rangle=\frac{m_{\rho}^{2}}{\gamma} \varepsilon_{\mu} \tag{6.25}
\end{equation*}
$$

with $\varepsilon_{\mu}$ the polarisation vector and $\gamma$ a dimensionless constant. On the other hand, using the explicit form of the electromagnetic current we find that

$$
\begin{align*}
\langle 0| J_{\mathrm{em} \mu}\left|\rho^{0}\right\rangle & =\langle 0| \frac{2}{3} \bar{u} \gamma_{\mu} u-\frac{1}{3} \bar{d} \gamma_{\mu} d\left|\rho^{0}\right\rangle=\langle 0| \underbrace{\frac{1}{2}\left(\bar{u} \gamma_{\mu} u-\bar{d} \gamma_{\mu} d\right)}_{I=1}+\underbrace{\frac{1}{6}\left(\bar{u} \gamma_{\mu} u+\bar{d} \gamma_{\mu} d\right)}_{I=0} \underbrace{\left|\rho^{0}\right\rangle}_{I=1}  \tag{6.26}\\
& =\frac{1}{\sqrt{2}}\langle 0| \frac{1}{\sqrt{2}}\left(\bar{u} \gamma_{\mu} u-\bar{d} \gamma_{\mu} d\right)\left|\rho^{0}\right\rangle=\frac{1}{\sqrt{2}}\langle 0| \bar{u} \gamma_{\mu} d\left|\rho^{-}\right\rangle=\frac{1}{\sqrt{2}} g_{\rho} \varepsilon_{\mu}
\end{align*}
$$

where on the second line we used the fact that the isovector part of the electromagnetic current and the charged weak current belong to the same isomultiplet. Comparing Eqs. (6.25) and (6.26) we find

$$
\begin{equation*}
\frac{g_{\rho}}{m_{\rho}}=\sqrt{2} \frac{m_{\rho}}{\gamma} \tag{6.27}
\end{equation*}
$$

The amplitude square summed over spins is

$$
\begin{align*}
\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{fi}}\right|^{2}\right\rangle\right\rangle & =\left(\frac{4 \pi \alpha_{\mathrm{em}}}{\gamma}\right)^{2} \operatorname{tr}\left[\gamma^{\mu}\left(\not p_{e^{+}}-m_{e}\right) \gamma^{\nu}\left(\not p_{e^{-}}+m_{e}\right)\right]\left(-\eta_{\mu \nu}+\frac{q_{\mu} q_{\nu}}{m_{\rho}^{2}}\right)^{2} \\
& =4\left(\frac{4 \pi \alpha_{\mathrm{em}}}{\gamma}\right)^{2}\left(2 p_{e^{+}} \cdot p_{e^{-}}+4 m_{e}^{2}+\frac{1}{m_{\rho}^{2}}\left(2 q \cdot p_{e^{+}} q \cdot p_{e^{-}}-q^{2} p_{e^{+}} \cdot p_{e^{-}}\right)-\frac{m_{e}^{2}}{m_{\rho}^{2}} q^{2}\right) \\
& =4\left(\frac{4 \pi \alpha_{\mathrm{em}}}{\gamma}\right)^{2}\left(p_{e^{+}} \cdot p_{e^{-}}+3 m_{e}^{2}+\frac{2}{m_{\rho}^{2}} q \cdot p_{e^{+}} q \cdot p_{e^{-}}\right) \tag{6.28}
\end{align*}
$$

Since $m_{e}^{2} \ll m_{\rho}^{2}$ we can treat the leptons as massless, so that $m_{\rho}^{2}=2 p_{e^{+}} \cdot p_{e^{-}}$and $m_{\rho}^{2}=2 q \cdot p_{e^{ \pm}}$, and so

$$
\begin{equation*}
\left.\left\langle\left.\langle | \mathcal{M}_{\mathrm{fi}}\right|^{2}\right\rangle\right\rangle=4\left(\frac{4 \pi \alpha_{\mathrm{em}}}{\gamma}\right)^{2}\left(\frac{m_{\rho}^{2}}{2}+\frac{m_{\rho}^{2}}{2}\right)=4 m_{\rho}^{2}\left(\frac{4 \pi \alpha_{\mathrm{em}}}{\gamma}\right)^{2} . \tag{6.29}
\end{equation*}
$$

For the decay width we then obtain $\left(\Phi^{(2)}=\left|\vec{p}_{e^{ \pm}}\right| /\left(4 \pi m_{\rho}\right)=\left|E_{e^{ \pm}}\right| /\left(4 \pi m_{\rho}\right)=\left(m_{\rho} / 2\right) /\left(4 \pi m_{\rho}\right)=\right.$ $1 /(8 \pi)$ )

$$
\begin{equation*}
\Gamma=\frac{1}{3} \frac{1}{2 m_{\rho}} \Phi^{(2)} 4 m_{\rho}^{2}\left(\frac{4 \pi \alpha_{\mathrm{em}}}{\gamma}\right)^{2}=\frac{4 \pi}{3} \frac{\alpha_{\mathrm{em}}^{2}}{\gamma^{2}} m_{\rho}=\frac{4 \pi}{3} \frac{\alpha_{\mathrm{em}}^{2}}{m_{\rho}}\left(\frac{m_{\rho}}{\gamma}\right)^{2} \tag{6.30}
\end{equation*}
$$

and so

$$
\begin{equation*}
\left(\frac{m_{\rho}}{\gamma}\right)^{2}=\frac{3}{4 \pi} \frac{m_{\rho}}{\alpha_{\mathrm{em}}^{2}} \Gamma\left(\rho^{0} \rightarrow e^{+} e^{-}\right) \tag{6.31}
\end{equation*}
$$

Comparison with experiment yields

$$
\begin{equation*}
\frac{\gamma^{2}}{4 \pi}=\frac{\alpha_{\mathrm{em}}^{2}}{3} \frac{\Gamma\left(\rho^{0} \rightarrow e^{+} e^{-}\right)}{m_{\rho}}=2.1 \div 2.36 \tag{6.32}
\end{equation*}
$$

## 7 Heavy quarks

In this section we briefly discuss the heavy quarks charm $(c)$, bottom $(b)$, and top $(t)$.

### 7.1 Decay of charmed particles

Charmed particles are those with nonzero charm $C$ (number of charm quarks minus number of charm antiquarks). The lightest such particles are the pseudoscalar $D$ and $D_{s}$ mesons,

$$
\begin{align*}
D^{+} & =c \bar{d}, & & D^{0}=c \bar{u} \\
\bar{D}^{0} & =u \bar{c}, & & D^{-}=d \bar{c}  \tag{7.1}\\
D_{s}^{+} & =c \bar{s}, & & D_{s}^{-}
\end{align*}=s \bar{c} .
$$

An older notation for the $D_{s}^{ \pm}$mesons is $F^{ \pm}$(e.g., in Ref. [6]). The $D$ mesons form two isospin doublets, $\left(D^{+}, D^{0}\right)$ and ( $\left.\bar{D}^{0}, D^{-}\right)$, while the $D_{s}^{ \pm}$mesons are isosinglets. The masses of these particles are $m_{D^{ \pm}}=1.870 \mathrm{GeV}, m_{D^{0}, \bar{D}^{0}}=1.865 \mathrm{GeV}$, and $m_{D_{s}^{ \pm}}=1.968 \mathrm{GeV}$.

The relevant product of currents involved in charmed particles decay is (in the two-family approximation)

$$
\begin{equation*}
\left(\cos \theta_{C} \bar{s} \mathcal{O}_{L}^{\alpha} c-\sin \theta_{C} \bar{d} \mathcal{O}_{L}^{\alpha} c\right)\left(\sum_{\ell} \bar{\nu}_{\ell} \mathcal{O}_{L \alpha} \ell+\cos \theta_{C} \bar{u} \mathcal{O}_{L \alpha} d+\sin \theta_{C} \bar{u} \mathcal{O}_{L \alpha} s\right)+\text { h.c. } \tag{7.2}
\end{equation*}
$$

from which we can read off the couplings relevant to semi-leptonic decays, and to the various possible non-leptonic decays.

Semi-leptonic decays The terms of Eq. (7.2) involved in semi-leptonic decays are

$$
\begin{align*}
\cos \theta_{C} \bar{s} \mathcal{O}_{L}^{\alpha} c \bar{\nu}_{\ell} \mathcal{O}_{L}^{\alpha} \ell & \Rightarrow c \rightarrow s \nu_{\ell} \ell^{+}, \\
-\sin \theta_{C} \bar{d} \mathcal{O}_{L}^{\alpha} c \bar{\nu}_{\ell} \mathcal{O}_{L}^{\alpha} \ell & \Rightarrow c \rightarrow d \nu_{\ell} \ell^{+}, \tag{7.3}
\end{align*}
$$

where we have written also the quark decay process to which they correspond. The related decay widths are proportional to

$$
\begin{array}{ll}
c \rightarrow s \nu_{\ell} \ell^{+}, & \Gamma \propto \cos ^{2} \theta_{c} \\
c \rightarrow d \nu_{\ell} \ell^{+}, & \Gamma \propto \sin ^{2} \theta_{c} \tag{7.4}
\end{array}
$$

Processes of the first type satisfy the selection rule $\Delta C=\Delta S$, and are dominant compared to those of the second type, which are suppressed by the tangent squared of the Cabibbo angle, $\left(\sin \theta_{C} / \cos \theta_{C}\right)^{2} \simeq 0.05$. Examples of decays of the first and second type are

$$
\begin{array}{lll}
\text { first type }(\Delta C=\Delta S): & D^{+} \rightarrow \ell^{+} \nu_{\ell} \bar{K}^{0}, & D_{s}^{+} \rightarrow \ell^{+} \nu_{\ell} \\
\text { second type }(\Delta S=0): & D^{+} \rightarrow \ell^{+} \nu_{\ell}, & D_{s}^{+} \rightarrow \ell^{+} \nu_{\ell} K^{0} \tag{7.5}
\end{array}
$$

Non-leptonic decays The four types of non-leptonic decays correspond to the following products of currents,

$$
\begin{array}{rr}
\cos ^{2} \theta_{C} \bar{s} \mathcal{O}_{L}^{\alpha} c \bar{u} \mathcal{O}_{L \alpha} d, & \cos \theta_{C} \sin \theta_{C} \bar{s} \mathcal{O}_{L}^{\alpha} c \bar{u} \mathcal{O}_{L \alpha} s, \\
-\sin \theta_{C} \cos \theta_{C} \bar{d} \mathcal{O}_{L}^{\alpha} c \bar{u} \mathcal{O}_{L \alpha} d, & -\sin ^{2} \theta_{C} \bar{d} \mathcal{O}_{L}^{\alpha} c \bar{u} \mathcal{O}_{L \alpha} s . \tag{7.6}
\end{array}
$$

The corresponding quark process, decay width and selection rules are

$$
\begin{array}{lll}
c \rightarrow s u \bar{d}, & \Gamma \propto \cos ^{4} \theta_{C}, & \Delta C=\Delta S \\
c \rightarrow s u \bar{s}, d u \bar{d}, & \Gamma \propto \sin ^{2} \theta_{C} \cos ^{2} \theta_{C}, & \Delta C=-1, \Delta S=0,  \tag{7.7}\\
c \rightarrow d u \bar{s}, & \Gamma \propto \sin ^{4} \theta_{C}, & \Delta C=-\Delta S .
\end{array}
$$

The dominant processes are again those with $\Delta C=\Delta S$, where a $c$ turns into an $s$, and a $u \bar{d}$ pair is produced. If a $u \bar{s}$ pair is produced the decay rate is suppressed, as it is if $c$ turns into $d$ and a $u \bar{d}$ pair is produced; if in the latter case a $u \bar{s}$ pair is produced instead, the rate is doubly suppressed. Due to the (trivial) colour structure of the charged weak current, the extra $q$ and $\bar{q}$ always have the same colour. For this reason, approximating $\cos ^{2} \theta_{C} \simeq 1, \sin ^{2} \theta_{C} \simeq 0$, one has

$$
\begin{equation*}
\Gamma(c \rightarrow s u \bar{d}) \simeq N_{c} \Gamma\left(c \rightarrow s \ell^{+} \nu_{\ell}\right)=3 \Gamma\left(c \rightarrow s \ell^{+} \nu_{\ell}\right) \tag{7.8}
\end{equation*}
$$

where since $m_{c} \gg m_{\mu} \gg m_{e}$ and $m_{c} \gg m_{u, d, s}$ the leptons and lighter quarks can be treated as massless.

Charmed particles creation The creation of charmed particles would be preferably done by shining muonic neutrino beams on strange-quark-rich targets, but due to the lack of such things one has to settle for second-best and use down-quark-rich targets (i.e., essentially anything). Creation of charmed particles via neutrino beams and their subsequent semi-leptonic decay results in dileptonic events, i.e.,

$$
\begin{align*}
\nu_{\mu} s \rightarrow & c \mu^{-} \\
& \left\llcorner s \ell^{+} \nu_{\ell} .\right. \tag{7.9}
\end{align*}
$$

Leptonic decays Purely leptonic decays are also possible, including into a $\tau$ lepton. The width for leptonic decays of charmed mesons is obtained with a calculation completely analogous to the one for pion decays: the relevant hadronic matrix elements have the same structure (although, obviously, involving different constants), since the only vector available is the charmed particle four-momentum (incidentally, only the axial current contributes since $D$ and $D_{s}$ mesons are pseudoscalars). For the ratio of semi-leptonic decay widths the unknown constants cancel out and all that matters is the mass dependence. One easily finds

$$
\begin{align*}
& \frac{\Gamma\left(D^{+} \rightarrow \tau^{+} \nu_{\tau}\right)}{\Gamma\left(D^{+} \rightarrow \mu^{+} \nu_{\mu}\right)}=\frac{m_{\tau}^{2}\left(1-\frac{m_{\tau}^{2}}{m_{D^{+}}}\right)^{2}}{m_{\mu}^{2}\left(1-\frac{m_{\mu}^{2}}{m_{D^{+}}^{2}}\right)^{2}} \simeq 2.5, \\
& \frac{\Gamma\left(D_{s}^{+} \rightarrow \tau^{+} \nu_{\tau}\right)}{\Gamma\left(D_{s}^{+} \rightarrow \mu^{+} \nu_{\mu}\right)}=\frac{m_{\tau}^{2}\left(1-\frac{m_{\tau}^{2}}{m_{D_{s}^{+}}^{2}}\right)^{2}}{m_{\mu}^{2}\left(1-\frac{m_{\mu}^{2}}{m_{D_{s}^{+}}^{2}}\right)^{2}} \simeq 9.6 . \tag{7.10}
\end{align*}
$$

Notice that the processes appearing in the first row are Cabibbo-suppressed. The origin of the factor $\left(m_{\tau} / m_{\mu}\right)^{2}$ lies in the chiral nature of the charged weak interaction, which makes it counterintuitively more likely for the charmed mesons to decay into taus than into lighter leptons.

### 7.2 The third quark family and the CKM matrix

The existence of a third quark family was suggested in 1973 by Kobayashi and Maskawa. As we will see below, this allows for violations of $C P$ in the weak Lagrangian.

The $b$ quark The lighter element of the third family, the bottom quark $b$, was discovered in 1977 at Fermilab by Lederman and collaborators. Using a proton beam against a platinum target, they studied the process

$$
\begin{equation*}
p p \rightarrow \ell^{+} \ell^{-} X, \tag{7.11}
\end{equation*}
$$

where a pair of charged leptons $\ell^{ \pm}$is created among other things (everything else is bundled into the symbol $X$ ). They were mostly looking for muons, which due to their penetrating power could survive the necessary filtering required to remove all the uninteresting hadrons from the final products. Looking at the invariant mass of the $\mu^{+} \mu^{-}$system, they found a narrow bump at $\sqrt{s}=9.46 \mathrm{GeV}$. This was identified as the $\Upsilon$ (upsilon), a bound state of a new type of quark, the $b$ quark. The $\Upsilon$ is the ground state of bottomonium, i.e., the bound state $b \bar{b}$ of a bottom and antibottom quarks. The $\Upsilon$ can also be seen in lepton collider via the process $e^{+} e^{-} \rightarrow \Upsilon \rightarrow$ hadrons. The $\Upsilon$ resonance is characterised by $s_{\Upsilon}=1, m_{\Upsilon}=9.46 \mathrm{GeV}$ and $\Gamma_{\Upsilon}=44.3 \mathrm{keV}$, corresponding to a lifetime $\tau_{\Upsilon}=1.2 \cdot 10^{-20} s$. The narrow width and corresponding long lifetime are due to the smallness of the coupling of the $\Upsilon$ to hadrons (cf. the OZI rule); this is however still stronger than the electromagnetic coupling, which leads to the observed lifetime being intermediate between typical hadronic and typical electromagnetic lifetimes. The bottom quark has $m_{b}=4.2 \mathrm{GeV}$ i.e., half the mass of the $\Upsilon$ ), and electric charge $q_{b}=-\frac{1}{3}$. The lightest mesons with nonzero "bottomness" are the $B$ mesons $B^{0}=d \bar{b}$ and
$B^{+}=u \bar{b}$, with masses $m_{B} \simeq 5.3 \mathrm{GeV}$. The $B^{0}-\bar{B}^{0}$ system has many similarities with the $K^{0}-\bar{K}^{0}$ system, e.g., it displays oscillations of "bottomness".

The $t$ quark The last quark discovered so far is the top quark $t$, observed in 1995 by the CDF and $\mathrm{D} \emptyset$ collaborations at Fermilab. The top quark is extremely heavy, $m_{t}=173 \mathrm{GeV}$, which makes it the heaviest elementary particle, and has electric charge $q_{t}=\frac{2}{3}$. No top-antitop bound states have been observed: in fact, the top quark decays in about $10^{-25} s$, so having not enough time to hadronise.

The CKM matrix The inclusion of a third family of quarks has important consequences for the symmetries of the weak interactions ${ }^{39}$ As we show now, three is the minimal number of families for which $C P$ violating effects can appear explictly in the Lagrangian.

Recall that in general, assuming the existence of $n$ families of quarks and universality of the charged weak interactions, one would write for the charged hadronic current

$$
\begin{equation*}
J_{h}^{\alpha}=\sum_{f, f^{\prime}=1}^{n} \bar{\alpha}_{f} \mathcal{O}_{L}^{\alpha} \kappa_{f^{\prime}} V_{f f^{\prime}}, \tag{7.12}
\end{equation*}
$$

where the quark fields $\alpha_{f}$ and $\kappa_{f}$ from the $f$ th family,

$$
\begin{equation*}
\binom{\alpha_{f}}{k_{f}}, \tag{7.13}
\end{equation*}
$$

and $V$ is a $n \times n$ unitary matrix, $V^{\dagger} V=\mathbf{1}$. Unitarity is required so that there are appropriate linear combinations,

$$
\begin{equation*}
\kappa_{f}^{\prime}=\sum_{f^{\prime}=1}^{n} V_{f f^{\prime}} \kappa_{f^{\prime}}, \tag{7.14}
\end{equation*}
$$

of $\kappa$-fields that interact with the corresponding field $\alpha_{f}$ in the same way that leptons and corresponding neutrinos do.

Let us now determine the number of physically relevant parameters contained in $V$. A general complex $n \times n$ matrix contains $2 n^{2}$ real parameters. Unitarity implies

$$
\begin{equation*}
\sum_{k} V_{i k} V_{j k}^{*}=\delta_{i j} \tag{7.15}
\end{equation*}
$$

and so $n$ real relations $\sum_{k}\left|V_{i k}\right|^{2}=1$ and $n(n-1) / 2$ complex relations $\sum_{k} V_{i k} V_{j k}^{*}=0$ for $i \neq j$, corresponding to $n(n-1)$ real relations. All in all there are $n+n(n-1)=n^{2}$ relations, reducing the number of real parameters in $V$ to $2 n^{2}-n^{2}=n^{2}$. Not all of these are physically meaningful: in fact, it is possible to redefine the phases of the $\alpha$ and $\kappa$ fields independently (without affecting the QCD Lagrangian underlying the hadronic interactions, which has a $U(1)^{2 n}$ flavour symmetry). Redefining $\alpha_{f} \rightarrow e^{i \phi_{f}} \alpha_{f}$ and $\kappa_{f} \rightarrow e^{i \psi_{f}} \kappa_{f}$, we have that changing

$$
\begin{equation*}
V_{j k} \rightarrow e^{-i\left(\phi_{j}-\psi_{k}\right)} V_{j k} \tag{7.16}
\end{equation*}
$$

[^25]has no observable physical effect. This can be used to set to zero the phase of a certain number of matrix elements. The number of independent phase factors $e^{-i\left(\phi_{j}-\psi_{k}\right)}$ can be determined by writing
\[

$$
\begin{equation*}
\phi_{j}-\psi_{k}=\left(\phi_{1}-\psi_{1}\right)+\left(\phi_{j}-\phi_{1}\right)-\left(\psi_{k}-\psi_{1}\right) . \tag{7.17}
\end{equation*}
$$

\]

One can choose the first term on the right-hand side arbitrarily; there are then $n-1$ independent differences $\phi_{j}-\phi_{1}$ and $n-1$ independent differences $\psi_{k}-\psi_{1}$. This results in $1+2(n-1)=2 n-1$ independent phase factors. Another way to see this is that one can change all the $2 n$ phases $\phi_{j}$ and $\psi_{k}$ by the same amount without changing the phase differences entering Eq. (7.17). This reduces the number of independent phase factors from $2 n$ to $2 n-1$. Either way, the number of physically meaningful real parameters in $V$ is $n^{2}-2 n+1=(n-1)^{2}$. In general, the space of $n \times n$ unitary matrices contains the space of $n \times n$ real orthogonal matrices as a subspace. A general orthogonal matrix in $n$ dimensions depends on $n(n-1) / 2$ angles, so among the $n^{2}$ real parameters of a unitary matrix there are $n(n-1) / 2$ angles; the remaining $n^{2}-n(n-1) / 2=n(n+1) / 2$ are phases. In our case not all the phases are physical, but only $n(n+1) / 2-2 n+1=(n-1)(n-2) / 2$ are.

Summarising, there are $n(n-1) / 2$ quark mixing angles and $(n-1)(n-2) / 2$ physical phases. Let us check a few cases:

$$
\begin{array}{ll}
n=1 \text { family: } & 0 \text { angles, } 0 \text { phases (nothing to mix); } \\
n=2 \text { families: } & 1 \text { angles, } 0 \text { phases }(V \text { is real); }  \tag{7.18}\\
n=3 \text { families: } & 3 \text { angles, } 1 \text { phase }(V \text { complex; } C P \text { violation possible). }
\end{array}
$$

The matrix $V$ is known as the Kobayashi-Maskawa or Cabibbo-Kobayashi-Maskawa matrix. A possible parameterisation is obtained by treating the quarks $(d, s, b)$ as coordinates $(z, y, x)$ and using Euler angles $\theta_{1,2,3}$ : the most general three-dimensional rotation is obtained by a rotation of $\theta_{3}$ around the $z$ axis, followed by a rotation of $\theta_{1}$ around the new $x$ axis, in turn followed by a further rotation of $\theta_{2}$ around the new $z$ axis. Denoting $c_{i}=\cos \theta_{i}$ and $s_{i}=\sin \theta_{i}$, we have

$$
\left(\begin{array}{ccc}
1 & 0 & 0  \tag{7.19}\\
0 & c_{2} & s_{2} \\
0 & -s_{2} & c_{2}
\end{array}\right)\left(\begin{array}{ccc}
c_{1} & s_{1} & 0 \\
-s_{1} & c_{1} & 0 \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & c_{3} & s_{3} \\
0 & -s_{3} & c_{3}
\end{array}\right) .
$$

To make the remaining phase surely physical we can include it in the $(3,3)$ element of the second factor in Eq. 7.19,

$$
\left(\begin{array}{ccc}
1 & 0 & 0  \tag{7.20}\\
0 & c_{2} & s_{2} \\
0 & -s_{2} & c_{2}
\end{array}\right)\left(\begin{array}{ccc}
c_{1} & s_{1} & 0 \\
-s_{1} & c_{1} & 0 \\
0 & 0 & e^{i \delta}
\end{array}\right)\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & c_{3} & s_{3} \\
0 & -s_{3} & c_{3}
\end{array}\right)
$$

since in this way it cannot be removed by a phase redefinition. Carrying out the multiplications we find

$$
V=\left(\begin{array}{ccc}
V_{u d} & V_{u s} & V_{u b}  \tag{7.21}\\
V_{c d} & V_{c s} & V_{c b} \\
V_{t d} & V_{t s} & V_{t b}
\end{array}\right)=\left(\begin{array}{ccc}
c_{1} & s_{1} c_{3} & s_{1} s_{3} \\
-s_{1} c_{2} & c_{1} c_{2} c_{3}-e^{i \delta} s_{2} s_{3} & c_{1} c_{2} s_{3}+e^{i \delta} s_{2} c_{3} \\
s_{1} s_{2} c_{2} & -c_{1} s_{2} c_{3}-e^{i \delta} c_{2} s_{3} & -c_{1} s_{2} s_{3}+e^{i \delta} c_{2} c_{3}
\end{array}\right) .
$$

It is an experimental fact that the $d$ and $s$ quarks have small mixing with the $b$ quark. One then expects $\theta_{2,3}$ to be small, in which case $V$ reduces to

$$
V \simeq\left(\begin{array}{ccc}
c_{1} & s_{1} & 0  \tag{7.22}\\
-s_{1} & c_{1} & 0 \\
0 & 0 & e^{i \delta}
\end{array}\right)
$$

It is then easy to identify $\theta_{1} \simeq \theta_{C}$. More generally, retaining the leading contributions, one finds

$$
V \simeq\left(\begin{array}{ccc}
1 & s_{1} & s_{1} s_{3}  \tag{7.23}\\
-s_{1} & 1 & s_{3}+e^{i \delta} s_{2} \\
s_{1} s_{2} & -s_{2}-e^{i \delta} s_{3} & e^{i \delta}
\end{array}\right)
$$

which shows that the main processes involve the transitions $d \leftrightarrow u, s \leftrightarrow c$ and $b \leftrightarrow t$. This make the concept of families physically meaningful, as the dominant decays involve transitions within families.

Experimental results for the first row of the CKM matrix are

$$
\begin{equation*}
\left|V_{u d}\right|=0.97420(21), \quad\left|V_{u s}\right|=0.2243(5), \quad\left|V_{u b}\right|=3.94(36) \cdot 10^{-3} \tag{7.24}
\end{equation*}
$$

From this one finds $\left|V_{u d}\right|^{2}+\left|V_{u s}\right|^{2}+\left|V_{u b}\right|^{2}=0.9994(5)$, in good agreement with the expected unitarity. From $\left|V_{u b}\right| /\left|V_{u d}\right|=\left|s_{3}\right| \simeq 2 \cdot 10^{-2}$ it follows that indeed $\theta_{3}$ is small. Similarly, $\left|V_{t d}\right|$ is found to be small, resulting in small $\left|s_{2}\right|$ and so small $\theta_{2}$.

## 8 Towards the Standard Model

In this section we describe in some detail the Standard Model (SM) of particle physics. After discussing the limitations of the four-fermion theory we introduce the tools required for the formulation of the SM, namely the theory of spontanteously broken symmetries, gauge fields, and the Higgs mechanism.

### 8.1 Limitations of the four-fermion theory

The most evident limitation of the four-fermion theory is its lack of renormalisability: due to the presence of a coupling with negative mass dimension, one keeps encountering new types of divergences as one increases the perturbative order, which requires the introduction of infinitely many counterterms, and thus results ultimately in a lack of predictivity of the theory. This last statement requires qualification: one can in fact treat the four-fermion theory as an effective theory, valid only up to a certain energy scale. Once the theory has been renormalised up to, say, $n$ counterterms, it will be predictive until the effects of the $(n+1)$ th type of divergence become phenomenologically relevant.

When does the effective theory break down? One can show that problems with unitarity are present at high energy already at tree level. These can be cured only going to higher orders of perturbation theory, which brings us back to the problem of non-renormalisability. To see when the effective theory fails, one can look at $e \nu_{e}$ elastic scattering. Since the amplitude is necessarily a polynomial in the four-momenta, it is polynomial in $\cos \theta_{\mathrm{CM}}$, where $\theta_{\mathrm{CM}}$ is the angle between the incoming and outgoing trajectories in the centre-of-mass frame. This means that in a partial wave expansion of the amplitude, only a finite number of partial waves $f_{J}$ will appear. On the other hand, simple dimensional analysis shows that the total cross section behaves as $\sigma_{\text {tot }} \sim G^{2} s$ at high energy ${ }^{40}$ But since $\sigma_{\text {tot }} \propto p^{-1} \sum(2 J+1)\left|f_{J}\right|^{2}$ (with $p$ the magnitude of the initial spatial momenta in the centre-of-mass frame), and since $\left|f_{J}\right|^{2} \leq \operatorname{Im} f_{J}$ due to unitarity of the $S$-matrix, it follows that at some point at least one of the partial wave amplitudes will violate the unitarity

[^26]bound. Looking for simplicity at the contribution of the charged current only, one find for the $J=0$ partial wave $f_{0}=G s /(2 \sqrt{2} \pi)$. The unitarity bound implies $\left|\operatorname{Re} f_{0}\right| \leq 1 / 2$, and since the tree-level amplitude is real one finds the bound
\[

$$
\begin{equation*}
\frac{G s}{\sqrt{2} \pi} \leq 1 \tag{8.1}
\end{equation*}
$$

\]

which means that unitarity will be violated above $\sqrt{s}=\sqrt{\sqrt{2} \pi / G} \simeq 600 \mathrm{GeV}$, signaling the breakdown of the theory.

How could one improve the situation? A possibility (already considered by Yukawa in the 1930s) is to replace the four-fermion interaction by the exchange of an intermediate massive vector boson. One finds for the differential cross section ${ }^{41}$

$$
\begin{equation*}
\left.\frac{d \sigma}{d|t|}\right|_{\mathrm{FF}}=\left.\frac{G^{2}}{\pi} \longrightarrow \frac{d \sigma}{d|t|}\right|_{\mathrm{IVB}}=\frac{G^{2}}{\pi} \frac{m_{W}^{4}}{\left(m_{W}^{2}+|t|\right)^{2}} \tag{8.2}
\end{equation*}
$$

where "FF" and "IVB" stand for "four fermion" and "intermediate vector boson", respectively, and $m_{W}$ is the mass of the intermediate boson. The effect of boson exchange shows up in the propagator factor $1 /\left(m_{W}^{2}+|t|^{2}\right)$, which implies that all partial waves are present, and that cuts off the contribution of large transferred momentum. In fact, since $0 \leq|t| \leq s$ (we ignore the electron mass here), one has

$$
\begin{equation*}
\left.\sigma\right|_{\mathrm{FF}}=\left.\int_{0}^{s} d|t| \frac{d \sigma}{d|t|}\right|_{\mathrm{FF}}=\left.\frac{G^{2}}{\pi} s \longrightarrow \sigma\right|_{\mathrm{IVB}}=\left.\int_{0}^{s} d|t| \frac{d \sigma}{d \mid t t}\right|_{\mathrm{IVB}}=\frac{G^{2} m_{W}^{2}}{\pi} \frac{s}{s+m_{W}^{2}} \tag{8.3}
\end{equation*}
$$

In the IVB case, the total cross section rises linearly with $s$ at low energy, while approaching a constant at high energy. Unitarity is therefore respected, and the unitarity bound becomes a bound on $m_{W}$.

An important point is that the coupling constant in the IVB theory leading to Eq. (8.2) is $g_{W}^{2}=G m_{W}^{2}$, which is dimensionless. This raises the hope that the theory might be renormalisable. Unfortunately, this is not the case. The massive vector propagator in fact contains a term $p_{\mu} p_{\nu} / p^{2} m_{W}^{2}$ which is of order $\mathcal{O}(1)$ at high energies, thus bringing back a problematic high-energy behaviour of the theory. This term would not be problematic if the IVB were coupled to a conserved current, since it would give no contribution, but this is not the case here: neither the vector nor the axial-vector currents are conserved, the first one because of the mass difference between electron neutrino, and the second one since they are not both massless. Even worse, even if the leptons were massless and the weak current therefore conserved, the massive vector bosons we need must be electrically charged, and their electromagnetic interaction is not renormalisable. One has moreover problems with unitarity showing up in the boson-boson cross section, which could however be cured if a further, neutral boson were also introduced.

One way out of this problem would be to use massless rather than massive vector bosons as the exchanged particles. In this case the theory would be renormalisable, even including electromagnetic interactions. This seems however a non-starter: weak interactions are known to be short-ranged, while massless bosons would lead to long-range intaractions, like with the photon in electromagnetic interactions; and as soon as we add a mass term to the Lagrangian, non-renormalisability kicks back. There is however one more trick that can be used: if the vector bosons are not given a mass "by hand", but they acquire it dinamically due to the spontaneous breaking of a symmetry, then renormalisability remains.

[^27]
### 8.2 Massive vector bosons

While the discussion above was qualitative, in this subsection we give a detailed analysis of the problems with massive vector bosons. The Lagrangian describing free massive vector particles (the Proca Langrangian) reads

$$
\begin{equation*}
\mathscr{L}_{\text {Proca }}=-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}+\frac{1}{2} m^{2} W_{\mu} W^{\mu}, \quad F_{\mu \nu}=\partial_{\mu} W_{\nu}-\partial_{\nu} W_{\mu} . \tag{8.4}
\end{equation*}
$$

The corresponding equations of motion are

$$
\begin{align*}
\frac{\partial \mathscr{L}_{\text {Proca }}}{\partial W_{\mu}}-\partial_{\nu} \frac{\partial \mathscr{L}_{\text {Proca }}}{\partial\left(\partial_{\nu} W_{\mu}\right)} & =0, \\
m^{2} W^{\mu}-\partial_{\nu}\left(-F^{\nu \mu}\right) & =0,  \tag{8.5}\\
\left(\square+m^{2}\right) W^{\mu}-\partial^{\mu} \partial_{\nu} W^{\nu} & =0 .
\end{align*}
$$

Taking the divergence $\partial_{\mu}$ of this equation we find

$$
\begin{equation*}
\left(\square+m^{2}\right) \partial_{\mu} W^{\mu}-\square \partial_{\nu} W^{\nu}=m^{2} \partial_{\mu} W^{\mu}=0 \Rightarrow \partial_{\mu} W^{\mu}=0 \tag{8.6}
\end{equation*}
$$

which we can plug back into Eq. 8.5 to get the system of equations

$$
\begin{align*}
\left(\square+m^{2}\right) W^{\mu} & =0, \\
\partial_{\mu} W^{\mu} & =0 . \tag{8.7}
\end{align*}
$$

These equations are most easily solved in momentum space. One finds

$$
\begin{equation*}
W^{\mu}(x)=\int d \Omega_{p} \sum_{j=1}^{3}\left\{\varepsilon_{j}^{\mu}(\vec{p}) e^{-i p \cdot x} a_{j}(\vec{p})+\varepsilon_{j}^{\mu *}(\vec{p}) e^{i p \cdot x} b_{j}^{\dagger}(\vec{p})\right\}, \tag{8.8}
\end{equation*}
$$

where $p^{0}=\sqrt{\vec{p}^{2}+m^{2}}$ and $d \Omega_{p}$ is the invariant phase-space measure. The polarisation vectors $\varepsilon_{j}^{\mu}(\vec{p})$ satisfy $p \cdot \varepsilon_{j}(\vec{p})=0$, as the second equation demands. There are three independent solutions, that we can choose as follows:

$$
\begin{align*}
\varepsilon_{1,2}^{\mu} & =\left(0, \vec{s}_{1,2}\right), \quad \vec{p} \cdot \vec{s}_{1,2}=0, \quad \vec{s}_{i} \cdot \vec{s}_{j}=\delta_{i j}, \quad i, j=1,2  \tag{8.9}\\
\varepsilon_{3}^{\mu} & =\frac{1}{m}\left(|\vec{p}|, p^{0} \hat{p}\right) .
\end{align*}
$$

Notice that

$$
\begin{equation*}
\varepsilon_{3}^{\mu}=\frac{p^{\mu}}{m}+\frac{m}{p^{0}+|\vec{p}|}(-1, \hat{p}) . \tag{8.10}
\end{equation*}
$$

With this choice $\varepsilon_{i}^{\mu}=\varepsilon_{i}^{\mu_{*}}$. In general one chooses them so that

$$
\begin{equation*}
\varepsilon_{i} \cdot \varepsilon_{j}^{*}=-\delta_{i j}, \quad i, j=1,2,3 . \tag{8.11}
\end{equation*}
$$

These vectors form an orthonormal basis of the three-dimensional space transverse to the fourmomentum $p$, hence they satisfy

$$
\begin{equation*}
\sum_{j=1}^{3} \varepsilon_{j}^{\mu}(\vec{p}) \varepsilon_{j}^{\nu *}(\vec{p})=-\eta^{\mu \nu}+\frac{p^{\mu} p^{\nu}}{m^{2}} \tag{8.12}
\end{equation*}
$$

as one can verify explicitly for our choice Eq. 8.9. .
A large class of interacting theories can be obtained adding a term of the form $-W_{\mu} j^{\mu}$ to the Lagrangian, coupling the massive vector boson to a current $j^{\mu}$. The equations of motion are obtained replacing $m^{2} W^{\mu} \rightarrow m^{2} W^{\mu}-j^{\mu}$ in the last equation in Eq. (8.5), and read

$$
\begin{equation*}
\left(\square+m^{2}\right) W^{\mu}-\partial^{\mu} \partial_{\nu} W^{\nu}=j^{\mu} . \tag{8.13}
\end{equation*}
$$

Taking the divergence we find this time

$$
\begin{equation*}
m^{2} \partial_{\mu} W^{\mu}=\partial_{\mu} j^{\mu} \tag{8.14}
\end{equation*}
$$

that can be plugged back into Eq. 8.13) to obtain

$$
\begin{equation*}
\left(\square+m^{2}\right) W^{\mu}=\left(\eta^{\mu \nu}+\frac{\partial^{\mu} \partial^{\nu}}{m^{2}}\right) j_{\nu} \tag{8.15}
\end{equation*}
$$

One can easily read off the Green's function, or propagator, $D^{\mu \nu}(x)$, that connects the solution of the equation to the current,

$$
\begin{equation*}
W^{\mu}(x)=\int d^{4} y D^{\mu \nu}(x-y) j_{\nu}(y) \tag{8.16}
\end{equation*}
$$

One has

$$
\begin{equation*}
D^{\mu \nu}=\left(\eta^{\mu \nu}+\frac{\partial^{\mu} \partial^{\nu}}{m^{2}}\right) \frac{1}{\square+m^{2}} \tag{8.17}
\end{equation*}
$$

In momentum space this reads

$$
\begin{equation*}
\tilde{D}^{\mu \nu}=\frac{-\eta^{\mu \nu}+\frac{p^{\mu} p^{\nu}}{m^{2}}}{p^{2}-m^{2}} \tag{8.18}
\end{equation*}
$$

ignoring the choice of prescription to deal with the pole at $p^{2}=m^{2}$. The second term in the numerator can lead to bad high energy behaviour: in momentum space the propagator couples to the Fourier transform $\tilde{j}^{\mu}$ of $j^{\mu}$, and the second term originates a factor $p_{\mu} \tilde{j}^{\mu}$. If the current is conserved, $\partial_{\mu} j^{\mu}=0$, then $p_{\mu} \tilde{j}^{\mu}=0$ and the potentially dangerous term has no effect. Moreover, only the second term of the longitudinal polarisation vector $\varepsilon_{3}$ in Eq. 8.10) contributes to Feynman diagrams, and no troublesome high-energy behaviour come from it. This is the reason why one can give the photon a mass without spoiling the renormalisability of the theory despite the loss of gauge invariance. On the other hand, if $p_{\mu} \tilde{J}^{\mu} \neq 0$ then the $p^{\mu} p^{\nu}$ term cannot be dropped, and makes the theory non renormalisable due to its bad high-energy behaviour. For this reason, one cannot give mass "by hand" to the intermediate vector bosons that one wants to use in the description of weak interactions.

### 8.3 Spontaneous symmetry breaking and the Goldstone theorem

The trick that will be used to give a mass to the intermediate vector boson combines gauge invariance with the appearance of massless scalar particles in a theory with a spontaneously broken symmetry. In this section we discuss what the spontaneous breaking of a symmetry is and why massless particles appear in the spectrum. Such particles, known as Goldstone bosons, result from the breaking of a global continuous symmetry due to the non-invariance of the vacuum. In fact, even if the equation of motion of a system show a certain symmetry, this does
not mean that every solution should show it as well. If this happens to the solution of minimal energy, and the symmetry is continuous, then this solution cannot be unique, and moving from one such solution to another costs no energy, hence giving rise to massless modes. The Goldstone theorem states that there is one such massless mode for every generator of the symmetry which is broken by the vacuum. If the symmetry that is "broken" is a gauge (local) symmetry ${ }^{42}$ then the "would-be" Goldstone modes are absorbed by the gauge bosons as longitudinal (zero helicity) modes, thus making the bosons massive. More precisely, the gauge bosons corresponding to the broken generators acquire a mass, while the unbroken ones remain massless. The reason why Goldstone theorem is not applicable when the symmetry is local is simply that one of its hypothesis is violated. In fact, quantisation of a gauge theory requires to choose a gauge, thus breaking explicitly the local symmetry, and if one asks that the Hilbert space contains only physical states then the gauge choice cannot respect Lorentz covariance, which is required by the theorem. Examples of "physical" gauges are the Coulomb gauge (i.e., $\vec{\nabla} \cdot \vec{A}=0$ ) or the axial gauge (i.e., $A_{3}=0$ ). Covariant gauges exist, like the Lorenz gauge (i.e., $\partial_{\mu} A^{\mu}=0$ ), in which the theorem must therefore apply. On the other hand, Lorenz gauge contains unphysical, negative-norm states corresponding to the remaining gauge modes (gauge fixing is only partial), and the Goldstone mode happens to be a gauge mode decoupled from the physical states.

Let us now discuss in some detail how Goldstone modes appear. Consider a system of $N$ scalar fields $\phi_{i}(x)$ described by the following Lagrangian density,

$$
\begin{equation*}
\mathscr{L}=\frac{1}{2} \partial_{\mu} \phi_{i} \partial^{\mu} \phi_{i}-\mathscr{U}(\phi), \tag{8.19}
\end{equation*}
$$

where the potential $\mathscr{U}(\phi)$ is some polynomial which includes the mass (quadratic) terms, and has to be at most of order four to ensure renormalisability. We will treat the fields as classical, having in mind path-integral quantisation. Assume that the set of scalar fields provides a basis for the representation space of some $N$-dimensional unitary representation of some Lie group $G$,

$$
\begin{equation*}
\phi_{i}(x) \rightarrow(g \phi)_{i}(x)=D_{i j}(g) \phi_{j}(x) \tag{8.20}
\end{equation*}
$$

with $D(g) N \times N$ unitary matrices satisfying $D\left(g_{1}\right) D\left(g_{2}\right)=D\left(g_{1} g_{2}\right)$ for all $g_{1}, g_{2} \in G$. Such matrices can be written as

$$
\begin{equation*}
D(g)=e^{\epsilon_{a}(g) T^{a}}, \quad \epsilon_{a}(g) \in \mathbb{R} \tag{8.21}
\end{equation*}
$$

where summation over $a=1, \ldots n=\operatorname{dim} G$ is understood, and $T^{a}$ are $N \times N$ real antisymmetric matrices providing a representation of the group algebra $\left[T^{a}, T^{b}\right]=-f^{a b c} T^{c}[43$ Since we are dealing with real fields, the representation must be real and therefore orthogonal. This setting is fully general, since any set of complex scalar fields $\varphi_{i}(x)$ can always be reduced to Eq. 8.19) by separating their real and imaginary parts, $\varphi_{i}=\varphi_{i}^{R}+i \varphi_{i}^{I}$, that can be collected in a vector of $2 N$ real fields $\phi_{i}=\varphi_{i}^{R}, i=1, \ldots, N, \phi_{i}=\varphi_{i-N}^{I}, i=N+1, \ldots, 2 N$. The general unitary representation under which the $\varphi_{i}$ transform gives rise to an orthogonal representation under which the $\phi_{i}$ transform. Explicitly, if $t^{a}$ are the Hermitian generators of the representation for the complex fields, $\delta \varphi=i \epsilon_{a} t^{a} \varphi$, then $\delta \phi=\epsilon_{a} T^{a} \phi$ where the real antisymmetric matrices $T^{a}$ are given by

$$
T^{a}=\left(\begin{array}{cc}
-\operatorname{Im} t^{a} & -\operatorname{Re} t^{a}  \tag{8.22}\\
\operatorname{Re} t^{a} & -\operatorname{Im} t^{a}
\end{array}\right)
$$

[^28]Assume further that the Lagrangian density is invariant under group transformations, which amounts to ask that $\mathscr{U}(g \phi)=\mathscr{U}(\phi)$. In other words, the system has an internal symmetry corresponding to $G$. The energy functional corresponding to Eq. (8.19) is

$$
\begin{equation*}
E[\phi]=\int d^{3} x\left[\frac{1}{2} \partial_{0} \phi_{i} \partial_{0} \phi_{i}+\frac{1}{2} \vec{\nabla} \phi_{i} \cdot \vec{\nabla} \phi_{i}+\mathscr{U}(\phi)\right] . \tag{8.23}
\end{equation*}
$$

This functional is bounded from below if $\mathscr{U}$ is. We assume that this is the case, and set its minimum to zero by adding to it an irrelevant constant. In this way we ensure that $E[\phi] \geq 0$. The ground state of the system (also called the vacuum state in the context of relativistic quantum field theories) is the one with minimal energy (i.e., zero), which is easily seen to correspond to a constant field configuration (so that there is no contribution from the derivative terms) which minimises the potential $\mathscr{U}$. Since by construction $\min \mathscr{U}=0$, the ground state is given by $\phi_{i}(x)=\phi_{0 i}$ with $\mathscr{U}\left(\phi_{0}\right)=0$. However, due to the symmetry of the system under $G$ one has for any $g$ that $\mathscr{U}\left(g \phi_{0}\right)=\mathscr{U}\left(\phi_{0}\right)=0$, and since in general it can happen that $g \phi_{0} \neq \phi_{0}$, more than one ground state can exist. One then defines the manifold $\mathcal{M}$ of ground states,

$$
\begin{equation*}
\mathcal{M}=\left\{\phi_{0} \mid \mathscr{U}\left(\phi_{0}\right)=0\right\}, \tag{8.24}
\end{equation*}
$$

which by construction is left invariant by the action of $G$, i.e., $G \mathcal{M}=\mathcal{M}$. If $\mathcal{M}$ contains more than one state, we say that the symmetry $G$ is broken, since any ground state in $\mathcal{M}$ will not be left invariant by a generic symmetry transformation. Given $\phi_{0} \in \mathcal{M}$, its $G$-orbit is the set $\left\{g \phi_{0} \mid g \in G\right\}$. We assume that any ground state can be reached from any other by means of a symmetry transformation $4^{44}$ the cases we will be considering are all of this type. This is tantamount to saying that $\mathcal{M}$ is equal to the $G$-orbit of any ground state. We further define the stability group $H$ as

$$
\begin{equation*}
H=\left\{h \in G \mid h \phi_{0}=\phi_{0}\right\} . \tag{8.25}
\end{equation*}
$$

Clearly, $H$ is a subgroup of $G \sqrt{45}$ the one that leaves the ground state invariant. The group $H$ is the unbroken part of the symmetry group $G$.

Strictly speaking, one should define $H\left(\phi_{0}\right)$ as the stability group of the ground state $\phi_{0}$. However, since we assumed that $\mathcal{M}$ is equal to the $G$-orbit of $\phi_{0}$, for any other ground state $\phi_{0}^{\prime}$ we have $\phi_{0}^{\prime}=g \phi_{0}$. If $h \in H\left(\phi_{0}\right)$, then $g^{-1} \phi_{0}^{\prime}=\phi_{0}=h \phi_{0}=h g^{-1} \phi_{0}^{\prime}$, so that $g h g^{-1} \phi_{0}^{\prime}=\phi_{0}^{\prime}=h^{\prime} \phi_{0}^{\prime}$ with $h^{\prime}=g h g^{-1}$. Conversely, given $h^{\prime} \in H\left(\phi_{0}^{\prime}\right)$, one shows that $h=g^{-1} h^{\prime} g \in H\left(\phi_{0}\right)$. The stability groups $H\left(\phi_{0}\right)$ are therefore all isomorphic, and we can simply denote with $H$ the corresponding equivalence class.

We finally define the (right) cosets as the sets $g H=\{g h \mid h \in H\}$. These sets are clearly left invariant by right multiplication with any element of $H$. Each coset corresponds uniquely to an equivalence class with respect to the relation $g_{1} \sim g_{2}$ if $g_{1}=g_{2} h$ for some $h \in H$, or in other words to the elements of $G$ modulo elements of $H$. The set of cosets/equivalence classes is the (right) coset space, denoted $G / H$. Choosing some $\phi_{0}$, any element of $\mathcal{M}$ can be written as $g \phi_{0}$, but the choice of $g$ is not unique since $g h \phi_{0}=g \phi_{0}$ if $h \in H$ belongs to the stability group. On the other hand, $g \phi_{0}$ corresponds uniquely to a coset $g H$, and therefore one has that $\mathcal{M}=G / H{ }^{46}$ One distinguishes three cases:

[^29]- $H=G$ : the ground state is invariant under the whole symmetry group, so $\phi_{0}$ is unique, $\mathcal{M}=\left\{\phi_{0}\right\}$, and the symmetry group $G$ is unbroken;
- $H=\{e\}$ (the neutral element): the ground state is not invariant under any transformation, and $G$ is completely broken;
- $H \subset G$ is a proper subgroup: $G$ is broken down to $H$.

In what follows it is convenient to choose the generators $\left\{T^{1}, \ldots T^{n}\right\}$ of $G$ in such a way as to contain also the generators of $H$. More precisely, one chooses the first $n^{\prime}=\operatorname{dim} H$ generators $\left\{T^{1}, \ldots, T^{n^{\prime}}\right\}$ to be the generators of $H$ (i.e., they span the Lie algebra of $H$ which is a subalgebra of the Lie algebra of $G$ ), and the remaining $n-n^{\prime}$ generators $\left\{T^{n^{\prime}+1}, \ldots, T^{n}\right\}$ to span the rest, thus "generating" $G / H{ }^{[7]}$ Since $H \phi_{0}=\phi_{0}$, one has that $T^{a} \phi_{0}=0$ for $a=1, \ldots, n^{\prime}$, while $T^{a} \phi_{0} \neq 0$ if $a=n^{\prime}+1, \ldots, n$. Also, no nontrivial linear combination $c_{a} T^{a}, a=n^{\prime}+1, \ldots, n$ can annihilate $\phi_{0}$, for otherwise $c_{a} T^{a}$ would belong to the algebra of $H$, against the hypothesis.

Example 1: $G=\mathrm{SO}(2)$ Let the symmetry group be $\mathrm{SO}(2)$, and let the scalar multiplets belong to the defining representation (i.e., $N=2$ ). Since for the group $\operatorname{SO}(N)$ of proper rotations in $N$ dimensions one has $\operatorname{dimSO}(N)=\frac{N(N-1)}{2}$, we have $n=1$. For $n=1, N=2$ the group generator is

$$
T=\left(\begin{array}{cc}
0 & 1  \tag{8.26}\\
-1 & 0
\end{array}\right) .
$$

Let the potential be

$$
\begin{equation*}
\mathscr{U}(\phi)=\frac{\lambda}{2}\left(\phi_{1}^{2}+\phi_{2}^{2}-a^{2}\right)^{2}, \tag{8.27}
\end{equation*}
$$

where $\lambda, a \in \mathbb{R}$ and $\lambda>0$. A potential of the type Eq. (8.26) is known as Mexican-hat potential for obvious reasons (try to draw it). The ground state manifold is defined by $\mathscr{U}(\phi)=0$, and easily found to be

$$
\begin{equation*}
\mathcal{M}=\left\{\phi \mid \phi_{1}^{2}+\phi_{2}^{2}=a^{2}\right\} \sim S^{1} \tag{8.28}
\end{equation*}
$$

i.e., it is the two-dimensional sphere (circle). The whole $\mathcal{M}$ is obtained as $G \phi_{0}$ for any $\phi_{0} \in \mathcal{M}$. No point in $\mathcal{M}$ is left invariant by any rotation, so that $H=\{e\}$ and the symmetry is completely broken. In fact, $G$ has no proper subgroups, so the symmetry is either unbroken or completely broken. The only solution to $T \phi=0$, which is equivalent to asking invariance under $\mathrm{SO}(2)$, is the point $\phi_{1}=\phi_{2}=0 \notin \mathcal{M}$.

Example 2: $G=\mathrm{SU}(2) \quad$ Consider a doublet $\Psi$ of complex fields,

$$
\Psi=\binom{\psi_{1}}{\psi_{2}}, \quad\left\{\begin{array}{l}
\psi_{1}=\phi_{1}+i \phi_{2}  \tag{8.29}\\
\psi_{2}=\phi_{3}+i \phi_{4}
\end{array}\right.
$$

transforming under the defining representation of $\operatorname{SU}(2)(N=4, n=3)$. The Mexican-hat potential in this case is

$$
\begin{equation*}
\mathscr{U}(\Psi)=\lambda\left(\Psi^{\dagger} \Psi-a^{2}\right)^{2}=\lambda\left(\psi_{1}^{*} \psi_{1}+\psi_{2}^{*} \psi_{2}-a^{2}\right)^{2}=\lambda\left(\sum_{i=1}^{4} \phi_{i}^{2}-a^{2}\right)^{2}, \tag{8.30}
\end{equation*}
$$

[^30]with $\lambda, a \in \mathbb{R}$ and $\lambda>0$. A generic $\mathrm{SU}(2)$ matrix reads
\[

g=\left($$
\begin{array}{cc}
c & d  \tag{8.31}\\
-d^{*} & c^{*}
\end{array}
$$\right), \quad|c|^{2}+|d|^{2}=1
\]

so that $\mathrm{SU}(2) \sim S^{3}$ (the four-dimensional sphere). Clearly $(g \Psi)^{\dagger}(g \Psi)=\Psi^{\dagger} \Psi$, since this is the very definition of unitary matrix. The ground state manifold is

$$
\begin{equation*}
\mathcal{M}=\left\{\phi \mid \sum_{i=1}^{4} \phi_{i}^{2}=a^{2}\right\} \sim S^{3} \tag{8.32}
\end{equation*}
$$

so diffeomorphic to the group $G$. No element of $\mathcal{M}$ can be left invariant by any subgroup of $G$ : in fact, $G=\mathcal{M}=G / H$, so $H=\{e\}$ and the symmetry is completely broken.

Example 3: $G=\mathrm{SO}(3)$ Consider a triplet of real fields transforming under the defining representation of $\mathrm{SO}(3)$, or equivalently the adjoint representation of $\mathrm{SU}(2)$,

$$
\phi=\left(\begin{array}{l}
\phi_{1}  \tag{8.33}\\
\phi_{2} \\
\phi_{3}
\end{array}\right)
$$

and let

$$
\begin{equation*}
\mathscr{U}(\phi)=\lambda\left(\sum_{i=1}^{3} \phi_{i}^{2}-a^{2}\right)^{2} . \tag{8.34}
\end{equation*}
$$

The ground state manifold is now the three-dimensional sphere,

$$
\begin{equation*}
\mathcal{M}=\left\{\phi \mid \sum_{i=1}^{3} \phi_{i}^{2}=a^{2}\right\} \sim S^{2} \tag{8.35}
\end{equation*}
$$

Choosing $\phi_{0}=(0,0, a)$, it is easy to identify the stability group $H=\{h(\alpha), \alpha \in[0,2 \pi)\}$ as comprising the rotations

$$
h(\alpha)=\left(\begin{array}{ccc}
\cos \alpha & \sin \alpha & 0  \tag{8.36}\\
-\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{array}\right)
$$

There is therefore a nontrivial $\mathrm{SO}(2) \sim U(1)$ stability group, and indeed the ground state manifold is $G / H=\mathrm{SO}(3) / \mathrm{SO}(2)=S^{2}=\mathcal{M}$. This example show how not only the group but also the choice of representation plays an important role in the symmetry breaking pattern.

We now discuss the main result.
Goldstone theorem: if $G$ with $\operatorname{dim} G=n$ is broken down to $H$ with $\operatorname{dim} H=n^{\prime}$, then there are $n-n^{\prime}$ massless bosons (Goldstone bosons) in the spectrum, one per "generator" of the coset space.

Proof: By assumption, $\mathscr{U}(\phi)=\mathscr{U}(g \phi)$. Under an infinitesimal transformation $g=e^{\epsilon \cdot T} \simeq$ $1+\epsilon \cdot T$,

$$
\begin{equation*}
\mathscr{U}(\phi)=\mathscr{U}(g \phi)=\mathscr{U}(\phi+\epsilon \cdot T \phi) \simeq \mathscr{U}(\phi)+\frac{\partial \mathscr{U}}{\partial \phi_{i}}(\phi)(\epsilon \cdot T)_{i j} \phi_{j}, \tag{8.37}
\end{equation*}
$$

from which it follows due to the arbitrariness of $\epsilon_{a}$ that

$$
\begin{equation*}
\frac{\partial \mathscr{U}}{\partial \phi_{i}}(\phi) T_{i j}^{a} \phi_{j}=0, \tag{8.38}
\end{equation*}
$$

for any field configuration $\phi$. We can take one more derivative with respect to $\phi_{k}$, and find

$$
\begin{equation*}
\frac{\partial^{2} \mathscr{U}}{\partial \phi_{k} \partial \phi_{i}}(\phi) T_{i j}^{a} \phi_{j}+\frac{\partial \mathscr{U}}{\partial \phi_{i}}(\phi) T_{i k}^{a}=0 . \tag{8.39}
\end{equation*}
$$

Setting now $\phi=\phi_{0} \in \mathcal{M}$, we have that $\frac{\partial \mathscr{U}}{\partial \phi_{i}}\left(\phi_{0}\right)=0$ since it minimises the potential and so

$$
\begin{equation*}
\frac{\partial^{2} \mathscr{U}}{\partial \phi_{k} \partial \phi_{i}}\left(\phi_{0}\right) T_{i j}^{a} \phi_{0 j}=0 . \tag{8.40}
\end{equation*}
$$

The matrix

$$
\begin{equation*}
M_{k i}^{2} \equiv \frac{\partial^{2} \mathscr{U}}{\partial \phi_{k} \partial \phi_{i}}\left(\phi_{0}\right) \tag{8.41}
\end{equation*}
$$

is the matrix of the coefficients of the quadratic part of the potential $\mathscr{U}$, and as such it is the mass matrix of the fluctuations $\tilde{\phi}$ of $\phi=\phi_{0}+\tilde{\phi}$ around the ground state $\phi_{0}$ :

$$
\begin{equation*}
\mathscr{U}(\phi)=\mathscr{U}\left(\phi_{0}+\tilde{\phi}\right)=\frac{1}{2} \tilde{\phi}_{k} M_{k i}^{2} \tilde{\phi}_{i}+\text { higher orders } . \tag{8.42}
\end{equation*}
$$

Eq. 8.40 therefore reads

$$
\begin{equation*}
M_{k i}^{2} T_{i j}^{a} \phi_{0 j}=0 \tag{8.43}
\end{equation*}
$$

This tells us that as long as it is nonzero $T^{a} \phi_{0}$ is an eigenvector of $M^{2}$ with eigenvalue zero, i.e., a massless fluctuation. For $a=1, \ldots, n^{\prime}$ one has $T^{a} \phi_{0}=0$, so Eq. 8.43) is trivially satisfied. For $a=n^{\prime}+1, \ldots, n$ one has instead that $T^{a} \phi_{0} \neq 0$ are linearly independent, which proves the existence of $n-n^{\prime}$ massless modes, one per broken generator of $G$.

The Goldstone modes can be taken to be the linear combinations $\tilde{\phi}_{i} T_{i j}^{a} \phi_{0 j}, a=n^{\prime}+1, \ldots, n$ : in fact, taking the scalar product of the fluctuations $\tilde{\phi}$ with any of the $T^{a} \phi_{0}$ automatically removes the contributions of massive modes, since these correspond to eigenvectors of the symmetric matrix $M^{2}$ with nonzero eigenvalue and are therefore orthogonal to the massless modes. These combinations are linearly independent: if $\tilde{\phi}_{i} c_{a} T_{i j}^{a} \phi_{0 j}=0, a=n^{\prime}+1, \ldots, n$ for all $\tilde{\phi}$, then $c_{a} T^{a} \phi_{0}=0$ which contradicts our hypotheses. Notice that if for a given configuration $\tilde{\phi}_{i} T_{i j}^{a} \phi_{0 j}=$ 0 for all $a$ then there is no contribution from the Goldstone modes.

Example: $G=\mathrm{SO}(2)$, doublet of real scalars Consider the case discussed above in example 1, corresponding to a doublet of real scalar fields with potential given by Eq. 8.27). Choosing as ground state

$$
\begin{equation*}
\phi_{0}=\binom{a}{0} \tag{8.44}
\end{equation*}
$$

one can recast the Lagrangian as

$$
\begin{align*}
\mathscr{L} & =\frac{1}{2} \partial_{\mu} \tilde{\phi}_{1} \partial^{\mu} \tilde{\phi}_{1}+\frac{1}{2} \partial_{\mu} \tilde{\phi}_{2} \partial^{\mu} \tilde{\phi}_{2}-\frac{\lambda}{2}\left(\left(a+\tilde{\phi}_{1}\right)^{2}+\tilde{\phi}_{2}^{2}-a^{2}\right)^{2} \\
& =\frac{1}{2} \partial_{\mu} \tilde{\phi}_{1} \partial^{\mu} \tilde{\phi}_{1}+\frac{1}{2} \partial_{\mu} \tilde{\phi}_{2} \partial^{\mu} \tilde{\phi}_{2}-\frac{\lambda}{2}\left(\tilde{\phi}_{1}\left(2 a+\tilde{\phi}_{1}\right)+\tilde{\phi}_{2}^{2}\right)^{2}  \tag{8.45}\\
& =\frac{1}{2} \partial_{\mu} \tilde{\phi}_{1} \partial^{\mu} \tilde{\phi}_{1}+\frac{1}{2} \partial_{\mu} \tilde{\phi}_{2} \partial^{\mu} \tilde{\phi}_{2}-\frac{\lambda}{2}\left(4 a^{2} \tilde{\phi}_{1}^{2}+4 a \tilde{\phi}_{1}\left(\tilde{\phi}_{1}^{2}+\tilde{\phi}_{2}^{2}\right)+\left(\tilde{\phi}_{1}^{2}+\tilde{\phi}_{2}^{2}\right)^{2}\right)
\end{align*}
$$

from which one reads of the masses $m_{1,2}$ of the fluctuations $\tilde{\phi}_{1,2}$ to be $m_{1}^{2}=4 \lambda a^{2}$ and $m_{2}^{2}=0$. Notice that $\tilde{\phi}_{i} T_{i j} \phi_{0 j}=-a \tilde{\phi}_{2}$ is precisely the Goldstone mode. Perhaps a more transparent way to see the appearance of the Goldstone mode is to set $\varphi=\frac{\phi_{1}+i \phi_{2}}{\sqrt{2}}$ and recast the Lagrangian as the Langrangian of a single complex field with a $\mathrm{U}(1)$ internal symmetry,

$$
\begin{equation*}
\mathscr{L}=\partial_{\mu} \varphi^{*} \partial^{\mu} \varphi-\frac{\lambda}{2}\left(2 \varphi^{*} \varphi-a^{2}\right)^{2} \tag{8.46}
\end{equation*}
$$

Parameterising the fluctuations around the ground state $\varphi_{0}=\frac{a}{\sqrt{2}}$ as

$$
\begin{equation*}
\varphi(x)=\frac{1}{\sqrt{2}} \rho(x) e^{i \frac{\theta(x)}{a}}=\frac{1}{\sqrt{2}}(a+\eta(x)) e^{i \frac{\theta(x)}{a}} \tag{8.47}
\end{equation*}
$$

one gets

$$
\begin{equation*}
\mathscr{L}=\frac{1}{2} \partial_{\mu} \eta \partial^{\mu} \eta-2 a^{2} \lambda \eta^{2}+\frac{1}{2} \partial_{\mu} \theta \partial^{\mu} \theta-\frac{\lambda}{2}\left(4 a \eta^{3}+\eta^{4}\right)+\left(\frac{\eta}{a}+\frac{\eta^{2}}{2 a^{2}}\right) \partial_{\mu} \theta \partial^{\mu} \theta \tag{8.48}
\end{equation*}
$$

The first three terms constitute the quadratic, free part of the Lagrangian, while the others describe the interactions. One clearly sees that the field $\eta$ is massive with $m_{\eta}^{2}=4 \lambda a^{2}$, while $\theta$ is the massless Goldstone mode. The symmetry under $\theta \rightarrow \theta+c$ for any constant $c$ guarantees that a mass term will not be generated in higher orders of perturbation theory. This parameterisations shows clearly the origin of the Goldstone mode. The ground state manifold is just $|\varphi|=\rho=a$, and so a change of phase as described by a fluctuation in the field $\theta$ corresponds to moving along the valley of minima of the potential, which comes at no cost in energy. A change in the amplitude $\rho=\frac{a+\eta}{\sqrt{2}}$ instead displaces the system from the minimum of the potential, and encounters an inertia which corresponds to nonzero mass.

### 8.4 Gauge theories

Gauge theories are characterised by the presence of a local symmetry, rather than just a global one. An example is the theory of electromagnetic interactions, which possesses a local $\mathrm{U}(1)$ symmetry under a local change of phase of the electron field, and thus of the electron (and positron) states:

$$
\begin{equation*}
\psi(x) \rightarrow e^{i \alpha(x)} \psi(x), \quad \bar{\psi}(x) \rightarrow e^{-i \alpha(x)} \bar{\psi}(x) \tag{8.49}
\end{equation*}
$$

What is the motivation for assuming the existence of local symmetries, and what are its consequences?

One of the basic tenets of a relativistic theory is locality: interactions do not propagate instantaneously, but rather take place locally between fields, and then propagate compatibly with the finiteness of the speed of light. No event can therefore affect anything outside of its future lightcone. We also know that the overall phase of the quantum state vector of a system
is experimentally unobservable, and can therefore be chosen arbitrarily. For example, we could decide that the phase of ever electron state be rotated by a factor $e^{i \varphi}$, and no experimental consequence would follow. If all observers agree on this phase redefinition, one speaks of a global $\mathrm{U}(1)$ transformation; since this leaves physics invariant, the system possesses a global $\mathrm{U}(1)$ symmetry. On the other hand, while we make this phase redefinition another experimenter outside our lightcone might want to change the phase of electron states in a different way, and combining the two statements above this should not have any experimental consequence neither for the other observer nor for us. It should therefore be possible to choose the phase of the electron field, which creates and destroys electrons and positrons anywhere in the Universe, in different way in different places. This amounts to asking invariance under the local $\mathrm{U}(1)$ transformations Eq. 88.49), i.e., that the system is invariant under a $\mathrm{U}(1)$ local or gauge symmetry. This symmetry principle is called the gauge principle. In the remainder of this subsection we discuss this principle in some detail.

Consider first a theory of scalar fields, either real or complex, invariant under a global symmetry group $G$,

$$
\begin{equation*}
\mathscr{L}(\phi)=\frac{1}{2} \partial_{\mu} \phi_{i} \partial^{\mu} \phi_{i}-\mathscr{U}(\phi), \quad \mathscr{L}(\phi)=\partial_{\mu} \phi_{i}^{*} \partial^{\mu} \phi_{i}-\mathscr{U}(\phi), \tag{8.50}
\end{equation*}
$$

with $\mathscr{L}(g \phi)=\mathscr{L}(\phi)$ for any $g \in G$, where

$$
\begin{equation*}
(g \phi)_{i}(x)=U_{i j}(g) \phi_{j}(x), \quad U_{i j}(g)=e^{i \varepsilon_{a}(g) T^{a}} \tag{8.51}
\end{equation*}
$$

with $U(g)$ providing a unitary representation of the group, therefore with real $\varepsilon_{a}$, and Hermitian $T^{a}$ representing the generators of the group, $\left[T^{a}, T^{b}\right]=i f_{a b c} T^{c}$. As already discussed before, in the case of real scalar fields the representation is orthogonal and $T^{a}$ are purely imaginary and antisymmetric. Invariance under $G$ amounts to asking $\mathscr{U}(g \phi)=\mathscr{U}(\phi)$. We are assuming the potential to be a function of $\phi$ only and not its derivatives.

The parameters $\varepsilon_{a}$ in Eq. (8.51) are independent of $x$, indicating that we are performing a global transformation. What happens if we promote it to a local one? Since the potential is a local function of $\phi(x)$ only, $\mathscr{U}=\mathscr{U}(\phi(x))$, it makes no difference whether the transformation is global or local, and so $\mathscr{U}(g(x) \phi(x))=\mathscr{U}(\phi(x))$. On the other hand, the kinetic term depends on the derivatives of the field, and is not left invariant by a local transformation:

$$
\begin{equation*}
\partial_{\mu} \phi_{i}(x) \rightarrow \partial_{\mu}\left(U_{i j}(x) \phi_{j}(x)\right)=U_{i j}(x) \partial_{\mu} \phi_{j}(x)+\partial_{\mu}\left(U_{i j}(x)\right) \phi_{j}(x) . \tag{8.52}
\end{equation*}
$$

The first term would cancel, but for a nontrivial $x$-dependence also the second term contributes and invariance is lost. In order to have invariance we are led to introduce a new set of fields, the gauge fields $A_{\mu}^{a}(x), a=1, \ldots, \operatorname{dim} G$, one for each generator of the local symmetry group $G$, to reabsorb the extra term in Eq. 8.52). These fields must be Lorentz vectors, transforming like $\partial_{\mu}$ under Lorentz transformations, and transform almost like adjoint objects under an internal $G$ transformation. Let us now replace in Eq. 8.50) the ordinary derivative $\partial_{\mu}$ with the covariant derivative $D_{\mu}$,

$$
\begin{equation*}
\left(D_{\mu} \phi\right)_{i} \equiv \partial_{\mu} \phi_{i}-i g T_{i j}^{a} A_{\mu}^{a} \phi_{j}, \tag{8.53}
\end{equation*}
$$

where $g$ is a dimensionless coupling constant, and ask for $A_{\mu}^{a}$ to transform in the appropriate way to make Eq. 8.50) invariant under the combined transformation $\phi(x) \rightarrow U(x) \phi(x), A_{\mu}^{a}(x) \rightarrow$ $A_{\mu}^{\prime a}(x)$. Denoting with $A_{\mu}=A_{\mu}^{a} T^{a}$ we have

$$
\begin{equation*}
\left(\partial_{\mu}-i g A_{\mu}\right) \phi \rightarrow U \partial_{\mu} \phi+\left(\partial_{\mu} U\right) \phi-i g A_{\mu}^{\prime} U \phi=U\left[\partial_{\mu}-i g\left(U^{-1} A_{\mu}^{\prime} U+\frac{i}{g} U^{-1} \partial_{\mu} U\right)\right] \phi, \tag{8.54}
\end{equation*}
$$

and asking for invariance we find

$$
\begin{equation*}
A_{\mu}=U^{-1} A_{\mu}^{\prime} U+\frac{i}{g} U^{-1} \partial_{\mu} U \Longrightarrow A_{\mu}^{\prime}=U A_{\mu} U^{-1}-\frac{i}{g}\left(\partial_{\mu} U\right) U^{-1} \tag{8.55}
\end{equation*}
$$

With this transformation rule

$$
\begin{equation*}
D_{\mu} \phi(x) \rightarrow U(x) D_{\mu} \phi(x), \tag{8.56}
\end{equation*}
$$

and since $U^{-1}=U^{\dagger}$ or $U^{-1}=U^{T}$ for unitary or orthogonal matrices, Eq. 8.50 with $\partial_{\mu} \rightarrow D_{\mu}$ becomes invariant under local transformations.

Gauge fields and the covariant derivative are analogous to the connections and the covariant derivative one encounters in general relativity, although there is an important difference, namely that the gauge connections act in some internal space, while the spacetime connections act on the tangent space of spacetime itself. The non-homogenous transformation rule Eq. 88.55) is also analogous to the transformation rule of the connections in general relativity. In the case at hand, the first term corresponds to the transformation rule for a multiplet of fields $A_{\mu}^{a}$ in the adjoint representation, as anticipated; the second term however spoils this property.

Notice that what is the particular representation $U$ of the group and $T^{a}$ of the generators of the algebra is not important, as the transformation rule can be expressed in terms of the gauge fields $A_{\mu}^{a}$ only. This is most easily done by considering infinitesimal transformations; any finite transformation can be obtained from these due to the Lie nature of the symmetry group. For infinitesimal $U(x)=1+i \varepsilon_{a}(x) T^{a}$ we find

$$
\begin{align*}
A_{\mu}^{a} T^{a} & =\left(\mathbf{1}+i \varepsilon_{b} T^{b}\right) A_{\mu}\left(\mathbf{1}-i \varepsilon_{c} T^{c}\right)-\frac{i}{g}\left(\partial_{\mu} i \varepsilon_{a} T^{a}\right)\left(\mathbf{1}-i \varepsilon_{b} T^{b}\right)  \tag{8.57}\\
& =A_{\mu}^{a} T^{a}+i \varepsilon_{b} A_{\mu}^{c}\left[T^{b}, T^{c}\right]+\frac{1}{g} \partial_{\mu} \varepsilon_{a} T^{a}=A_{\mu}^{a} T^{a}-\varepsilon_{b} A_{\mu}^{c} f_{b c a} T^{a}+\frac{1}{g} \partial_{\mu} \varepsilon_{a} T^{a} .
\end{align*}
$$

For semi-simple groups (see below) one has cyclic, totally antisymmetric structure constants $f_{a b c}$ and so $f_{b c a}=f_{a b c}$. We then find

$$
\begin{equation*}
\delta A_{\mu}^{a} \equiv A_{\mu}^{\prime a}-A_{\mu}^{a}=-f_{a b c} \varepsilon_{b} A_{\mu}^{c}+\frac{1}{g} \partial_{\mu} \varepsilon_{a} . \tag{8.58}
\end{equation*}
$$

Notice that as anticipated there is no reference to the representation under which the scalar fields are assumed to transform, and that there are both a $g$-independent homogenous term, and a $g$-dependent inhomogenous one.

The symmetry groups, or gauge groups, we are interested in are direct products of simple groups and Abelian groups, i.e., $G=\times_{i} G_{i}$ with $G_{i}$ either simple or Abelian. For example, the special unitary groups $\mathrm{SU}(N)$ are simple; the special orthogonal groups $\mathrm{SO}(N)$ are simple; the groups $\mathrm{SU}(N) \times \mathrm{SU}(N)$ are semisimple; the groups $\mathrm{U}(1)$ and $\mathrm{SO}(2)$ are Abelian.

A few definitions. An Abelian group is such that all its elements commute with each other. Its algebra is correspondingly generated by commuting elements, and is thus the direct sum of onedimensional commuting algebras. If there are non-commuting elements the group (and its algebra) are called non-Abelian. A simple Lie group $G$ is one with a simple Lie algebra $\mathfrak{g}$, which in turn is a non-Abelian Lie algebra with no nontrivial ideal. An ideal $\mathfrak{a}$ is a subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$, i.e., $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a}$, which is left invariant by the whole algebra, i.e., $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$. For a simple Lie algebra the only ideals are $\{0\}$ and the whole algebra. A semisimple Lie group is one with a semisimple Lie algebra, which in turn is such that it has no nontrivial Abelian ideal; equivalently, it is the direct sum of simple algebras.

It is an important property of non-Abelian gauge theories that there is a single, unique coupling constant for each simple factor in the gauge group: if we have several multiplets of matter fields, they will all be coupled with the same coupling to the non-Abelian gauge fields. This follows from Eq. 8.58): since the coupling constant to the matter fields enters the transformation properties of the gauge field to ensure gauge invariance, then it is uniquely defined by these and must therefore be the same for every matter field. The difference with the Abelian case is that there one can reabsorb the coupling constant by redefining the transformation law of the matter fields, so that these become $\phi \rightarrow e^{i g \alpha} \phi$ (we focus on the $\mathrm{U}(1)$ group case for definitess), while for the gauge field $A_{\mu} \rightarrow A_{\mu}-i \partial_{\mu} \alpha$. At this point nothing prevents us from choosing different couplings for different fields. In the non-Abelian case this trick would not work, since the coupling constant would reappear in the homogenous term in the transformation law of the gauge fields, and would still be constrained to be unique.

Proper (non-projective) irreducible representation of the group $\mathrm{U}(1)$ are of the form $e^{i n \alpha}$ with $n \in$ $\mathbb{Z}$. Asking for these representations only would then impose the existence of a single fundamental $\mathrm{U}(1)$ coupling constant, with any other coupling an integer multiple of this. This is actually what is observed in Nature (the fundamental coupling would be $e / 3$ ).

Gauge symmetry allows dynamics for the gauge fields $A_{\mu}^{a}$, similar to that provided by the Riemann tensor for spacetime connections. Consider the double covariant derivative,

$$
\begin{equation*}
D_{\mu} D_{\nu} \phi=\partial_{\mu} \partial_{\nu} \phi-i g A_{\mu} \partial_{\nu} \phi-i g A_{\nu} \partial_{\mu} \phi-i g\left(\partial_{\mu} A_{\nu}\right) \phi+(-i g)^{2} A_{\mu} A_{\nu} \phi \tag{8.59}
\end{equation*}
$$

The first three term give an object symmetric under $\mu \leftrightarrow \nu$, and so

$$
\begin{equation*}
\left[D_{\mu}, D_{\nu}\right] \phi=-i g\left(\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}-i g\left[A_{\mu}, A_{\nu}\right]\right) \phi \equiv-i g F_{\mu \nu} \phi \tag{8.60}
\end{equation*}
$$

The field strength tensor $F_{\mu \nu}=F_{\mu \nu}^{a} T^{a}$ measures the curvature of the internal space with gauge connection $A_{\mu}^{a}$. One has explicitly

$$
\begin{equation*}
F_{\mu \nu}^{a}=\partial_{\mu} A_{\nu}^{a}-\partial_{\nu} A_{\mu}^{a}+g f_{a b c} A_{\mu}^{b} A_{\nu}^{c} \tag{8.61}
\end{equation*}
$$

The first term is familiar from QED, while the second one is typical of non-Abelian gauge groups, and shows that in this case the gauge fields are self-interacting, even in the absence of matter. Moreover, the coupling constant is once again the same. The transformation properties of $F_{\mu \nu}$ can be obtained by direct calculation from those of $A_{\mu}$, using the simple fact that $U\left(\partial_{\mu} U^{-1}\right)=-\left(\partial_{\mu} U\right) U^{-1}$, and turn out ot be very simple:

$$
\begin{equation*}
F_{\mu \nu} \rightarrow F_{\mu \nu}^{\prime}=U F_{\mu \nu} U^{-1} \tag{8.62}
\end{equation*}
$$

The fields $F_{\mu \nu}^{a}$ thus transform properly as an adjoint multiplet, with no inhomogenous term. For infinitesimal transformations

$$
\begin{equation*}
\delta F_{\mu \nu}^{a}=-f_{a b c} \varepsilon_{b} F_{\mu \nu}^{c} \tag{8.63}
\end{equation*}
$$

It is now easy to build a gauge-invariant kinetic term for the gauge fields:

$$
\begin{equation*}
\mathscr{L}_{\mathrm{YM}}=-\frac{1}{4} F_{\mu \nu}^{a} F^{a \mu \nu}=-\frac{1}{2} \operatorname{tr}_{F} F_{\mu \nu} F^{\mu \nu} \tag{8.64}
\end{equation*}
$$

where $\operatorname{tr}_{F}$ denotes the trace in the fundamental representation and we have used the usual normalisation of the corresponding generators $\operatorname{tr} t_{F}^{a} t_{F}^{b}=\frac{1}{2} \delta^{a b}$. The Lagrangian Eq. (8.64)is known as the Yang-Mills(-Shaw) Lagrangian.

The last ingredient of a realistic gauge theory are fermion multiplets. To promote the usual globally invariant Dirac Lagrangian to a locally invariant one, one follows the same procedure as with the scalar fields, replacing the derivative with the covariant derivative,

$$
\begin{equation*}
\bar{\psi}(i \not \partial-m) \psi \rightarrow \bar{\psi}(i \not D-m) \psi, \tag{8.65}
\end{equation*}
$$

where $\not \mathscr{D}=\partial_{\mu} \gamma^{\mu}, \not D=D_{\mu} \gamma^{\mu}$, and

$$
\begin{equation*}
D_{\mu}=\partial_{\mu}-i g A_{\mu}^{a} t^{a}, \tag{8.66}
\end{equation*}
$$

with the same $g$ but not necessarily the same representation $t^{a}$ of the group generators, $\left[t^{a}, t^{b}\right]=$ $i f_{a b c} t^{c}$. All in all, the Lagrangian of a general gauge theory reads

$$
\begin{equation*}
\mathscr{L}=-\frac{1}{4} F_{\mu \nu}^{a} F^{a \mu \nu}+\frac{1}{2}\left(D_{\mu} \phi\right)_{i}\left(D^{\mu} \phi\right)_{i}-\mathscr{U}(\phi)+\bar{\psi}(i \not D-m) \psi=\mathscr{L}_{\mathrm{YM}}+\mathscr{L}_{\text {matter }} \tag{8.67}
\end{equation*}
$$

where scalar fields are taken to be real without loss of generality. The first term contains the $A-A$ interactions, the second one the $\phi-A$ interactions and the last one the $\psi-A$ interactions, all with the same coupling constant. In particular, the $F F$ term reads explicitly

$$
\begin{align*}
F_{\mu \nu}^{a} F^{a \mu \nu}= & \left(\partial_{\mu} A_{\nu}^{a}-\partial_{\nu} A_{\mu}^{a}\right)\left(\partial^{\mu} A^{a \nu}-\partial^{\nu} A^{a \mu}\right) \\
& +2 g\left(\partial_{\mu} A_{\nu}^{a}-\partial_{\nu} A_{\mu}^{a}\right) f_{a b c} A^{b \mu} A^{c \nu}  \tag{8.68}\\
& +g^{2} f_{a b c} f_{a d e} A_{\mu}^{b} A_{\nu}^{c} A^{d \mu} A^{e \nu}
\end{align*}
$$

so that both cubic and quartic interactions are present. In Eq. 8.67) a mass term $m^{2} A_{\mu} A^{\mu}$ is forbidden by gauge invariance. A term $\epsilon_{\mu \nu \rho \sigma} F^{a \mu \nu} F^{b \sigma \rho}$ is allowed by gauge invariance but is forbidden by parity. Furthermore, it is a total derivative which does not affect the equations of motion 48

Equations of motion are derived in the usual way. One has

$$
\begin{align*}
\frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu} A_{\nu}^{a}\right)} & =-\frac{1}{2} \frac{\partial F_{\rho \sigma}^{m}}{\partial\left(\partial_{\mu} A_{\nu}^{a}\right)} F^{m \rho \sigma}=-F_{\mu \nu}^{a} \\
\frac{\partial \mathscr{L}_{\mathrm{YM}}}{\partial A_{\nu}^{a}} & =-\frac{1}{2} \frac{\partial F_{\rho \sigma}^{m}}{\partial A_{\nu}^{a}} F^{m \rho \sigma}=-g f_{m a b} A_{\mu}^{b} F^{m \nu \mu}=g f_{a b m} A_{\mu}^{b} F^{m \mu \nu}=-i g\left[A_{\mu}, F^{\mu \nu}\right]^{a}  \tag{8.69}\\
\frac{\partial \mathscr{L}_{\text {matter }}}{\partial A_{\nu}^{a}} & \equiv-J^{a \nu}
\end{align*}
$$

from which it follows in matrix notation

$$
\begin{equation*}
\partial_{\mu} F^{\mu \nu}-i g\left[A_{\mu}, F^{\mu \nu}\right]=J^{\nu} . \tag{8.70}
\end{equation*}
$$

Explicitly, denoting with

$$
\begin{equation*}
\left(D_{\mu}^{(A)}\right)_{b}^{a}=\delta_{b}^{a} \partial_{\mu}+g f_{a c b} A_{\mu}^{c}=\delta_{b}^{a} \partial_{\mu}-i g\left(-i f_{c a b}\right) A_{\mu}^{c}=\delta_{b}^{a} \partial_{\mu}-i g T_{a b}^{(A) c} A_{\mu}^{c} \tag{8.71}
\end{equation*}
$$

the covariant derivative in the adjoint representation, then Eq. (8.70) reads

$$
\begin{equation*}
\left(D_{\mu}^{(A)}\right)_{b}^{a} F^{b \mu \nu}=J^{a \nu} . \tag{8.72}
\end{equation*}
$$

[^31]Gauge invariance imposes that the gauge fields be massless vector bosons. We now show that massless vector fields have only two physical degrees of freedom. To this end we consider the free case $g=0$, in which case the equations of motion for the gauge fields reduce to

$$
\begin{align*}
\partial_{\mu}\left(\partial^{\mu} A^{a \nu}-\partial^{\nu} A^{a \mu}\right) & =0,  \tag{8.73}\\
\square A^{a \nu}-\partial^{\nu} \partial \cdot A^{a} & =0 .
\end{align*}
$$

Differently from the massive case, taking the divergence of this equation does not lead to nontrivial constraints, since it gives zero identically. In fact, we are free to replace $A_{\mu}^{a} \rightarrow A_{\mu}^{a}+\partial_{\mu} \Lambda^{a}$ for arbitrary $\Lambda^{a}$ and we would still get a solution of Eq. 8.73), which tells us that $\partial \cdot A$ is not determined. This redundancy of the field variables is a consequence of gauge invariance, and further conditions must be imposed to obtain a unique solution to Eq. 8.73) (besides including initial conditions at some time $t$ ). The general solution of Eq. (8.73) is most easily obtained in momentum space: setting

$$
\begin{equation*}
A_{\mu}^{a}(x)=\int \frac{d^{4} p}{(2 \pi)^{4}} e^{-i p \cdot x} \tilde{A}_{\mu}^{a}(p), \quad \tilde{A}_{\mu}^{a}(p)=\int d^{4} x e^{i p \cdot x} A_{\mu}^{a}(x), \tag{8.74}
\end{equation*}
$$

the differential equation Eq. 8.73) turns into an algebraic equation,

$$
\begin{equation*}
\left(p^{2} \delta^{\nu}{ }_{\mu}-p^{\nu} p_{\mu}\right) \tilde{A}^{a \mu}(p)=0 . \tag{8.75}
\end{equation*}
$$

The solution of Eq. (8.75) can be decomposed on a complete basis of four-vectors and must be of the form

$$
\begin{equation*}
\tilde{A}^{a \mu}(p)=a^{i}(p) \varepsilon_{i}^{\mu}(p)+b(p) \tilde{p}^{\mu}+c(p) p^{\mu} \tag{8.76}
\end{equation*}
$$

where

$$
\begin{align*}
p^{\mu} & =(E, \vec{p}), \\
\tilde{p}^{\mu} & =(-E, \vec{p}),  \tag{8.77}\\
\varepsilon_{i}^{\mu}(p) & =\left(0, \vec{\varepsilon}_{i}(\vec{p})\right), \quad \vec{\varepsilon}_{i}(\vec{p}) \cdot \vec{p}=0, \quad \vec{\varepsilon}_{i}(\vec{p}) \cdot \vec{\varepsilon}_{j}(\vec{p})=\delta_{i j} .
\end{align*}
$$

No relation is assumed for the time being between $E$ and $\vec{p}$. Clearly $p \cdot \varepsilon_{i}=\tilde{p} \cdot \varepsilon_{i}=0$, and furthermore $p$ and $\tilde{p}$ are linearly independent, although $p \cdot \tilde{p}=-\left(E^{2}+\vec{p}^{2}\right)$. Imposing Eq. 8.76) we find

$$
\begin{align*}
p^{2}\left(a^{i}(p) \varepsilon_{i}^{\nu}(p)+b(p) \tilde{p}^{\nu}+c(p) p^{\nu}\right)-p^{\nu}\left(b(p) p \cdot \tilde{p}+c(p) p^{2}\right) & =0,  \tag{8.78}\\
p^{2}\left(a^{i}(p) \varepsilon_{i}^{\nu}(p)+b(p) \tilde{p}^{\nu}\right)-p \cdot \tilde{p} p^{\nu} b(p) & =0 .
\end{align*}
$$

The term proportional to $c(p)$ drops out of the equation, showing that it is completely arbitrary and therefore unphysical. Contracting Eq. (8.78) with $\tilde{p}$ we find

$$
\begin{align*}
{\left[p^{2} \tilde{p}^{2}-(p \cdot \tilde{p})^{2}\right] b(p) } & =0, \\
{\left[\left(E^{2}-\vec{p}^{2}\right)^{2}-\left(E^{2}+\vec{p}^{2}\right)^{2}\right] b(p) } & =0,  \tag{8.79}\\
4 E^{2} \vec{p}^{2} b(p) & =0
\end{align*}
$$

This is solved by $b(p)=0$ for arbitrary $p 4$ Contracting with either of the $\varepsilon_{i}$ yields instead

$$
\begin{equation*}
p^{2} a^{i}(p)=0, \tag{8.80}
\end{equation*}
$$

[^32]which is solved by arbitrary $a^{i}(p)$ if $p^{2}=0$ (or otherwise by $a^{i}=0$ if $p^{2} \neq 0$, but this would give a trivially vanishing field). The solution of the equations of motions is thus
\[

$$
\begin{equation*}
\tilde{A}^{a \mu}(p)=a^{i}(p) \varepsilon_{i}^{\mu}(p)+c(p) p^{\mu}, \quad p^{2}=0 \tag{8.81}
\end{equation*}
$$

\]

for arbitrary $a^{i}$ and $c$. Since $c$ is unphysical, there are only two degrees of freedom, corresponding to the transverse polarisations $\varepsilon_{i}^{\mu}(\vec{p})$.

### 8.5 Spontaneously broken gauge theories: the Higgs mechanism

It is now time to combine gauge theories with spontaneous symmetry breaking, to find that in this case there are no Goldstone bosons. We discuss first a few examples, before presenting the general theory.

Example 1: $G=\mathrm{SO}(2)$ or $G=\mathrm{U}(1) \quad$ It is more practical to use the $\mathrm{U}(1)$ version, where the two real scalar fields are combined into a single complex field. The Lagrangian reads

$$
\begin{equation*}
\mathscr{L}=-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}+\left(D_{\mu} \varphi\right)^{*}\left(D_{\mu} \varphi\right)-\frac{\lambda}{2}\left(2 \varphi^{*} \varphi-a^{2}\right)^{2}, \quad \lambda>0, \tag{8.82}
\end{equation*}
$$

where

$$
\begin{equation*}
F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\mu} A_{\nu}, \quad D_{\mu}=\partial_{\mu}+i e A_{\mu}, \quad \varphi=\frac{1}{\sqrt{2}}\left(\varphi_{1}+i \varphi_{2}\right) \tag{8.83}
\end{equation*}
$$

This is an Abelian gauge theory of a complex field with charge $-e$. The minimal energy is achieved with $F_{\mu \nu}=0$, which implies $A_{\mu}=0$ (up to gauge transformations), and with $\varphi(x)=\varphi_{0}$ with $\varphi_{0}^{*} \varphi_{0}=\frac{a^{2}}{2}$. We now choose the ground state to be $\varphi_{0}=\frac{a}{\sqrt{2}}, A_{\mu}=0$, and parameterise

$$
\begin{equation*}
\varphi(x)=\frac{1}{\sqrt{2}}(a+\eta(x)) e^{i \frac{\theta(x)}{a}} . \tag{8.84}
\end{equation*}
$$

In order to extract physical statements, either classically by solving the Cauchy problem or in the framework of quantum field theory by quantising the system), we need to fix the gauge. By means of a $\mathrm{U}(1)$ transformation we can always set

$$
\begin{equation*}
\varphi(x) \rightarrow e^{-i \frac{\theta(x)}{a}} \varphi(x)=\frac{1}{\sqrt{2}}(a+\eta(x)), \tag{8.85}
\end{equation*}
$$

i.e., impose that $\varphi$ be real. For the gauge field

$$
\begin{equation*}
A_{\mu}(x) \rightarrow A_{\mu}(x)-\frac{i}{-e}\left(\partial_{\mu} e^{-i \frac{\theta(x)}{a}}\right) e^{i \frac{\theta(x)}{a}}=A_{\mu}(x)+\frac{1}{e a} \partial_{\mu} \theta(x) \equiv B_{\mu} . \tag{8.86}
\end{equation*}
$$

Denoting for clarity $F_{\mu \nu}(B)=\partial_{\mu} B_{\nu}-\partial_{\nu} B_{\mu}$ and $D_{\mu}(B)=\partial_{\mu}+i e B_{\mu}$, we find that

$$
\begin{align*}
\mathscr{L}_{\text {gauge fixed }} & =-\frac{1}{4} F_{\mu \nu}(B) F^{\mu \nu}(B)+\left(D_{\mu}(B) \frac{a+\eta}{\sqrt{2}}\right)^{*}\left(D^{\mu}(B) \frac{a+\eta}{\sqrt{2}}\right)-\frac{\lambda}{2}\left((a+\eta)^{2}-a^{2}\right)^{2} \\
& =-\frac{1}{4} F_{\mu \nu}(B) F^{\mu \nu}(B)+\frac{1}{2}\left(\partial_{\mu} \eta+i e B_{\mu}(a+\eta)\right)^{*}\left(\partial^{\mu} \eta+i e B^{\mu}(a+\eta)\right)-\frac{\lambda}{2}\left(\eta^{2}+2 a \eta\right)^{2} \\
& =-\frac{1}{4} F_{\mu \nu}(B) F^{\mu \nu}(B)+\frac{1}{2} \partial_{\mu} \eta \partial^{\mu} \eta+\frac{1}{2} e^{2} B_{\mu} B^{\mu}(a+\eta)^{2}-\frac{\lambda}{2}\left(\eta^{2}+2 a \eta\right)^{2} . \tag{8.87}
\end{align*}
$$

There are two remarkable aspects in Eq. (8.87). First of all, a mass term has appeared for the gauge boson $B_{\mu}$, with $m_{B}=e a$. Moreover, the would-be Goldstone mode, $\theta(x)$, has disappeared, becoming the longitudinal component of the massive vector boson $B_{\mu}$. The remaining scalar
field $\eta$, the Higgs field, is massive, with $m_{\eta}=2 a \sqrt{\lambda}$. The gauge defined by Eq. 8.85) is called unitarity gauge, and allows to read explicitly the spectrum of the theory. In fact, the gauge-fixed Lagrangian Eq. 8.87) can be quantised without the appearance of unphysical modes, and its particle content is transparent. In terms of the real field multiplet $\phi=\left(\varphi_{1}, \varphi_{2}\right)$ the vacuum choice and the gauge choice Eq. 88.85 correspond to imposing

$$
\begin{equation*}
\phi_{0}=\binom{a}{0}, \quad \phi=\binom{a+\eta}{0} \equiv \phi_{0}+\tilde{\phi} \tag{8.88}
\end{equation*}
$$

which in terms of the $\mathrm{SO}(2)$ generator $T$ (see Eq. 8.26) reads

$$
\begin{equation*}
\tilde{\phi}^{T} T \phi_{0}=\tilde{\phi}_{i} T_{i j} \phi_{0 j}=0 . \tag{8.89}
\end{equation*}
$$

To see that nothing got lost along the way, let us count the number of degrees of freedom before and after symmetry breaking:

| before |  | after |  |
| :---: | :---: | :---: | :---: |
| 2 real scalars | $\eta, \theta=2$ | 1 real scalar | $\eta=1$ |
| 1 massless vector | $A_{\mu}=2$ | 1 massive vector | $B_{\mu}=3$ |

so that the totals match.

Example 2: $G=\mathrm{SU}(2) \quad$ Consider a doublet $\psi$ of complex scalar fields as in Eq. 8.29), coupled to the non-Abelian $\mathrm{SU}(2)$ gauge fields through $D_{\mu} \psi$,

$$
\begin{equation*}
D_{\mu}=\partial_{\mu}-i g A_{\mu}^{a} \frac{\sigma^{a}}{2}, \quad a=1,2,3 . \tag{8.91}
\end{equation*}
$$

The field strength tensor reads

$$
\begin{equation*}
F_{\mu \nu}^{a}=\partial_{\mu} A_{\nu}^{a}-\partial_{\nu} A_{\mu}^{a}+g \varepsilon_{a b c} A_{\mu}^{b} A_{\nu}^{c} \tag{8.92}
\end{equation*}
$$

and the Lagrangian of the model is

$$
\begin{equation*}
\mathscr{L}=-\frac{1}{4} F_{\mu \nu}^{a} F^{a \mu \nu}+\left(D_{\mu} \psi\right)^{\dagger}\left(D^{\mu} \psi\right)-\lambda\left(\psi^{\dagger} \psi-a^{2}\right)^{2}, \quad \lambda>0 . \tag{8.93}
\end{equation*}
$$

The ground state is $A_{\mu}^{a}=0$ and $\psi_{0}^{\dagger} \psi_{0}=a^{2}$. We choose $\psi_{0}=(0, a)$ and we write

$$
\begin{equation*}
\psi(x)=e^{i \frac{\theta^{a}(x)}{a} \frac{\sigma^{a}}{2}}\binom{0}{a+\frac{\eta(x)}{\sqrt{2}}} . \tag{8.94}
\end{equation*}
$$

This is the most general field configuration, expressed in terms of a real fluctuation $\eta$ around the vacuum $\psi_{0}$ and of an $\operatorname{SU}(2)$ rotation, parameterised by three real fields $\theta^{a}$. If we now choose the gauge by imposing that $\psi$ only has a real $\psi_{2}$ component, i.e., if we make the gauge transformation $\Omega(x)$ with $\Omega(x)=e^{-i \frac{\theta^{a}(x)}{a} \frac{\sigma^{a}}{2}} \in \mathrm{SU}(2)$,

$$
\begin{align*}
\psi^{\prime}(x) & =\Omega(x) \psi(x) \\
A_{\mu}^{\prime}(x) & =\Omega(x) A_{\mu}(x) \Omega(x)^{\dagger}-\frac{i}{g}\left(\partial_{\mu} \Omega(x)\right) \Omega(x)^{\dagger} \tag{8.95}
\end{align*}
$$

where $A_{\mu}=A_{\mu}^{a} \frac{\sigma^{a}}{2}$, then

$$
\begin{align*}
\left(D_{\mu} \psi^{\prime}\right)^{\dagger}\left(D^{\mu} \psi^{\prime}\right)= & \left(\frac{1}{\sqrt{2}}\left(0, \partial_{\mu} \eta\right)+i g\left(0, a+\frac{\eta(x)}{\sqrt{2}}\right) A_{\mu}^{\prime}\right)\left(\begin{array}{c}
\frac{1}{\sqrt{2}}\binom{0}{\partial^{\mu} \eta}-i g A^{\prime \mu}\binom{0}{a+\frac{\eta(x)}{\sqrt{2}}}
\end{array}\right) \\
= & \frac{1}{2} \partial_{\mu} \eta \partial^{\mu} \eta+\frac{i g}{\sqrt{2}}\left(\left(0, a+\frac{\eta(x)}{\sqrt{2}}\right) A_{\mu}^{\prime}\binom{0}{\partial^{\mu} \eta}-\left(0, \partial_{\mu} \eta\right) A^{\prime \mu}\binom{0}{a+\frac{\eta(x)}{\sqrt{2}}}\right)  \tag{8.96}\\
& +g^{2}\left(0, a+\frac{\eta(x)}{\sqrt{2}}\right) A_{\mu}^{\prime} A^{\prime \mu}\binom{0}{a+\frac{\eta(x)}{\sqrt{2}}}
\end{align*}
$$

The middle term vanishes: in fact, since only the lower components of the row and column vectors are nonzero, the contributions of $A_{\mu}^{\prime 1,2} \sigma^{1,2}$ are identically zero and only $A_{\mu}^{\prime 3} \sigma^{3}$ could contribute, but in this case the two terms in brackets cancel each other out. Moreover,

$$
\begin{equation*}
A_{\mu}^{\prime} A^{\prime \mu}=\frac{1}{4} A_{\mu}^{\prime a} A^{\prime b \mu} \sigma^{a} \sigma^{b}=\frac{1}{4} A_{\mu}^{\prime a} A^{\prime b \mu}\left(\delta^{a b}+i \varepsilon^{a b c} \sigma^{c}\right)=\frac{1}{4} A_{\mu}^{\prime a} A^{\prime a \mu} \tag{8.97}
\end{equation*}
$$

We conclude

$$
\begin{equation*}
\left(D_{\mu} \psi^{\prime}\right)^{\dagger}\left(D^{\mu} \psi^{\prime}\right)=\frac{1}{2} \partial_{\mu} \eta \partial^{\mu} \eta+\frac{g^{2}}{4} A_{\mu}^{\prime a} A^{\prime a \mu}\left(a+\frac{\eta(x)}{\sqrt{2}}\right)^{2} \tag{8.98}
\end{equation*}
$$

Dropping the primes, we find for the gauge-fixed Lagrangian

$$
\begin{equation*}
\mathscr{L}_{\text {gauge fixed }}=-\frac{1}{4} F_{\mu \nu}^{a} F^{a \mu \nu}+\frac{1}{2} \partial_{\mu} \eta \partial^{\mu} \eta+\frac{g^{2}}{4} A_{\mu}^{a} A^{a \mu}\left(a+\frac{\eta(x)}{\sqrt{2}}\right)^{2}-\lambda \frac{\eta^{2}}{2}\left(2 a+\frac{\eta}{\sqrt{2}}\right)^{2} \tag{8.99}
\end{equation*}
$$

From the quadratic part one can easily read off the degrees of freedom: there is one real massive scalar $\eta$ with mass $m_{\eta}=2 a \sqrt{\lambda}$, and three massive vectors $A_{\mu}^{a}$ with mass $m_{A}=\frac{g a}{\sqrt{2}}$. The $\mathrm{SU}(2)$ symmetry is completely broken, and all three gauge bosons acquire a mass. None of the wouldbe massless modes $\theta^{a}$ is contained in the physical spectrum, as they have been absorbed as the longitudinal component of the massive vector fields. This is again transparent in the unitarity gauge, Eq. 8.95). In terms of a quartet of real scalars $\phi=\left(\phi_{1}, \phi_{3}, \phi_{2}, \phi_{4}\right)$ (see Eq. 8.29) , the gauge group representation is built out of the generators $\Sigma^{a} / 2$,

$$
i \Sigma^{a}=\left(\begin{array}{cc}
-\operatorname{Im} \sigma^{a} & -\operatorname{Re} \sigma^{a}  \tag{8.100}\\
\operatorname{Re} \sigma^{a} & -\operatorname{Im} \sigma^{a}
\end{array}\right)
$$

The vacuum $\phi_{0}$ and the fluctuations $\tilde{\phi}$ around the vacuum in unitarity gauge read respectively

$$
\phi_{0}=\left(\begin{array}{l}
0  \tag{8.101}\\
a \\
0 \\
0
\end{array}\right), \quad \tilde{\phi}=\frac{1}{\sqrt{2}}\left(\begin{array}{l}
0 \\
\eta \\
0 \\
0
\end{array}\right)
$$

so the unitarity gauge condition reads in this case

$$
\begin{equation*}
\tilde{\phi}^{T} i \Sigma^{a} \phi_{0}, \quad a=1,2,3 \tag{8.102}
\end{equation*}
$$

This is precisely the request that the Goldstone modes be zero (see after Eq. (8.43)). Let us check again the counting of degrees of freedom:

| before |  | after |  |
| :---: | :---: | :---: | :---: |
| 4 real scalars | $\eta, \theta^{a}=4$ | 1 real scalar | $\eta=1$ |
| 3 massless vectors | $A_{\mu}^{a}=6$ | 3 massive vectors | $A_{\mu}^{a}=9$ |

which again match before and after symmetry breaking.

Example 3: $G=\mathrm{SO}(3)$ Let us finally consider gauge group $\mathrm{SU}(2)$ with a triplet of adjoint scalars, or equivalently a triplet of fundamental scalars with gauge group $\mathrm{SO}(3)$. The Lagrangian reads

$$
\begin{equation*}
\mathscr{L}=-\frac{1}{4} F_{\mu \nu}^{a} F^{a \mu \nu}+\left(D_{\mu} \phi\right)^{T}\left(D^{\mu} \phi\right)-\lambda\left(\phi^{T} \phi-\Lambda^{2}\right)^{2}, \quad \lambda>0, \tag{8.104}
\end{equation*}
$$

with $\phi_{i}, i=1,2,3$ real fields. The covariant derivative reads

$$
\begin{equation*}
D_{\mu}=\partial_{\mu}-i g A_{\mu}^{a} T^{a}, \quad a=1,2,3, \quad\left(T^{a}\right)_{b c}=-i \varepsilon_{a b c} . \tag{8.105}
\end{equation*}
$$

The representation is in this case orthogonal, with $\left(T^{a}\right)^{T}=-T^{a}$. We take as vacuum configuration $A_{\mu}^{a}=0$ and $\phi_{i}=\Lambda \delta_{i 3}$. The most general field configuration can be written as

$$
\phi(x)=e^{i\left(\theta^{1}(x) T^{1}+\theta^{2}(x) T^{2}\right)}\left(\begin{array}{c}
0  \tag{8.106}\\
0 \\
\Lambda+\eta(x)
\end{array}\right)=U(x)\left(\begin{array}{c}
0 \\
0 \\
\Lambda+\eta(x)
\end{array}\right) .
$$

No term proportional to $T^{3}$ appears in the exponent, since rotations around $T^{3}$ leave the column vector in Eq. 8.106 invariant. In fact, such rotations constitute the $\mathrm{SO}(2)$ stability group of the vacuum, to which the symmetry group breaks down. Unitarity gauge is reached by transforming $\phi \rightarrow \phi^{\prime}=U^{T} \phi$, and $A_{\mu}^{a}$ accordingly. In term of the new fields (we drop primes for notational clarity)

$$
\begin{equation*}
\left(D_{\mu} \phi\right)^{T}\left(D^{\mu} \phi\right)=\partial_{\mu} \phi^{T} \partial^{\mu} \phi+i g A_{\mu}^{a}\left(\phi^{T} T^{a} \partial^{\mu} \phi-\partial^{\mu} \phi^{T} T^{a} \phi\right)+g^{2} A_{\mu}^{a} A^{b \mu} \phi^{T} T^{a} T^{b} \phi \tag{8.107}
\end{equation*}
$$

Since $\phi_{a}=\delta_{a 3} \phi_{3}$,

$$
\begin{equation*}
\left(\phi^{T} T^{a} \partial^{\mu} \phi-\partial^{\mu} \phi^{T} T^{a} \phi\right)=T_{b c}^{a}\left(\phi^{b} \partial^{\mu} \phi^{c}-\partial^{\mu} \phi^{b} \phi^{c}\right)=T_{33}^{a}\left(\phi^{3} \partial^{\mu} \phi^{3}-\partial^{\mu} \phi^{3} \phi^{3}\right)=0 \tag{8.108}
\end{equation*}
$$

and

$$
\begin{equation*}
\phi^{T} T^{a} T^{b} \phi=\left(\phi_{3}\right)^{2}\left(T^{a} T^{b}\right)_{33}=-\left(\phi_{3}\right)^{2} \varepsilon_{a 3 m} \varepsilon_{b m 3}=\left(\phi_{3}\right)^{2}\left(\delta_{a b} \delta_{33}-\delta_{a 3} \delta_{b 3}\right), \tag{8.109}
\end{equation*}
$$

we conclude

$$
\begin{equation*}
\left(D_{\mu} \phi\right)^{T}\left(D^{\mu} \phi\right)=\partial_{\mu} \eta \partial^{\mu} \eta+g^{2}(\Lambda+\eta)^{2}\left(A_{\mu}^{1} A^{1 \mu}+A_{\mu}^{2} A^{2 \mu}\right) . \tag{8.110}
\end{equation*}
$$

The gauge fixed Lagrangian is thus

$$
\begin{equation*}
\mathscr{L}_{\text {gauge fixed }}=\frac{1}{2} \partial_{\mu} \eta \partial^{\mu} \eta-\lambda \eta^{2}(2 \Lambda+\eta)^{2}-\frac{1}{4} F_{\mu \nu}^{a} F^{a \mu \nu}+\frac{g^{2}}{2}(\Lambda+\eta)^{2}\left(A_{\mu}^{1} A^{1 \mu}+A_{\mu}^{2} A^{2 \mu}\right) . \tag{8.111}
\end{equation*}
$$

This time only two of the gauge fields acquire a mass $m_{1,2}=g \Lambda$, while $A_{\mu}^{3}$ remains massless, corresponding to the fact that the symmetry generated by $T^{3}$ is not broken by the vacuum. The remaining scalar field is also massive, with $m_{\eta}=2 \Lambda \sqrt{2 \lambda}$. Counting degrees of freedom we find
before

| 3 real scalars | $\eta, \theta^{1,2}=3$ | 1 real scalar | $\eta=1$ |
| :---: | :---: | :---: | :---: |
| 3 massless vectors | $A_{\mu}^{1,2,3}=6$ | 2 massive vectors | $A_{\mu}^{1,2}=6$ |
|  |  | 1 massless vector | $A_{\mu}^{3}=2$ |

again matching before and after the symmetry breaking.

Let us now discuss the general case. Consider a gauge theory with gauge group $G, \operatorname{dimG}=n$, broken by a set of scalar fields with a Mexican-hat type potential down to $H, \operatorname{dim} H=n^{\prime}$. In the case in which $G$ is only a global symmetry, this situation gives rise to $n-n^{\prime}$ massless Goldstone bosons. As we will show now, when the symmetry $G$ is local what happens is that the $n^{\prime}$ gauge bosons corresponding to the generators of $H$ remain massless, the $n-n^{\prime}$ gauge bosons corresponding to the broken generators acquire a mass, and no masslees scalars (Goldstone bosons) appear in the spectrum. The relevant part of the most general Lagrangian for the situation under discussion reads

$$
\begin{equation*}
\mathscr{L}=-\frac{1}{4} F_{\mu \nu}^{a} F^{a \mu \nu}+\left(D_{\mu} \phi\right)^{T}\left(D^{\mu} \phi\right)-\mathscr{U}(\phi), \tag{8.113}
\end{equation*}
$$

with positive potential $\mathscr{U}(\phi) \geq 0$, and real scalar fields $\phi$. In a real representation, $D_{\mu}=$ $\partial_{\mu}-i g A_{\mu}^{a} T^{a}$ with $i T^{a}$ real and antisymmetric. Assume now that $\exists \phi_{0} \neq 0$ for which $\mathscr{U}\left(\phi_{0}\right)=0$, build the vacuum manifold $\mathcal{M}=\left\{G \phi_{0}\right\}$ (we assume that it is made up of a single $G$-orbit), and identify the stability group $H, H \phi_{0}=\phi_{0}$. Setting $\phi=\phi_{0}+\tilde{\phi}$ with $\tilde{\phi}$ the fluctuations around the vacuum, we impose the unitarity gauge condition

$$
\begin{equation*}
\tilde{\phi}_{i} T_{i j}^{a} \phi_{0 j}=0, \quad a=n^{\prime}+1, \ldots, n, \tag{8.114}
\end{equation*}
$$

with $\left\{T^{a} \mid a=1, \ldots, n^{\prime}\right\}$ spanning the algebra of $H$. (For $a=1, \ldots, n^{\prime}$ Eq. (8.114) is trivially satisfied.) The condition Eq. 8.114) amounts precisely to setting the would-be Goldstone modes to zero, and it can be shown that it is an admissible gauge condition 50 The gauge fixed Lagrangian contains the term

$$
\begin{equation*}
\left(D_{\mu} \phi\right)^{T}\left(D^{\mu} \phi\right)=\partial_{\mu} \phi_{i} \partial^{\mu} \phi_{i}-i g A_{\mu}^{a}\left(\partial^{\mu} \phi_{i} T_{i k}^{a} \phi_{k}-\phi_{i} T_{i k}^{a} \partial^{\mu} \phi_{k}\right)+g^{2} \phi_{i}\left(T^{a} T^{b}\right)_{i j} \phi_{j} A_{\mu}^{a} A^{b \mu} \tag{8.115}
\end{equation*}
$$

and using the unitarity gauge condition Eq. (8.114) one finds

$$
\begin{align*}
\partial^{\mu} \phi_{i} T_{i k}^{a} \phi_{k}-\phi_{i} T_{i k}^{a} \partial^{\mu} \phi_{k} & =\partial^{\mu}\left(\tilde{\phi}_{i} T_{i k}^{a} \phi_{0 k}-\phi_{0 i} T_{i k}^{a} \tilde{\phi}_{k}\right)+\partial^{\mu} \tilde{\phi}_{i} T_{i k}^{a} \tilde{\phi}_{k}-\tilde{\phi}_{i} T_{i k}^{a} \partial^{\mu} \tilde{\phi}_{k}  \tag{8.116}\\
& =\partial^{\mu} \tilde{\phi}_{i} T_{i k}^{a} \tilde{\phi}_{k}-\tilde{\phi}_{i} T_{i k}^{a} \partial^{\mu} \tilde{\phi}_{k}
\end{align*}
$$

From this term originates a cubic interaction term $\tilde{\phi} \tilde{\phi} A$. The quadratic part of Eq. 8.115) reads

$$
\begin{equation*}
\left.\left(D_{\mu} \phi\right)^{T}\left(D^{\mu} \phi\right)\right|_{\text {quadratic part }}=\partial_{\mu} \tilde{\phi}_{i} \partial^{\mu} \tilde{\phi}_{i}++g^{2} \phi_{0 i}\left(T^{a} T^{b}\right)_{i j} \phi_{0 j} A_{\mu}^{a} A^{b \mu} \tag{8.117}
\end{equation*}
$$

[^33]and contains a mass term for the gauge fields. The mass matrix reads
\[

$$
\begin{equation*}
M_{a b}^{2}=g^{2} \phi_{0 i}\left(T^{a} T^{b}\right)_{i j} \phi_{0 j}=-g^{2}\left\langle T^{a} \phi_{0}, T^{b} \phi_{0}\right\rangle, \tag{8.118}
\end{equation*}
$$

\]

where angular brackets denote the standard (real) scalar product $\langle v, w\rangle=\sum_{i} v_{i} w_{i}$. The mass matrix $M^{2}$ is positive definite: in fact for any set of $v_{a} \in \mathbb{R}$

$$
\begin{equation*}
v_{a} v_{b} M_{a b}^{2}=\left\langle i g v_{a} T^{a} \phi_{0}, i g v_{b} T^{b} \phi_{0}\right\rangle \geq 0 \tag{8.119}
\end{equation*}
$$

since $i g v_{a} T^{a}$ is a real matrix. Since $T^{a} \phi_{0}=0$ for $a=1, \ldots, n^{\prime}$,

$$
M^{2}=\left(\begin{array}{cc}
\mathbf{0}_{n^{\prime} \times n^{\prime}} & \mathbf{0}_{n^{\prime} \times\left(n-n^{\prime}\right)}  \tag{8.120}\\
\mathbf{0}_{\left(n-n^{\prime}\right) \times n^{\prime}} & \tilde{M}_{\left(n-n^{\prime}\right) \times\left(n-n^{\prime}\right)}^{2}
\end{array}\right),
$$

i.e., there are still $n^{\prime}$ massless vector bosons in the spectrum. The $\left(n-n^{\prime}\right) \times\left(n-n^{\prime}\right)$ block $\tilde{M}^{2}$ can be diagonalised and the masses of the remaining gauge bosons, corresponding to the generators of the complement of the algebra of $H$, determined.

As we have already remarked, in unitarity gauge the spectrum of the theory is transparent. On the other hand, the fate of renormalisability (a gauge theory is renormalisable when the symmetries are intact) is unclear in this gauge. However, as shown by 't Hooft and others, there exist gauges in which renormalisability becomes apparent, at the cost of a less clear particle spectrum. On the other hand, gauge invariance means that the physics is independent of the particular choice one makes, so if the theory is renormalisabile in a certain gauge then it is just renormalisable; and if Goldstone bosons are absent in a gauge, then they are just unphysical (gauge) modes. We have then obtained a renormalisable way to give mass to gauge bosons.

## 9 The Standard Model

We have now all the tools to attemps a description of weak interactions in terms of the exchange of massive vector bosons, taking these as the gauge bosons of a spontanteously broken gauge theory in order to have renormalisability. Building the appropriate model requires three steps:

1. find the right gauge group $G$ and unbroken subgroup $H$;
2. find a set of scalar fields that realises the desired symmetry breaking pattern $G \rightarrow H$;
3. choose the representation multiplets of the physical fields.

The phenomenologically successful model is the Glashow-Salam-Weinberg model, which is the minimal model unifying electromagnetism and weak interactions using a spontaneously broken gauge theory with group $G=\mathrm{SU}(2)_{L} \times \mathrm{U}(1)_{Y}$ broken to $H=\mathrm{U}(1)_{\text {EM }}$ (two different $\mathrm{U}(1)$ groups appear in $G$ and $H$ ).

### 9.1 Finding the gauge group

Consider a model world with only $e, \nu_{e}$, electromagnetic and weak interactions. The phenomenologically known currents are

$$
\begin{equation*}
j_{\mu}^{W}=\bar{\nu}_{e} \mathcal{O}_{L \mu} e=\bar{\nu}_{e} \gamma_{\mu}\left(1-\gamma^{5}\right) e, \quad j_{\mu}^{E M}=-\bar{e} \gamma_{\mu} e \tag{9.1}
\end{equation*}
$$

In a gauge theory, such currents are coupled to gauge bosons, leading to

$$
\begin{equation*}
\mathscr{L}_{W}=g\left(j_{\mu}^{W} W^{\mu}+j_{\mu}^{W \dagger} W^{\mu \dagger}\right), \quad \mathscr{L}_{E M}=e j_{\mu}^{E M} A^{\mu} \tag{9.2}
\end{equation*}
$$

with $g$, e coupling constants, $W^{\mu}$ the $W$-boson field and $A^{\mu}$ the photon field. There are therefore at least three gauge bosons, with associated vector currents that are conserved due to the symmetry under global transformation. The corresponding charges are also conserved. These read

$$
\begin{align*}
T_{+}(t) & =\frac{1}{2} \int d^{3} x j_{0}^{W}(t, \vec{x})=\frac{1}{2} \int d^{3} x\left(\nu_{e}^{\dagger}\left(1-\gamma^{5}\right) e\right)(t, \vec{x})=\int d^{3} x\left(\nu_{e L}{ }^{\dagger} e_{L}\right)(t, \vec{x}) \\
T_{-}(t) & =\frac{1}{2} \int d^{3} x j_{0}^{W}(t, \vec{x})^{\dagger}=T_{+}(t)^{\dagger}  \tag{9.3}\\
Q(t) & =-\int d^{3} x j_{0}^{E M}(t, \vec{x})=\int d^{3} x\left(e^{\dagger} e\right)(t, \vec{x})
\end{align*}
$$

where the factors $\frac{1}{2}$ are conventional. Conserved charges are the generators of the symmetry group, and are part of a Lie algebra. Taking the commutator of $T_{+}$and $T_{-}$we then obtain another element of the symmetry algebra. Using the canonical anticommutation relations for fermion fields,

$$
\begin{equation*}
\left\{\psi_{i \alpha}{ }^{\dagger}(t, \vec{x}), \psi_{j \beta}(t, \vec{y})\right\}=\delta_{i j} \delta_{\alpha \beta} \delta^{(3)}(\vec{x}-\vec{y}), \tag{9.4}
\end{equation*}
$$

where $i, j$ denoted the field type and $\alpha, \beta$ are the Dirac indices, one finds

$$
\begin{align*}
T_{3} & \equiv \frac{1}{2}\left[T_{+}, T_{-}\right]=\frac{1}{2} \int d^{3} x \int d^{3} y\left[\nu_{e L}{ }^{\dagger} e_{L}(t, \vec{x}), e_{L}^{\dagger} \nu_{e L}(t, \vec{y})\right] \\
& =\frac{1}{2} \int d^{3} x\left[\nu_{e L}{ }^{\dagger} \nu_{e L}-e_{L}^{\dagger} e_{L}\right](t, \vec{x}) \tag{9.5}
\end{align*}
$$

This charge commutes with $Q,\left[T_{3}, Q\right]=0$, but it is independent of $T_{ \pm}$and $Q$ (it contains the neutrino field, and is an axial vector). This requires the introduction of a third gauge boson for the weak interactions. Further commutators do not give rise to new charges, so we are led to a four-dimensional gauge group ${ }^{51}$ Instead of $Q$ it is more convenient to use

$$
\begin{equation*}
Y=2\left(Q-T_{3}\right), \tag{9.6}
\end{equation*}
$$

since this combination commutes with $T_{1,2,3}$, having derived the Hermitian generators $T_{1,2}$ from the relations $T_{ \pm}=T_{1} \pm i T_{2}$. The gauge group is thus the direct product

$$
\begin{align*}
G & =\mathrm{SU}(2)_{L} \times \mathrm{U}(1)_{Y}, \\
\mathrm{SU}(2)_{L} & \Rightarrow\left\{T_{a} \mid a=1,2,3\right\}  \tag{9.7}\\
\mathrm{U}(1)_{Y} & \Rightarrow Y=2\left(Q-T_{3}\right) .
\end{align*}
$$

where the subscript $L$ refers to the fact that this part of the gauge group acts only on the lefthanded component of the matter fields, while the $\mathrm{U}(1)_{Y}$ part acts on both the left-handed and

[^34]the right-handed parts ${ }^{52}$ The group $\mathrm{SU}(2)_{L}$ is the weak isospin symmetry group, while $\mathrm{U}(1)_{Y}$ is the weak hypercharge symmetry group, both entirely unrelated to the isospin and hypercharge symmetry of the strong interactions in the $\mathrm{SU}(3)$ quark model. Since we have now four gauge bosons but only one long range force associated to the conserved electric charge $Q$, the group $G$ must break down to $\mathrm{U}(1)_{Q}$ if we want to obtain a realistic theory. We will denote the original gauge bosons as $W_{\mu}^{a}, a, 1,2,3$, associated to $T^{1,2,3}$, and $B_{\mu}$ associated to $Y$.

### 9.2 Scalar fields

From previous experience we know that a doublet of complex scalar fields plus a Mexican-hat potential break $\mathrm{SU}(2)$ completely when a vacuum expectation value is developed. We also want this set of fields to preserve the $\mathrm{U}(1)$ subgroup associated to $Q$ when the vacuum expectation value appears. Since $Q=T_{3}+\frac{Y}{2}$ and $T_{3}= \pm \frac{1}{2}$ for the upper and lower component of the doublet, respectively, we choose $Y=1$ and writ ${ }^{533}$

$$
\begin{equation*}
\phi=\binom{\varphi^{+}}{\varphi^{0}} . \tag{9.8}
\end{equation*}
$$

Clearly $\left[Q, \varphi^{+}\right]=\varphi^{+}$and $\left[Q, \varphi^{0}\right]=0$. Since we want to preserve electromagnetic gauge symmetry, it will be the lower component the one that develops a nonzero vacuum expectation value. The covariant derivative is

$$
\begin{equation*}
D_{\mu}=\partial_{\mu}-i g t^{a} W_{\mu}^{a}-\frac{i}{2} g^{\prime} Y B_{\mu} \tag{9.9}
\end{equation*}
$$

where $g, g^{\prime}$ are dimensionless coupling constants, $t^{a}$ are generators of $\mathrm{SU}(2)$ in the appropriate representation, and the factor $\frac{1}{2}$ is conventional. Acting on $\phi$ it reads

$$
\begin{equation*}
D_{\mu} \phi=\left(\partial_{\mu}-i g \frac{\tau^{a}}{2} W_{\mu}^{a}-\frac{i}{2} g^{\prime} B_{\mu}\right) \phi, \tag{9.10}
\end{equation*}
$$

where $\tau^{a}$ are the usual Pauli matrices, appropriate for a weak isospin doublet. The potential, up to an irrelevant constant, reads

$$
\begin{equation*}
V(\phi)=-\mu^{2} \phi^{\dagger} \phi+\lambda\left(\phi^{\dagger} \phi\right)^{2}, \quad \lambda, \mu^{2}>0 . \tag{9.11}
\end{equation*}
$$

The ground state is chosen to be

$$
\begin{equation*}
\langle\phi\rangle_{0} \equiv\langle 0| \phi|0\rangle=\binom{0}{\frac{v}{\sqrt{2}}}, \tag{9.12}
\end{equation*}
$$

where $v^{2}=\frac{\mu^{2}}{\lambda}$ corresponds to the minimum of $V, V_{\min }=-\frac{\mu^{4}}{4 \lambda}$. Having in mind to impose the unitarity gauge condition, we parameterise the most general field configuration as

$$
\begin{equation*}
\binom{\varphi^{+}}{\varphi^{0}}=\mathscr{U}^{-1}(\vec{\xi})\binom{0}{\frac{v+\eta(x)}{\sqrt{2}}}, \quad \mathscr{U}(\vec{\xi})=e^{i \frac{\vec{\xi}(x)}{v} \cdot \frac{\vec{\tau}}{2}} \in \mathrm{SU}(2) \tag{9.13}
\end{equation*}
$$

[^35]with $\langle\eta\rangle_{0}=0$ and $\langle\vec{\xi}\rangle_{0}=0$ corresponding to the choice Eq. (9.12) for the vacuum. We then choose unitarity gauge by making the gauge transformation
\[

$$
\begin{equation*}
\phi(x) \rightarrow \mathscr{U}(\vec{\xi}(x)) \phi(x)=\frac{v+\eta(x)}{\sqrt{2}}\binom{0}{1}=\frac{v+\eta(x)}{\sqrt{2}} \chi . \tag{9.14}
\end{equation*}
$$

\]

The potential reads in this gauge

$$
\begin{equation*}
V=\mu^{2} \eta^{2}+\lambda v \eta^{3}+\frac{\lambda}{4} \eta^{4}-\frac{\mu^{4}}{4 \lambda} \tag{9.15}
\end{equation*}
$$

displaying a mass term and (renormalisable) cubic and quartic self interactions of the Higgs field $\eta$. The kinetic term becomes instead

$$
\begin{align*}
\left(D_{\mu} \phi\right)^{\dagger}\left(D^{\mu} \phi\right)= & \frac{1}{2}\left(D_{\mu}(v+\eta) \chi\right)^{\dagger}\left(D^{\mu}(v+\eta) \chi\right) \\
= & \frac{1}{2} \chi^{\dagger} \chi \partial_{\mu} \eta \partial^{\mu} \eta \\
& +\frac{1}{2}(v+\eta) \chi^{\dagger}\left(i g \frac{\tau^{a}}{2} W_{\mu}^{a}+\frac{i}{2} g^{\prime} B_{\mu}\right) \chi \partial^{\mu} \eta-\frac{1}{2} \partial^{\mu} \eta\left(i g \frac{\tau^{a}}{2} W_{\mu}^{a}+\frac{i}{2} g^{\prime} B_{\mu}\right) \chi(v+\eta) \\
& +\frac{1}{2}(v+\eta)^{2} \chi^{\dagger}\left(i g \frac{\tau^{a}}{2} W_{\mu}^{a}+\frac{i}{2} g^{\prime} B_{\mu}\right)\left(-i g \frac{\tau^{a}}{2} W_{\mu}^{a}-\frac{i}{2} g^{\prime} B_{\mu}\right) \chi \\
= & \frac{1}{2} \chi^{\dagger} \chi \partial_{\mu} \eta \partial^{\mu} \eta+\frac{1}{2}(v+\eta)^{2} \chi^{\dagger}\left(g \frac{\tau^{a}}{2} W_{\mu}^{a}+\frac{1}{2} g^{\prime} B_{\mu}\right)\left(g \frac{\tau^{a}}{2} W_{\mu}^{a}+\frac{1}{2} g^{\prime} B_{\mu}\right) \chi . \tag{9.16}
\end{align*}
$$

The second term vanishes due to the unitarity gauge choice. The third term is the $(2,2)$ component of the matrix sandwiched between $\chi^{\dagger}$ and $\chi$,

$$
\begin{align*}
\text { mass term }= & \chi^{\dagger}\left(g \frac{\tau^{a}}{2} W_{\mu}^{a}+\frac{1}{2} g^{\prime} B_{\mu}\right)\left(g \frac{\tau^{a}}{2} W_{\mu}^{a}+\frac{1}{2} g^{\prime} B_{\mu}\right) \chi \\
= & \frac{g^{2}}{4}\left(W_{\mu}^{1} W^{1 \mu}+W_{\mu}^{2} W^{2 \mu}\right)+\frac{1}{4}\left(g W_{\mu}^{3}-g^{\prime} B_{\mu}\right)\left(g W^{3 \mu}-g^{\prime} B^{\mu}\right) \\
= & \frac{g^{2}}{2}\left(\frac{W_{\mu}^{1}-i W_{\mu}^{2}}{\sqrt{2}}\right)\left(\frac{W^{1 \mu}+i W^{2 \mu}}{\sqrt{2}}\right)  \tag{9.17}\\
& +\frac{g^{2}+g^{\prime 2}}{4}\left(\frac{g}{\sqrt{g^{2}+g^{\prime 2}}} W_{\mu}^{3}-\frac{g^{\prime}}{\sqrt{g^{2}+g^{\prime 2}}} B_{\mu}\right)\left(\frac{g}{\sqrt{g^{2}+g^{\prime 2}}} W^{3 \mu}-\frac{g^{\prime}}{\sqrt{g^{2}+g^{\prime 2}}} B^{\mu}\right) .
\end{align*}
$$

We now set

$$
\begin{align*}
\cos \theta_{W} & =\frac{g}{\sqrt{g^{2}+g^{\prime 2}}}, & \sin \theta_{W} & =\frac{g^{\prime}}{\sqrt{g^{2}+g^{\prime 2}}} \\
W_{\mu}^{ \pm} & =\frac{W_{\mu}^{1} \pm i W_{\mu}^{2}}{\sqrt{2}}, & Z_{\mu} & =\cos \theta_{W} W_{\mu}^{3}-\sin \theta_{W} B_{\mu} \tag{9.18}
\end{align*}
$$

and write

$$
\begin{equation*}
\text { mass term }=\frac{g^{2}}{2} W_{\mu}^{-} W^{+\mu}+\frac{g^{2}+g^{\prime 2}}{4} Z_{\mu} Z^{\mu}=\frac{g^{2}}{2} W_{\mu}^{-} W^{+\mu}+\frac{g^{2}}{4 \cos ^{2} \theta_{W}} Z_{\mu} Z^{\mu} \tag{9.19}
\end{equation*}
$$

All in all, the mass terms coming from the covariant derivative and the potential terms read

$$
\begin{equation*}
\mu^{2} \eta^{2}+\frac{v^{2}}{2}\left[\frac{g^{2}}{2} W_{\mu}^{-} W^{+\mu}+\frac{g^{2}}{4 \cos ^{2} \theta_{W}} Z_{\mu} Z^{\mu}\right]=\mu^{2} \eta^{2}+\left(\frac{g v}{2}\right)^{2} W_{\mu}^{-} W^{+\mu}+\frac{1}{2}\left(\frac{g v}{2 \cos \theta_{W}}\right)^{2} Z_{\mu} Z^{\mu} . \tag{9.20}
\end{equation*}
$$

One then reads off

$$
\begin{equation*}
m_{\eta}=\sqrt{2} \mu, \quad m_{W}=\frac{g v}{2}, \quad m_{Z}=\frac{g v}{2 \cos \theta_{W}}=\frac{m_{W}}{\cos \theta_{W}} \geq m_{W} \tag{9.21}
\end{equation*}
$$

All the gauge bosons have $Y=0: W_{\mu}^{1,2,3}$ do not couple to $B_{\mu}$, and $B_{\mu}$ is Abelian and does not couple to itself. It follows that $Q=T_{3}$ for the gauge fields, and so $Q\left(W^{ \pm}\right)= \pm 1, Q\left(W^{3}\right)=$ $Q(B)=0$. The gauge bosons $W_{\mu}^{ \pm}$are electrically charged, while the neutral gauge boson $Z_{\mu}$, and the orthogonal combination

$$
\begin{equation*}
A_{\mu}=\cos \theta_{W} B_{\mu}+\sin \theta_{W} W_{\mu}^{3}, \tag{9.22}
\end{equation*}
$$

are electrically neutral. It will soon become clear that $A_{\mu}$, which remains massless after symmetry breaking, is nothing but the photon field.

### 9.3 The fermionic sector

We consider first a model world where the only fermions are $e, \nu_{e}$, and the $u$ and $d$ quarks. Since gauge interactions are chiral, left-handed and right-handed components are independent and count as different degrees of freedom. The number of chiral (Weyl) fermions is 15: the leptons $e_{L}, \nu_{e L}$ and $e_{R}$, and the quarks $u_{L}, d_{L}, u_{R}$, and $d_{R}$, which come in three colours each. The $\mathrm{SU}(2)_{L}$ gauge bosons couple to the left-handed fields, while the $\mathrm{U}(1)_{Y}$ couples to both kinds of handedness but with different couplings. Let us see this in detail. The conserved charges read now

$$
\begin{align*}
T_{+} & =\int d^{3} x\left(\nu_{e L}{ }^{\dagger} e_{L}+u_{L}^{\dagger} d_{L}\right) \\
T_{-} & =\int d^{3} x\left(e_{L}^{\dagger} \nu_{e L}+d_{L}^{\dagger} u_{L}\right)  \tag{9.23}\\
2 T_{3} & =\int d^{3} x\left(\nu_{e L}{ }^{\dagger} \nu_{e L}-e_{L}^{\dagger} e_{L}+u_{L}^{\dagger} u_{L}-d_{L}^{\dagger} d_{L}\right)
\end{align*}
$$

The choice of multiplets is guided by phenomenology and the known leptonic and hadronic charged weak currents. The left-handed leptons and quarks form two weak isospin doublets, while the right-handed electron, up and down are isosinglets

$$
\begin{equation*}
\ell_{L}=\binom{\nu_{e L}}{e_{L}}, \quad q_{L}=\binom{u_{L}}{d_{L}}, \quad e_{R}, \quad u_{R}, \quad d_{R} \tag{9.24}
\end{equation*}
$$

The electric charge reads

$$
\begin{equation*}
Q=\int d^{3} x\left(-e_{L}^{\dagger} e_{L}-e_{R}^{\dagger} e_{R}+\frac{2}{3} u_{L}^{\dagger} u_{L}+\frac{2}{3} u_{R}^{\dagger} u_{R}-\frac{1}{3} d_{L}^{\dagger} d_{L}-\frac{1}{3} d_{R}^{\dagger} d_{R}\right) \tag{9.25}
\end{equation*}
$$

Combining Eqs. 9.23) and (9.25),
$Y=2\left(Q-T_{3}\right)=\int d^{3} x\left(-\nu_{e L}{ }^{\dagger} \nu_{e L}-e_{L}{ }^{\dagger} e_{L}-2 e_{R}{ }^{\dagger} e_{R}+\frac{1}{3} u_{L}{ }^{\dagger} u_{L}+\frac{4}{3} u_{R}{ }^{\dagger} u_{R}+\frac{1}{3} d_{L}{ }^{\dagger} d_{L}-\frac{2}{3} d_{R}{ }^{\dagger} d_{R}\right)$.
Clearly $Y$ must be the same in each multiplet since $\left[T_{a}, Y\right]=0$, so for a left-handed doublet it can be computed as $Y_{L}=\frac{1}{2}\left[2\left(Q_{+}-\frac{1}{2}\right)+2\left(Q_{-}+\frac{1}{2}\right)\right]=Q_{+}+Q_{-}$, i.e., the total electric charge of the doublet. For singlets clearly $Y=2 Q$. This can be summarised as twice the average charge of each multiplet. We can now read off the various values:

$$
\begin{align*}
Y\left(\ell_{L}\right) & =-1, \tag{9.27}
\end{align*} \quad Y\left(q_{L}\right)=\frac{1}{3}, \quad ~\left(d_{R}\right)=-\frac{2}{3} .
$$

There is an important theoretical reason why a definite set of fermions with definite quantum numbers have to be considered together. It is a general fact in quantum field theory that certain symmetries of the action at the classical level do not survive quantisation. Such symmetries are called anomalous. A typical example of anomalous symmetries are chiral symmetries, like the one we are using here for our gauge theory. In the presence of an anomaly, the Noether current $J^{\mu}$ associated to the global symmetry is no more conserved. This obviously breaks the gauge symmetry as well, and we lose all the nice properties of a gauge theory. However, the contributions to $\partial_{\mu} J^{\mu}$ can cancel out for the right matter content of the theory: this is what happens if one choses fermion fields as in Eq. (9.24) and (9.27). The set of leptons and quarks listed there constitute one generation of fermions, which is anomaly-free.

The fermionic term in the Lagrangian are of the general form

$$
\begin{equation*}
\mathscr{L}_{\text {fermion }}=\bar{\psi} i \not D \psi-m \bar{\psi} \psi-\mathscr{L}_{\text {Yukawa }}(\phi, \psi, \bar{\psi}) . \tag{9.28}
\end{equation*}
$$

Here

$$
\begin{align*}
D_{\mu} \psi & =\left(\partial_{\mu}-i g T^{a} W^{\mu}-\frac{i}{2} g^{\prime} Y B_{\mu}\right) \psi, \\
T^{a} & =\left\{\begin{array}{llll}
\tau^{a} & \psi=\psi_{L}, \\
0 & \psi=\psi_{R},
\end{array} \quad Y=\left\{\begin{array}{llll}
-1 & \psi=\ell_{L}, & -2 & \psi=e_{R}, \\
+\frac{1}{3} & \psi=q_{L}, & +\frac{4}{3} & \psi=u_{R}, \\
& & -\frac{2}{3} & \psi=d_{R} .
\end{array}\right.\right. \tag{9.29}
\end{align*}
$$

An explicit mass term is forbidden by the chiral nature of the symmetry (already at the global level), since $\bar{\psi} \psi=\bar{\psi}_{L} \psi_{R}+\bar{\psi}_{R} \psi_{L}$. We have to set $m=0$, and rely on a different mechanism to provide masses to the elementary fermions. This is achieved by means of the so-called Yukawa terms of the form $\phi \bar{\psi} \psi$. More precisely, taking into account invariance under the gauge group $G$, we have

$$
\begin{equation*}
\mathscr{L}_{\text {Yukawa }}(\phi, \psi, \bar{\psi})=f_{\ell}\left(\bar{\ell}_{L} \phi\right) e_{R}+f_{d}\left(\bar{q}_{L} \phi\right) d_{R}+f_{u}\left(\bar{q}_{L} \tilde{\phi}\right) u_{R}+\text { h.c. } \tag{9.30}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{\phi}=i \tau^{2} \phi^{*}=\binom{\varphi^{0}}{-\varphi^{+}} \tag{9.31}
\end{equation*}
$$

This field is $i \tau^{2}$ the charge conjugate of $\phi$. The latter is an anti-doublet, transforming as $\phi^{*} \rightarrow U^{*} \phi^{*}$ under $\mathrm{SU}(2)_{L}$ transformations, but since $U^{*}=\tau^{2} U \tau^{2}$ one has $\tilde{\phi} \rightarrow U \tilde{\phi}$, i.e., a doublet of $\operatorname{SU}(2)_{L}$, with $Y(\tilde{\phi})=-1$. Its introduction is motivated by out desire to give mass to the $u$ quark. It is now straightforward to check that each of the three terms has total $Y=0$. The dimensionless quantities $f_{\ell, d, u}$ are known as Yukawa couplings. Going over to unitarity gauge we find

$$
\begin{equation*}
\phi=\binom{0}{\frac{v+\eta}{\sqrt{2}}}, \quad \tilde{\phi}=\binom{\frac{v+\eta}{\sqrt{2}}}{0} \tag{9.32}
\end{equation*}
$$

and so

$$
\begin{equation*}
\mathscr{L}_{\text {Yukawa, unitarity gauge }}=\frac{v+\eta}{\sqrt{2}} f_{\ell} \bar{e}_{L} e_{R}+\frac{v+\eta}{\sqrt{2}} f_{d} \bar{d}_{L} d_{R}+\frac{v+\eta}{\sqrt{2}} f_{u} \bar{u}_{L} u_{R}+\text { h.c. }, \tag{9.33}
\end{equation*}
$$

from which the masses of the fermions and their coupling to the Higgs field $\eta$ can be obtained straightforwardly,

$$
\begin{equation*}
m_{i}=\frac{v f_{i}}{\sqrt{2}}, \quad f_{i}=\frac{\sqrt{2} m_{i}}{v} \tag{9.34}
\end{equation*}
$$

It follows that the larger the mass, the stronger the coupling with the Higgs field. If $v$ is large, such couplings will be small. Notice that no mass was given to the neutrino.

We finally have to read off the couplings to the physical (mass eigenstates) gauge fields $W_{\mu}^{ \pm}$, $Z_{\mu}$, and $A_{\mu}$. The interaction part of the Lagrangian reads (up to a $-i$ factor)

$$
\begin{equation*}
i \mathscr{L}_{\mathrm{int}}=g \vec{J}_{\mu} \vec{W}^{\mu}+\frac{1}{2} g^{\prime} J_{\mu}^{y} B^{\mu} \tag{9.35}
\end{equation*}
$$

with

$$
\begin{align*}
& \vec{J}_{\mu}=\bar{\ell}_{L} \overrightarrow{\frac{\tau}{2}} \ell_{L}+\bar{q}_{L} \frac{\vec{\tau}}{2} q_{L},  \tag{9.36}\\
& J_{\mu}^{y}=-\bar{\ell}_{L} \ell_{L}+\frac{1}{3} \bar{q}_{L} q_{L}-2 \bar{e}_{R} e_{R}+\frac{4}{3} \bar{u}_{R} u_{R}-\frac{2}{3} \bar{d}_{R} d_{R} .
\end{align*}
$$

We now recast Eq. 9.35 in terms of physical fields. We have

$$
\begin{equation*}
i \mathscr{L}_{\mathrm{int}}=\frac{g}{\sqrt{2}}\left[\left(J_{\mu}^{1}+i J_{\mu}^{2}\right) \frac{W^{1 \mu}-i W^{2 \mu}}{\sqrt{2}}+\left(J_{\mu}^{1}-i J_{\mu}^{2}\right) \frac{W^{1 \mu}+i W^{2 \mu}}{\sqrt{2}}\right]+\left(g J_{\mu}^{3} W^{3 \mu}+\frac{1}{2} g^{\prime} J_{\mu}^{y} B^{\mu}\right), \tag{9.37}
\end{equation*}
$$

and taking into account that

$$
\binom{Z}{A}=\left(\begin{array}{cc}
\cos \theta_{W} & -\sin \theta_{W}  \tag{9.38}\\
\sin \theta_{W} & \cos \theta_{W}
\end{array}\right)\binom{W^{3}}{B} \Longrightarrow\binom{W^{3}}{B}=\left(\begin{array}{cc}
\cos \theta_{W} & \sin \theta_{W} \\
-\sin \theta_{W} & \cos \theta_{W}
\end{array}\right)\binom{Z}{A},
$$

we can write

$$
\begin{equation*}
g J_{\mu}^{3} W^{3 \mu}+\frac{1}{2} g^{\prime} J_{\mu}^{y} B^{\mu}=\left(g \cos \theta_{W} J_{\mu}^{3}-\frac{g^{\prime}}{2} \sin \theta_{W} J_{\mu}^{y}\right) Z^{\mu}+\left(g \sin \theta_{W} J_{\mu}^{3}+\frac{g^{\prime}}{2} \cos \theta_{W} J_{\mu}^{y}\right) A^{\mu} . \tag{9.39}
\end{equation*}
$$

Since $J_{\mu}^{y}=2\left(J_{\mu}^{E M}-J_{\mu}^{3}\right)$, we further have that

$$
\begin{align*}
& g J_{\mu}^{3} W^{3 \mu}+\frac{1}{2} g^{\prime} J_{\mu}^{y} B^{\mu} \\
& =\left[\left(g \cos \theta_{W}+g^{\prime} \sin \theta_{W}\right) J_{\mu}^{3}-g^{\prime} \sin \theta_{W} J_{\mu}^{E M}\right] Z^{\mu}  \tag{9.40}\\
& \quad+\left[\left(g \sin \theta_{W}-g^{\prime} \cos \theta_{W}\right) J_{\mu}^{3}+g^{\prime} \cos \theta_{W} J_{\mu}^{E M}\right] A^{\mu} .
\end{align*}
$$

But

$$
\begin{equation*}
g \sin \theta_{W}-g^{\prime} \cos \theta_{W}=0, \quad g \cos \theta_{W}+g^{\prime} \sin \theta_{W}=g\left(\cos \theta_{W}+\tan \theta \sin \theta_{W}\right)=\frac{g}{\cos \theta_{W}} \tag{9.41}
\end{equation*}
$$

and so

$$
\begin{align*}
g J_{\mu}^{3} W^{3 \mu}+\frac{1}{2} g^{\prime} J_{\mu}^{y} B^{\mu} & =\frac{g}{\cos \theta_{W}}\left(J_{\mu}^{3}-\sin ^{2} \theta_{W} J_{\mu}^{E M}\right) Z^{\mu}+g \sin \theta_{W} J_{\mu}^{E M} A^{\mu}  \tag{9.42}\\
& =\frac{g}{\cos \theta_{W}} J_{\mu}^{0} Z^{\mu}+g \sin \theta_{W} J_{\mu}^{E M} A^{\mu} .
\end{align*}
$$

Since it couples to $J_{\mu}^{E M}$, the field $A_{\mu}$ is identified with the photon field, and the combination

$$
\begin{equation*}
e=g \sin \theta_{W} \tag{9.43}
\end{equation*}
$$

with the electromagnetic coupling constant. Summarising,

$$
\begin{equation*}
i \mathscr{L}_{\text {int }}=\frac{g}{\sqrt{2}}\left(J_{\mu}^{+} W^{-\mu}+J_{\mu}^{-} W^{+\mu}\right)+\frac{g}{\cos \theta_{W}} J_{\mu}^{0} Z^{\mu}+e J_{\mu}^{E M} A^{\mu} . \tag{9.44}
\end{equation*}
$$

We have already found one relation between the phenomenologically accessible quantity $e$, and $g$ and $\theta_{W}$. Other parameters of the theory that we need to fix are $v$ and the Yukawa couplings $f_{i}$. These in turn are known once $v$ and the fermion masses are known. It is possible to relate
$v$ to the Fermi constant $G_{F}$, and $\sin ^{2} \theta_{W}$ to the elastic neutrino-electron cross section. In fact, assuming that $m_{W}$ is large one has that low-energy processes corresponding to a single $W$-boson exchange, which are given by the Feynman diagram (in the low energy approximation)

$$
\begin{equation*}
\left(\frac{i g}{\sqrt{2}}\right)^{2} \frac{i}{m_{W}^{2}}\langle f| J_{\mu}^{+} J^{-\mu}|i\rangle=-i \frac{g^{2}}{2 m_{W}^{2}}\langle f| J_{\mu}^{+} J^{-\mu}|i\rangle, \tag{9.45}
\end{equation*}
$$

are equally well described by the effective interaction

$$
\begin{equation*}
\mathscr{L}_{\mathrm{eff}}=-\frac{g^{2}}{2 m_{W}^{2}} J_{\mu}^{+} J^{-\mu}=-\frac{g^{2}}{8 m_{W}^{2}} j_{\mu}^{W} j^{W \mu \dagger}=-\frac{G_{F}}{\sqrt{2}} j_{\mu}^{+} j^{-\mu} \tag{9.46}
\end{equation*}
$$

where we took Eq. (9.1) into account to make contact with the phenomenological approach of the previous sections. We have

$$
\begin{equation*}
G_{F}=\frac{g^{2}}{4 m_{W}^{2} \sqrt{2}}=\frac{g^{2}}{4 \frac{g^{2} v^{2}}{4} \sqrt{2}}=\left(v^{2} \sqrt{2}\right)^{-1} \Longrightarrow v=2^{-\frac{1}{4}} G_{F}^{-\frac{1}{2}} \simeq 250 \mathrm{GeV} . \tag{9.47}
\end{equation*}
$$

This is a much larger scale than $m_{u, d, e}$, making the corresponding Yukawa couplings small. A reasoning similar to the one above shows that for low-energy processes involving the neutral weak current, i.e., one $Z$-boson exchange, the relevant Feynman diagram reads

$$
\begin{equation*}
\left(\frac{i g}{\cos \theta_{W}}\right)^{2} \frac{i}{m_{Z}^{2}}\langle f| \frac{1}{2} J_{\mu}^{0} J^{0 \mu}|i\rangle=-i \frac{g^{2}}{2 \cos \theta_{W}^{2} m_{Z}^{2}}\langle f| J_{\mu}^{0} J^{0 \mu}|i\rangle, \tag{9.48}
\end{equation*}
$$

where the factor $\frac{1}{2}$ is introduced to avoid double counting. This can be equivalently obtained from the low-energy effective Lagrangian

$$
\begin{equation*}
\mathscr{L}_{\mathrm{eff}}^{0}=-\frac{g^{2}}{2 \cos \theta_{W}^{2} m_{Z}^{2}} J_{\mu}^{0} J^{0 \mu}=-\frac{g^{2}}{2 m_{W}^{2}} J_{\mu}^{0} J^{0 \mu} \tag{9.49}
\end{equation*}
$$

which shows that the same coupling appears in the charge-current and neutral-current interactions. The neutral current reads explicitly

$$
\begin{align*}
J_{\mu}^{0} & =\sum_{i} g_{L}^{(i)} \bar{\psi}_{L}^{(i)} \gamma_{\mu} \psi_{L}^{(i)}+g_{R}^{(i)} \bar{\psi}_{R}^{(i)} \gamma_{\mu} \psi_{R}^{(i)},  \tag{9.50}\\
g_{L, R}^{(i)} & =T^{3}\left(\psi_{L, R}^{(i)}\right)-\sin ^{2} \theta_{W} Q\left(\psi_{L, R}^{(i)}\right),
\end{align*}
$$

(cfr. Eq. $\sqrt{1.22}$ ) and its contribution to elastic neutrino-electron scattering is proportional to the product of currents

$$
\begin{equation*}
\frac{1}{2} \bar{\nu}_{e L} \gamma_{\mu} \nu_{e L} \bar{e} \gamma_{\mu}\left(a+b \gamma^{5}\right) e, \tag{9.51}
\end{equation*}
$$

with

$$
\begin{equation*}
a=g_{R}^{(e)}+g_{L}^{(e)}=-\frac{1}{2}+2 \sin ^{2} \theta_{W}, \quad b=g_{R}^{(e)}-g_{L}^{(e)}=\frac{1}{2} . \tag{9.52}
\end{equation*}
$$

From experimental studies of the cross section of this process one can then determine $\sin ^{2} \theta_{W} \simeq$ $0.22 \div 0.23$. This leads to the following prediction for the $W$-boson mass,

$$
\begin{equation*}
m_{W}=\left|\frac{g v}{2}\right|=\left|\frac{e v}{2 \sin \theta_{W}}\right|=\left|\frac{2^{-\frac{5}{4}} e G_{F}^{-\frac{1}{2}}}{\sin \theta_{W}}\right|=\frac{37 \mathrm{GeV}}{\left|\sin \theta_{W}\right|} \rightarrow 81.8 \mathrm{GeV} \tag{9.53}
\end{equation*}
$$

in good agreement with experiments. Good agreement is obtained also for $m_{Z}$.
To summarise: in the one generation case there are 15 Weyl (2-component) fermion fields with definite chirality, 4 vector bosons (3 massive and 1 massless) and one Higgs field. The parameters in the Lagrangian are $g, g^{\prime}, \mu^{2}, \lambda, f_{e}, f_{u}, f_{d}$, corresponding to the phenomenological parameters $e, \sin \theta_{W}, m_{W}, m_{\eta}, m_{e}, m_{u}, m_{d}$. Unification is not complete like in the electromagnetic case, since there are still two independent coupling constants. By construction baryon and lepton number are conserved. The results of the model depend heavily on having a single complex doublet of scalar fields in the unbroken theory.

The generalisation to more generations of fermions is almost straightforward. In fact, by simply replicating the families one is sure to have an anomaly-free theory. One thus adds four doublets of left-handed fields and four singlets of right-handed fields, assigning hypercharge in the same way as with the lightest fermionic generation. However, one cannot exclude mixing of the various fermion species. To see this explicitly, let us introduce the fermion fields in the following form,

$$
\begin{equation*}
\tilde{\ell}_{A L}=\binom{\tilde{\nu}_{A}}{\tilde{e}_{A}}_{L}, \quad \tilde{q}_{A L}=\binom{\tilde{p}_{A}}{\tilde{n}_{A}}_{L}, \quad \tilde{e}_{A R}, \quad \tilde{p}_{A R}, \quad \tilde{n}_{A R}, \tag{9.54}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{e}_{A}=\tilde{e}, \tilde{\mu}, \tilde{\tau}, \quad \tilde{p}_{A}=\tilde{u}, \tilde{c}, \tilde{t}, \quad \tilde{n}_{A}=\tilde{d}, \tilde{s}, \tilde{b} \tag{9.55}
\end{equation*}
$$

and assign to them weak isospin and weak hypercharge as in the single generation case. These fields have definite transformation properties under gauge transformation, i.e., they are coupled to the gauge fields as follows,

$$
\begin{align*}
\bar{\Psi} \not D \Psi= & \overline{\tilde{\ell}}_{A L}\left(\not \partial-\frac{i}{2} g \vec{\tau} \overrightarrow{W_{H}}+\frac{i}{2} g^{\prime} \not \mathbb{B}_{\mu}\right) \tilde{\ell}_{A L}+\overline{\tilde{q}}_{A L}\left(\not \partial-\frac{i}{2} g \vec{\tau} \overrightarrow{W_{\mu}}-\frac{i}{6} g^{\prime} \not B_{\mu}\right) \tilde{q}_{A L} \\
& +\tilde{\tilde{\ell}}_{A R}\left(\not \partial+i g^{\prime} \not \mathbb{B}_{\mu}\right) \tilde{\ell}_{A R}+\overline{\tilde{p}}_{A R}\left(\not D-i \frac{2}{3} g^{\prime} \not \mathbb{B}_{\mu}\right) \tilde{p}_{A R}+\overline{\tilde{n}}_{A R}\left(\not \partial+i \frac{1}{3} g^{\prime} \mathbb{B}_{\mu}\right) \tilde{n}_{A R} . \tag{9.56}
\end{align*}
$$

For the Yukawa couplings we have to allow for the mixing of fields with the same quantum numbers. Quarks and leptons do not mix due to the different colour charges, and quarks of type $p$ and type $n$ do not mix because of the different electric charges, but any other mixing is allowed. The most general Yukawa term thus reads

$$
\begin{equation*}
\left.\mathscr{L}_{\text {Yukawa }}=f_{A B}^{(e)} \overline{\tilde{\ell}}_{A L} \phi\right) \tilde{e}_{A R}+f_{A B}^{(p)}\left(\overline{\tilde{q}}_{A L} \tilde{\phi}\right) \tilde{p}_{B R}+f_{A B}^{(n)}\left(\overline{\tilde{q}}_{A L} \phi\right) \tilde{n}_{B R}+\text { h.c. } \tag{9.57}
\end{equation*}
$$

After symmetry breaking in unitarity gauge the mass matrices read

$$
\begin{equation*}
M_{A B}^{(i)}=\frac{v}{\sqrt{2}} f_{A B}^{(i)}, \quad i=e, p, n \tag{9.58}
\end{equation*}
$$

These are in general complex $3 \times 3$ matrices without further structure. It is a general theorem than any such matrix can be transformed to a real positive diagonal matrix by means of a pair of unitary matrices, $S M T^{\dagger}=M_{\text {diag }}{ }^{54}$ One has then

$$
\begin{align*}
M_{A B}^{(i)} \bar{\psi}_{A L}^{(i)} \tilde{\psi}_{B R}^{(i)} & =\overline{\tilde{\psi}}_{L}^{(i)} M^{(i)} \tilde{\psi}_{R}^{(i)}=\overline{\tilde{\psi}}_{L}^{(i)} S^{(i) \dagger} M_{\text {diag }}^{(i)} T^{(i)} \tilde{\psi}_{R}^{(i)}=\bar{S}^{(i)} \tilde{\psi}_{L}^{(i)} M_{\text {diag }}^{(i)}\left(T^{(i)} \tilde{\psi}_{R}^{(i)}\right)  \tag{9.59}\\
& =\bar{\psi}_{L}^{(i)} M_{\text {diag }}^{(i)} \psi_{R}^{(i)} .
\end{align*}
$$

[^36]The fields $\psi^{(i)}$ have definite mass, although not definite gauge transformation properties anymore. For the quarks charged current one then finds

$$
\begin{align*}
J_{\mu}^{h+} & =\overline{\tilde{q}}_{A L} \tau^{+} \gamma_{\mu} \tilde{q}_{A L}=\overline{\tilde{p}}_{A L} \gamma_{\mu} \tilde{n}_{A L} \\
& =\bar{p}_{A L} \gamma_{\mu}\left[S^{(p)} S^{(n) \dagger}\right]_{A B} n_{B L}=\bar{p}_{A L} \gamma_{\mu} U_{A B} n_{B L}, \tag{9.60}
\end{align*}
$$

with $U_{A B}$ the unitary CKM matrix. One then sets

$$
\left(\begin{array}{l}
d^{\prime}  \tag{9.61}\\
s^{\prime} \\
b^{\prime}
\end{array}\right)=U\left(\begin{array}{l}
d \\
s \\
b
\end{array}\right),
$$

and recovers the phenomenological description of quark mixing. Of course the choice of rotating only the $Q=-\frac{1}{3}$ quarks is purely conventional. Repeating now the calculation for the leptonic current we find

$$
\begin{equation*}
J_{\mu}^{\ell+}=\overline{\tilde{e}}_{A L} \tau^{+} \gamma_{\mu} \tilde{e}_{A L}=\overline{\tilde{\nu}}_{A L} \gamma_{\mu} \tilde{e}_{A L}=\overline{\tilde{\nu}}_{A L} \gamma_{\mu} S_{A B}^{(e) \dagger} e_{B L}=\bar{S}^{(e)} \tilde{\nu}_{A L} \gamma_{\mu} e_{A L}, \tag{9.62}
\end{equation*}
$$

since the neutrinos are taken to be massless and no corresponding matrix $S^{(\nu)}$ appears. One can now simply define $\left(S^{(e)} \tilde{\nu}\right)_{A}=\nu_{A}$, and since all neutrinos are degenerate in mass one still has massless fields $\nu_{A}$, coupled to $e_{A}$ by the charged weak current, and both the $\nu_{A}$ and the $e_{A}$ still have definite gauge transformation properties. One then defines the fields $\nu_{A}$ and $e_{A}$ to be the neutrinos and the charged leptons with definite lepton flavour, which is then a conserved quantity. Non-mixing and exact lepton family number conservation would then be a consequence of mass-degeneracy of the neutrinos. Finally, for the neutral current one has

$$
\begin{align*}
J_{\mu}^{0} & =\sum_{i} g_{L}^{(i)} \overline{\tilde{\psi}}_{A L}^{(i)} \gamma_{\mu} \tilde{\psi}_{A L}^{(i)}+g_{R}^{(i)} \overline{\tilde{\psi}}_{A R}^{(i)} \gamma_{\mu} \tilde{\psi}_{A R}^{(i)} \\
& =\sum_{i} g_{L}^{(i)} \bar{\psi}_{A L}^{(i)} \gamma_{\mu}\left[S^{(i)} S^{(i) \dagger}\right]_{A B} \psi_{B L}^{(i)}+g_{R}^{(i)} \bar{\psi}_{A R}^{(i)} \gamma_{\mu}\left[T^{(i)} T^{(i) \dagger}\right]_{A B} \psi_{B R}^{(i)}  \tag{9.63}\\
& =\sum_{i} g_{L}^{(i)} \bar{\psi}_{A L}^{(i)} \gamma_{\mu} \psi_{A L}^{(i)}+g_{R}^{(i)} \bar{\psi}_{A R}^{(i)} \gamma_{\mu} \psi_{A R}^{(i)},
\end{align*}
$$

so it has the same form in terms of the mass eigenstates.

### 9.4 Summary

As a final summary, let us list here the properties of the electroweak sector of the Standard Model with three generations of fermions:

- gauge group $G=\mathrm{SU}(2)_{L} \times \mathrm{U}(1)_{Y}$;
- $3 \times 15=45$ Weyl fermion fields;
- 3 massive ( $W^{ \pm}$and $Z$ ) and 1 massless $(\gamma)$ vector particles;
- 1 Higgs scalar;
- 18 free parameters:
- 2 gauge couplings $e, \sin \theta_{W}$;
- 3 lepton and 6 quark masses (Yukawa couplings);
- 3 Cabibbo angles and 1 Kobayashi-Maskawa phase;
- the W-boson and Higgs boson masses $m_{W}$ and $m_{\eta}$ (corresponding to the vacuum expectation value $v$ and to the mass parameter $\mu$ );
- the Higgs self-coupling $\lambda$.

To this one should add the gauge group of Quantum Chromodynamics (QCD), to obtain the full gauge group $G_{\mathrm{SM}}=\mathrm{SU}(3)_{C} \times \mathrm{SU}(2)_{L} \times \mathrm{U}(1)_{Y}$. Each type of quark is a fundamental colour triplet, while all other matter particles are colour singlets. One has to add the eight massless gluon (gauge boson) fields of the $\mathrm{SU}(3)$ part of the group, and include 1 more parameter, the dimensionless strong fine structure constant $\alpha_{S}$ (or equivalently the mass dimension 1 QCD scale $\left.\Lambda_{\mathrm{QCD}}\right)$.

## 10 Beyond the Standard Model

Since the completion of the Standard model there has been attempts to extend it, but only one compelling reason to amend it. An interesting idea about extensions of the Standard Model is that of Grand Unification, i.e., the attempt to further unify electroweak and strong interactions. Such attempts have so far failed. While theoretical appealing, Grand Unified theories are not required to explain experimental results contraddicting the Standard Model predictions (and usually give prediction that disagree with the experimental evidence). On the other hand, the Standard Model with its massless neutrinos disagrees with the by now established fact that neutrinos are actually massive.

The two topics mentioned above are briefly discussed in this section, beginning with the subject of neutrino oscillations, and concluding with the first (failed) attempt at Grand Unification.

### 10.1 Neutrino oscillations and the need for neutrino masses

Neutrino masses have a long and complicated history. Initially, when proposed by Pauli, it was believed that the neutrino had to be very light, but there was no particular reason to believe it was massless. Later, with the two-component neutrino theory that postulated that this particle existed only with definite chirality and handedness, it was assumed that it was massless. At the same time, ideas about neutrino having finite masses and oscillating between the different flavours started to appear. These ideas go back to one of the most important neutrino physicists, namely Bruno Maximovich Pontecorvo. It was Pontecorvo who ideated the experimental technique later used by Raymond Davies to detect solar neutrinos, produced by the nuclear reactions in the Sun. This technique is based on the following neutrino-capture reaction,

$$
\begin{equation*}
\nu_{e}+{ }^{37} \mathrm{Cl} \rightarrow e^{-}+{ }^{37} \mathrm{Ar} \tag{10.1}
\end{equation*}
$$

However, the flux of solar neutrinos measured experimentally was significantly lower than the prediction of Bahcall et al.. This came to be known as the solar anomaly, later confirmed by KamiokaNDE and other experiments. Since only $\nu_{e}$ could be seen by the detectors, the most natural explanation of this deficit was that the electronic neutrinos were actually produced in
the Sun, but along the way turned into a different flavour and so escaped detection. Another anomaly was found in atmospheric neutrinos: given the charged pion decays,

$$
\begin{align*}
\pi^{+} \rightarrow & \mu^{+} \nu_{\mu} \\
& \left\llcorner e^{+} \nu_{e} \bar{\nu}_{\mu}\right.  \tag{10.2}\\
\pi^{-} \rightarrow & \mu^{-} \bar{\nu}_{\mu} \\
& \left\llcorner e^{-} \bar{\nu}_{e} \nu_{\mu}\right.
\end{align*}
$$

one expects the muonic and electronic fluxes to be in a ratio $2: 1$. This ratio however turned out to be sensitive to the direction in which the flux was measured: while the fluxes coming from above showed the expected ratio, those coming from below (after crossing the Earth) showed a ratio $1: 1$. This again can be explained away by neutrino oscillations. Finally, a third anomaly that can be explained by neutrino oscillations comes from measurements of the flux of electronic antineutrinos from nuclear reactors (e.g., KamLAND), which show a dependence of the flux on the distance from the reactor.

The quantum mechanical description of neutrino oscillations is not particularly complicated. Consider for simplicity only two families, and assume for generality that neutrinos have masses $m_{a, b}$. There is in general no reason to assume that the weak flavour eigenstates (i.e., the neutrino states that couple directly to $e, \mu$ and $\tau$ ) are also mass eigenstates, and so the former will be linear superpositions of the latter. When a neutrino is produced in a weak process, its state has a definite flavour, or lepton family number, as it comes together with a charged lepton. On the other hand, as they propagate in space, the evolution of their state is determined by their content in mass eigenstates: these are the ones the evolve simply under temporal evolution. Given the tiny chance of interacting with anything along the way, the temporal evolution can be treated as free. Finally, when neutrinos are detected, the state that is detected is again a flavour/lepton family number eigenstate, as detection is signalled by the production of a charged lepton. Denoting by $\left|\ell_{1,2}\right\rangle$ the lepton-family eigenstates and by $|a, b\rangle$ the mass eigenstates with masses $m_{a, b}$, the most general parameterisation of the lepton-family eigenstates is ${ }^{55}$

$$
\begin{align*}
& \left|\ell_{1}\right\rangle=\cos \theta|a\rangle+\sin \theta|b\rangle, \\
& \left|\ell_{2}\right\rangle=-\sin \theta|a\rangle+\cos \theta|b\rangle, \tag{10.3}
\end{align*}
$$

with $\theta$ the mixing angle. As is well known, the state of quantum mechanical system at time $t$ is determined by its state at $t=0$ as $|\psi(t)\rangle=e^{-i H t}|\psi(0)\rangle$. In our case $H$ is the free Hamiltonian, and we will assume an initial state $|\psi(0)\rangle=\left|\ell_{1}\right\rangle$ with definite momentum $\vec{p}$. Then

$$
\begin{equation*}
|\psi(t)\rangle=\cos \theta e^{-i E_{a} t}|a\rangle+\sin \theta e^{-i E_{b} t}|b\rangle, \quad E_{a, b}=\sqrt{\vec{p}^{2}+m_{a, b}^{2}} \tag{10.4}
\end{equation*}
$$

The probability to detect the same lepton-number eigenstate at time $t$ as the initial one is given by

$$
\begin{align*}
\left|\left\langle\ell_{1} \mid \psi(t)\right\rangle\right|^{2} & =\left|\cos \theta e^{-i E_{a} t}\left\langle\ell_{1} \mid a\right\rangle+\sin \theta e^{-i E_{b} t}\left\langle\ell_{1} \mid b\right\rangle\right|^{2}=\left|\cos ^{2} \theta e^{-i E_{a} t}+\sin ^{2} \theta e^{-i E_{b} t}\right|^{2} \\
& =\cos ^{4} \theta+\sin ^{4} \theta+2 \cos ^{2} \theta \sin ^{2} \theta \cos \left(E_{a}-E_{b}\right) t \tag{10.5}
\end{align*}
$$

[^37]For small $m_{a, b}$ the neutrinos are produced in an ultrarelativistic state, $m_{a, b} \ll|\vec{p}|$, and so

$$
\begin{equation*}
E_{a}-E_{b}=\frac{E_{a}^{2}-E_{b}^{2}}{E_{a}+E_{b}}=\frac{m_{a}^{2}-m_{b}^{2}}{E_{a}+E_{b}} \simeq \frac{m_{a}^{2}-m_{b}^{2}}{2|\vec{p}|}=\frac{\Delta m^{2}}{2|\vec{p}|} . \tag{10.6}
\end{equation*}
$$

Since they travel almost at the speed of light, $t \simeq x$, i.e., the distance covered in $t$ from the production process. Then since flux $1(x(t)) \propto\left|\left\langle\ell_{1} \mid \psi(t)\right\rangle\right|^{2}$ we find

$$
\begin{equation*}
\operatorname{flux}_{1}(x)=A+B \cos \frac{\Delta m^{2}}{2|\vec{p}|} x, \tag{10.7}
\end{equation*}
$$

with

$$
\begin{equation*}
\frac{A}{B}=\frac{\cos ^{4} \theta+\sin ^{4} \theta}{2 \cos ^{2} \theta \sin ^{2} \theta}=\frac{1+\cos ^{2} 2 \theta}{1-\cos ^{2} 2 \theta} . \tag{10.8}
\end{equation*}
$$

Oscillation of the neutrino flavour would explain the three anomalies discussed above, but it requires the non-degeneracy of the neutrino masses. This implies that at least one of the neutrinos must be massive. Since oscillations have been experimentally observed, it follows that we have to abandon the assumption that neutrinos are massless. Obviously, lepton family number is not conserved anymore.

The generalisation to three families involves the parameterisation of the mixing matrix in terms of three angle and one ineliminable phase, and three mass-square differences, but is otherwise straightforward. Here are recent experimental results about these quantities.

$$
\begin{align*}
\Delta m_{21}^{2} & =7.55_{-0.16}^{+0.20} \cdot 10^{-5} \mathrm{eV}^{2}, & \left|\Delta m_{31}^{2}\right| & =\left\{\begin{array}{lll}
2.50 \pm 0.03 \cdot 10^{-3} \mathrm{eV}^{2} & (\mathrm{NO}), \\
2.42_{-0.04}^{+0.03} \cdot 10^{-3} \mathrm{eV}^{2} & (\mathrm{IO}),
\end{array}\right. \\
\sin ^{2} \theta_{12} & =3.20_{-0.16}^{+0.20} \cdot 10^{-1}, & \sin ^{2} \theta_{23} & =\left\{\begin{array}{lll}
5.47_{-0.30}^{+0.20} \cdot 10^{-1} & (\mathrm{NO}), \\
5.51_{-0.30}^{+0.18} \cdot 10^{-1} & (\mathrm{IO}),
\end{array}\right. \\
\sin ^{2} \theta_{13} & =\left\{\begin{array}{lll}
2.160_{-0.069}^{+0.083} \cdot 10^{-2} & (\mathrm{NO}), \\
2.220_{-0.076}^{+0.074} \cdot 10^{-2} & (\mathrm{IO}),
\end{array}\right. & \frac{\delta_{C P}}{\pi} & =\left\{\begin{array}{lll}
1.32_{-0.15}^{+0.21} & (\mathrm{NO}), \\
1.56_{-0.15}^{+0.13} & (\mathrm{IO}) .
\end{array}\right. \tag{10.9}
\end{align*}
$$

Here NO stands for "normal ordering", in which case $\Delta m_{32}^{2}>0$, while IO stands for "inverted ordering", in which case $\Delta m_{32}^{2}<0$. Which ordering is realised is not determined by current experiments. With NO one would have $m_{1}<m_{2} \ll m_{3}$, while with IO one would have instead $m_{3} \ll m_{1}<m_{2}$.

How can one modify the Standard Model to account for neutrino masses? The simplest possibility is to add a mass term analogous to the other ones, i.e., a Yukawa coupling to the Higgs field,

$$
\begin{equation*}
f_{A B}^{(\nu)}\left(\overline{\tilde{\ell}}_{A}\right)_{L} \tilde{\phi}\left(\nu_{B}\right)_{R} \tag{10.10}
\end{equation*}
$$

with $\left(\nu_{A}\right)_{R}$ the right-handed (more precisely: negative chirality) neutrino fields, and were lepton mixing has been allowed. Right-handed fields are needed for a Dirac mass term, generated from the coupling above after spontaneous symmetry breaking. On the other hand, the corresponding particle, i.e., a right-handed neutrino, would not couple to any of the other particles in the Standard Model, and would therefore be sterile. The mass matrix reads

$$
\begin{equation*}
M_{A B}^{(\nu)}=\frac{v}{\sqrt{2}} f_{A B}^{(\nu)}=S^{(\nu) \dagger} M_{\mathrm{diag}}^{(\nu)} T^{(\nu)} \tag{10.11}
\end{equation*}
$$

If the leptonic current $\bar{\nu}_{A} \mathcal{O}_{L}^{\alpha} \ell_{A}$ is written in terms of definite-mass fields $\ell_{A}$ for the charged leptons, the corresponding neutrino fields $\nu_{A}$ have definite lepton family number (by definition). The definite-mass fields are obtained by means of a unitary transformation,

$$
\begin{equation*}
\nu_{L}=S^{(\nu)} \nu_{L}^{(\mathrm{mass})}, \quad \nu_{R}=T^{(\nu)} \nu_{R}^{(\mathrm{mass})} \tag{10.12}
\end{equation*}
$$

The matrix $S^{(\nu)}$ relating mass and lepton-family (left-handed) eigenstates is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, a $3 \times 3$ unitary matrix that can be parameterised (up to irrelevant, unphysical phases) in terms of three angles and one phase. Lepton number is still conserved, while lepton family number is not anymore. The right-handed neutrino field $\nu_{R}$ is a $\mathrm{SU}(2)_{L}$ singlet with vanishing $\mathrm{U}(1)_{Y}$ charge (since $-y(\ell)+y(\tilde{\phi})=1-1=0$ ), and so is invariant under the whole gauge group $G$; as such, it causes no problems with the anomaly.

While very simple, the Dirac mass term discussed above has the obviously annoying feature that it introduces essentially unobservable particles (which is exactly what Pauli regretted after his proposal of the neutrino hypothesis). It also has no chance to explain why the neutrino masses are so small. In addition to the Dirac mass term, the $G$-singlet field $\nu_{R}$ (with definite flavour), which as such is a truly neutral fermion, can take a Majorana mass term. Majorana's condition for neutrality of $\nu_{R}$ is $\left(\nu_{R}\right)^{c}=\nu_{R}$, where $\left(\nu_{R}\right)^{c}=C \bar{\nu}_{R}^{T}$ with $C=-i \gamma^{2} \gamma^{0}$. One then finds $\left(\nu_{R}\right)^{c}=-i \gamma^{2} \nu_{R}^{*}$. The Majorana mass term read ${ }^{56}$

$$
\begin{equation*}
\mathscr{L}_{\mathrm{Maj}}=\frac{1}{2} m_{M} \overline{\nu_{R}}\left(\nu_{R}\right)^{c}+\text { c.c. } . \tag{10.13}
\end{equation*}
$$

Such a term violates lepton number, but no other symmetry, and incidentally leads to predict neutrinoless double-beta decay processes, which would then provide an experimental signature (unobserved so far). It must be noted that $\left(\nu_{R}\right)^{c}$ is actually a left-handed field so that the Majorana neutrino, which is equally an antineutrino, appears with both chiralities. If one know puts together the Dirac and Majorana mass terms, one finds

$$
\begin{equation*}
\mathscr{L}=\frac{1}{2} \bar{\nu}^{c} M \nu, \quad \nu=\binom{\nu_{L}}{\left(\nu_{R}\right)^{c}} \tag{10.14}
\end{equation*}
$$

with $\nu$ a doublet of left-handed fields, and $M$ the mass matrix

$$
M=\left(\begin{array}{cc}
0 & m_{D}  \tag{10.15}\\
m_{D} & m_{M}
\end{array}\right) .
$$

This matrix is easily diagonalised yield the eigenvalues

$$
\begin{equation*}
m_{ \pm}=\frac{1}{2}\left(m_{M} \pm \sqrt{m_{M}^{2}+4 m_{D}^{2}}\right) . \tag{10.16}
\end{equation*}
$$

Since $m_{M}$ would be the origin of lepton-number breaking, it is natural to assume that it is a large mass scale, possibly related to new physics. In the limit $m_{M} \gg m_{D}$ one then find 5

$$
\begin{equation*}
m_{+} \simeq m_{M}, \quad m_{-} \simeq-\frac{m_{D}^{2}}{m_{M}}, \tag{10.17}
\end{equation*}
$$

[^38]and for the definite-mass fields
\[

$$
\begin{equation*}
N \simeq\left(\nu_{R}\right)^{c}+\frac{m_{D}}{m_{M}} \nu_{L}, \quad \nu \simeq \nu_{L}-\frac{m_{D}}{m_{M}}\left(\nu_{R}\right)^{c} . \tag{10.18}
\end{equation*}
$$

\]

The $N$ field has a large mass and small coupling to the active neutrino field $\nu_{L}$, so it corresponds to a heavy neutrino weakly interacting with other matter. The $\nu$ field instead is essentially the same as $\nu_{L}$, and has naturally a small mass: a reasonable expectation for $m_{D}$ would be for it to be of the same order of the mass of the corresponding charged lepton, and the factor $m_{D} / m_{M}$ leads to a strong suppression. The suppression mechanism described above is known as the see-saw mechanism.

### 10.2 Grand Unification Theories

The Standard Model has gauge group $G_{\mathrm{SM}}=\mathrm{SU}(3)_{c} \times \mathrm{SU}(2)_{L} \times \mathrm{U}_{y}$, with a different coupling associated to each factor of the group. It is then not truly unified, i.e., containing a single coupling constant governing all the types of interactions. To some, this is an unsatisfactory aspect. The idea of further unification is based on finding a bigger gauge group with a single coupling constant from which the Standard Model group $G_{\text {SM }}$ will be obtained via symmetry breaking. The minimal possibility to do this is to use the group $\mathrm{SU}(5)$. This group infact contains $G_{\text {SM }}$ and has rank 4, i.e., the same rank as $G_{\mathrm{SM}}$, meaning that it has four commuting generators that can be identified with $t_{3}, t_{8}, T_{3}, Y$. Moreover, $\mathrm{SU}(5)$ is the only rank- 4 group that admits complex representations (required by the chiral structure) which can accommodate the matter spectrum of the Standard Model (including their electric charge) without introducing new matter.

The group $\operatorname{SU}(5)$ is the 24 -dimensional Lie group of 5 -dimensional unitary unimodular matrices. Being a simple group, using it as the gauge group introduces a single coupling constant. Besides the known gauge bosons, this group would come with $24-(8+3+1)=12$ new ones. Among its diagonal generators there is

$$
\begin{equation*}
\lambda^{24}=\frac{1}{\sqrt{15}} \operatorname{diag}(2,2,2,-3-3) \tag{10.19}
\end{equation*}
$$

where the first three entries are proportional to the hypercharge $Y$ of $d_{L}^{c} \sim d_{R}^{*}$, and the last two to the hypercharge of $\ell_{L}$. Here $\psi_{L}^{c}=C \bar{\psi}_{R}^{T}=-i \gamma^{2} \gamma^{0} \bar{\psi}_{R}^{T}=-i \gamma^{2} \psi_{R}^{*}$. In other words, $\lambda^{24}$ represents the hypercharges of the Standard Model particles up to a common normalisation. Being diagonal in its upper $\operatorname{SU}(3)$ and lower $\operatorname{SU}(2)$ subgroups, if we embed the $\mathrm{SU}(3)_{c}$ and $\mathrm{SU}(2)_{L}$ factors of $G_{\mathrm{SM}}$ in the upper and lower corners,

$$
\left(\begin{array}{cc}
\mathrm{SU}(3) & 0  \tag{10.20}\\
0 & 0
\end{array}\right), \quad\left(\begin{array}{cc}
0 & 0 \\
0 & \mathrm{SU}(2)
\end{array}\right)
$$

then one finds $\left[\lambda^{24}, G_{\mathrm{SM}}\right]=0$. One needs now group representations for the matter particles, and a suitable symmetry-breaking pattern. Consider the $\overline{\mathbf{5}}_{F}$ (antifundamental) representation, and organise the three colours of the negatively-charged quark and the leptons of one generation of the Standard Model matter fields as follows,

$$
\left(\begin{array}{c}
d_{1}^{c}  \tag{10.21}\\
d_{2}^{c} \\
d_{3}^{c} \\
e^{-} \\
-\nu_{e}
\end{array}\right)_{L} .
$$

The first three components correspond to the $(\overline{\mathbf{3}}, \mathbf{1})$ representation of $\mathrm{SU}(3)_{c} \times \mathrm{SU}(2)_{L} \subset G_{\mathrm{SM}}$, the last two to the $(\mathbf{1}, \mathbf{2})$ representation. The corresponding $\mathbf{5}_{F}$ representation,

$$
\left(\begin{array}{c}
d_{1 R}  \tag{10.22}\\
d_{2 R} \\
d_{3 R} \\
e_{L}^{-c} \\
-\nu_{e L}^{c}
\end{array}\right),
$$

would contain instead right-handed fields and is not used. Next, organise the remaining matter fields in the $\mathbf{1 0}$ representation of $\mathrm{SU}(5)$, i.e., the antisymmetric part of $\mathbf{5}_{F} \otimes \mathbf{5}_{F}=\mathbf{1 0} \oplus \mathbf{1 5}$,

$$
\left(\begin{array}{ccccc}
0 & u_{3}^{c} & -u_{2}^{c} & u_{1} & d_{1}  \tag{10.23}\\
& 0 & u_{1}^{c} & u_{2} & d_{2} \\
& & 0 & u_{3} & d_{3} \\
& & & 0 & e^{+} \\
& & & & 0
\end{array}\right)_{L},
$$

where the entries below the diagonal are such that this matrix is antisymmetric. The top-left block is an $\mathrm{SU}(2)_{L}$ singlet and contains the antisymmetric part of the $\mathbf{3} \otimes \mathbf{3}=\overline{\mathbf{3}} \oplus \mathbf{6}$ representation of $\operatorname{SU}(3)_{c}$, i.e., the $\overline{\mathbf{3}}$. The top-right block transforms as $(\mathbf{3}, \mathbf{2})$, and the bottom-right block as $(\mathbf{1}, \mathbf{1})$. In fact, this block is the antisymmetric part of the $\mathbf{2} \otimes \mathbf{3}=\mathbf{1} \oplus \mathbf{3}$, i.e., the $\mathbf{1}$, and corresponds to the field $e_{R}$. Eq. 10.21) and (10.23) display precisely the matter content of one generation of the Standard Model. The number of generations would remain unexplained in this framework. Writing now

$$
T_{3}=\frac{1}{2}\left(\begin{array}{cc}
0 & 0  \tag{10.24}\\
0 & \tau^{3}
\end{array}\right)
$$

we have for the electric charge

$$
\begin{equation*}
Q=T_{3}+\frac{Y}{2}=T_{3}+c \frac{\lambda^{24}}{2}, \quad c=-\sqrt{\frac{5}{3}}, \tag{10.25}
\end{equation*}
$$

and we find

$$
\begin{align*}
& Y\left(\overline{\mathbf{5}}_{F}\right)=\left(+\frac{2}{3},+\frac{2}{3},+\frac{2}{3},-1,-1\right), \\
& Q\left(\overline{\mathbf{5}}_{F}\right)=\left(+\frac{1}{3},+\frac{1}{3},+\frac{1}{3},-1,0\right), \\
& Q(\mathbf{1 0})=\left(\begin{array}{ccccc}
0 & -\frac{2}{3} & -\frac{2}{3} & +\frac{2}{3} & -\frac{1}{3} \\
& 0 & -\frac{2}{3} & +\frac{2}{3} & -\frac{1}{3} \\
& & 0 & +\frac{2}{3} & -\frac{1}{3} \\
& & & 0 & 1 \\
& & & & 0
\end{array}\right), \tag{10.26}
\end{align*}
$$

where we used $Q(\mathbf{1 0})=Q_{\text {row }}(\mathbf{5})+Q_{\text {column }}(\mathbf{5})=-\left(Q_{\text {row }}(\overline{\mathbf{5}})+Q_{\text {column }}(\overline{\mathbf{5}})\right)$. We then find that $Y$ and $Q$ match those of the Standard Model, and moreover that the right representations of $\mathrm{SU}(3)_{c} \times \mathrm{SU}(2)_{L}$ are obtained. The gauge bosons transform under the $\mathbf{2 4}$ representation and
can be organised as follows ( $\mathbf{5} \otimes \overline{\mathbf{5}}=\mathbf{1} \oplus \mathbf{2 4}$ ),

$$
A_{\mu}=\left(\begin{array}{ccc|cc} 
& & & X_{\mu}^{1 c} & Y_{\mu}^{1 c}  \tag{10.27}\\
G_{\mu}^{i}{ }_{\mu}+\frac{2}{\sqrt{30}} B_{\mu} \delta^{i}{ }_{j} & X_{\mu}^{2 c} & Y_{\mu}^{2 c} \\
& & X_{\mu}^{3 c} & Y_{\mu}^{3 c} \\
\hline X_{\mu}^{1} & X_{\mu}^{2} & X_{\mu}^{3} & \frac{1}{\sqrt{2}} W_{\mu}^{3}-\sqrt{\frac{3}{10}} B_{\mu} & W_{\mu}^{+} \\
Y_{\mu}^{1} & Y_{\mu}^{2} & Y_{\mu}^{3} & W^{-} \mu & -\frac{1}{\sqrt{2}} W_{\mu}^{3}-\sqrt{\frac{3}{10}} B_{\mu}
\end{array}\right),
$$

where $G^{i}{ }_{j} \sim(\mathbf{8}, \mathbf{1})$ correspond to the gluons, $W^{ \pm, 3} \sim(\mathbf{1}, \mathbf{3})$ correspond to the intermediate vector bosons, $B \sim(\mathbf{1}, \mathbf{1})$ correspond to the hypercharge generator (eventually mixing with $W^{3}$ to yield the $Z^{0}$ and the photon), and $X, Y \sim(\overline{\mathbf{3}}, \mathbf{2})$ are twelve new gauge bosons ${ }^{58}$ Their electric charges can be read out of

$$
\begin{equation*}
Q^{\dagger} A_{\mu} Q=(Q(\overline{\mathbf{5}})+Q(\mathbf{5})) A_{\mu}, \tag{10.28}
\end{equation*}
$$

and read $Q_{X}=-\frac{1}{3}-1=-\frac{4}{3}$ and $Q_{Y}=-\frac{1}{3}+0=-\frac{1}{3}$.
As a first step to achieve the desired symmetry breaking pattern, we need to give mass to $X$ and $Y$. With the appropriate potential for an adjoint Higgs field $H$, transforming in the $\mathbf{2 4}$ representation, one gets

$$
\begin{equation*}
\langle H\rangle=v \lambda^{24} \tag{10.29}
\end{equation*}
$$

which breaks $\mathrm{SU}(5) \rightarrow G_{\mathrm{SM}}$ (recall $\left[\lambda^{24}, G_{\mathrm{SM}}\right]=0$ ). After that, we break $G_{\mathrm{SM}} \rightarrow \mathrm{U}(1)_{Q}$ as was done before, and we get back the Standard Model plus new bosons. The SU(5) covariant derivative reads

$$
\begin{equation*}
D_{\mu}=\partial_{\mu}+i g_{5} \sum_{a=1}^{24} A_{\mu}^{a} \frac{\lambda^{a}}{2} \tag{10.30}
\end{equation*}
$$

while the Standard Model one reads

$$
\begin{equation*}
D_{\mu}=\partial_{\mu}+i g_{3} \sum_{a=1}^{8} G_{\mu}^{a} \frac{\lambda_{\mathrm{GM}}^{a}}{2}+i g \sum_{a=1}^{3} W_{\mu}^{a} \frac{\tau^{a}}{2}+i g^{\prime} B_{\mu} \frac{Y}{2} \tag{10.31}
\end{equation*}
$$

Comparing the two equations, and recalling that $Y=-\sqrt{5 / 3} \lambda^{24}$, we find

$$
\begin{equation*}
g_{3}=g=g_{5}, \quad g^{\prime}=-\sqrt{\frac{3}{5}} g_{5} \tag{10.32}
\end{equation*}
$$

from which we get

$$
\begin{equation*}
\tan \theta_{W}=\frac{g^{\prime}}{g}=-\sqrt{\frac{3}{5}} \tag{10.33}
\end{equation*}
$$

From

$$
\begin{equation*}
\sin ^{2} x=\frac{\tan ^{2} x}{1+\tan ^{2} x} \tag{10.34}
\end{equation*}
$$

we then obtain $\sin ^{2} \theta_{W}=0.375$, which differs from $\sin ^{2} \theta_{W}^{\exp }=0.212023$. Moreover, this model predicts $g_{3}=g$, i.e., unified strong and weak coupling. This is known to be not true experimentally at low energies (recall that couplings "run" with energy, i.e., are energy dependent).

[^39]

Figure 13: $|\Delta B|=1$ processes in the $\mathrm{SU}(5)$ GUT.

However, the identification of the two couplings should be done at some high energy scale $M_{X}$ of the order of the masses of the new bosons, i.e., the scale at which the full $\mathrm{SU}(5)$ symmetry breaks down. The running of the Weinberg angle in the unified theory reads (for three Standard Model families)

$$
\begin{equation*}
\sin ^{2} \theta_{W}(\mu)=\frac{3}{8}-\frac{55}{24 \pi} \alpha(\mu) \ln \frac{M_{x}}{\mu} \tag{10.35}
\end{equation*}
$$

where $\alpha(\mu)$ is the running QCD coupling and $\mu$ the energy scale of the relevant process. Imposing $\sin ^{2} \theta_{W}=0.22-0.23$ at low energy (i.e., at $\mu \sim M_{Z}$ ), we find that $\theta_{W}(\mu)$ reaches the grand unification value at $\mu=M_{X} \sim 4 \cdot 10^{4} \mathrm{GeV}$.

A drawback of this model is that while $B-L$ is still conserved, $B$ and $L$ separately are not conserved anymore. In particular, the new bosons mediate proton decay at tree level: since leptons and quarks are in the same multiplet, this is to be expected (it is like $u \rightarrow d$ via $W$ boson). This makes, e.g., $p \rightarrow e^{+} \pi^{0}(\Delta B=-1, \Delta L=-1)$ allowed, while $n \rightarrow e^{-} \pi^{+}(\Delta B=-1, \Delta L=$ 1) remains forbidden. On the other hand, $\Delta B \neq 0$ processes are suppressed due to the large mass of the new bosons. In any case, so far no proton decay has been observed experimentally. The current bound on the proton lifetime resulting from this null result is $\tau_{\text {proton }}>10^{34}$ years. The lifetime predicted by the $\mathrm{SU}(5) \mathrm{GUT}$ model, $\tau \sim \frac{M_{X}^{4}}{m_{p}^{5}} 5^{59}$ is $\tau \approx 10^{30} \div 10^{31}$ years, so this model is experimentally disproved. Notice that $\tau$ is very sensitive to $M_{X}$, which is constrained by low-energy phenomenology (e.g., $\sin ^{2} \theta_{W}$ ). Variations (non-minimal, either supersymmetric or not) exist, and all predict proton decay; still, no proton decay has been observed so far.

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${ }^{59}$ At tree level $\Gamma \propto\left(\frac{g^{2}}{M_{X}^{2}}\right)^{2}$, so by dimensional reasons $\tau \propto \frac{M_{X}^{4}}{m_{p}^{5}}$.
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[^0]:    ${ }^{1}$ When $C P$ is violated, violation of $T$ is automatic if elementary particles are described by a Poincaré-invariant local quantum field theory, due to the $C P T$ theorem.
    ${ }^{2}$ If one insists on massless neutrinos, lepton family number is also a good symmetry. Also, tiny nonperturbative effects are expected to violate baryon $(B)$ and lepton $(L)$ number separately, leaving only $B-L$ as a symmetry [1].

[^1]:    ${ }^{3}$ In the presence of a magnetic field, $\alpha$ and $\beta$ rays where deflected in opposite directions, and $\gamma$ rays not at all. The $\alpha$ and $\gamma$ rays are now known to be helium nuclei, ${ }_{2}^{4} \mathrm{He}$, and highly energetic photons, respectively.
    ${ }^{4}$ We are running ahead of time and distinguishing neutrinos from antineutrinos, and electronic from muonic neutrinos. These distinctions were unknown to Fermi.

[^2]:    ${ }^{5}$ This is the case if lepton number is conserved. Since neutrinos actually have small but nonzero mass, it is not excluded that they are fermions of Majorana type, identical to their own antiparticle, which would lead to lepton number violations.
    ${ }^{6}$ Terms differing by a permutation of the fields $p(x), n(x), e(x)$ and $\nu(x)$ can be reduced to those in Eq. 1.2 making use of the so-called Fierz transformations. The matrices $M_{j}$ are the matrices $M^{j}$ with covariant indices.

[^3]:    ${ }^{7}$ Actually, parity violations in weak interactions are already borne out of the experiments of Cox (1928) and his student Chase (1930).

[^4]:    ${ }^{8}$ In Yukawa's intentions, this boson would have mediated both weak and strong interactions. Using different bosons for the two interactions does not change the argument.

[^5]:    ${ }^{10}$ There is no need to introduce a second mixing matrix, mixing the positive-charge quarks, as it could be reabsorbed in $V_{\text {CKM }}$.

[^6]:    ${ }^{11}$ It is called so due to the fact that it was introduced by Glashow (see "Arnol'd principle" (7).

[^7]:    ${ }^{12}$ Notice that $\vec{\eta} \cdot \frac{\vec{\sigma}}{2} \tilde{\varphi}_{s}=-i \vec{\eta} \cdot \frac{\vec{\sigma}}{2} \sigma_{2} \varphi_{s}^{*}=i \sigma_{2}\left(\vec{\eta} \cdot \frac{\vec{\sigma}}{2} \varphi_{s}\right)^{*}=-s\left(-i \sigma_{2} \varphi_{s}^{*}\right)=-s \tilde{\varphi}_{s}$. However, $\tilde{\varphi}_{s}$ corresponds to an antiparticle state with spin component $+s$. The reason is that while $u_{s}$ is paired with an annihilation operator, $v_{s}$ is paired with a creation operator: taking particles and antiparticles to transform in the same representation under rotations, this requires that the two wave functions transform in representations that are one the complex conjugate of the other for $\psi$ to have a simple transformation law.

[^8]:    ${ }^{13}$ The first equality must be understood as a matrix equality, not a tensor equality.
    ${ }^{14}$ Since $\bar{f}=f^{\dagger} \gamma^{0}$, one has $\bar{f} \gamma^{\alpha} P_{-}=f^{\dagger} \gamma^{0} \gamma^{\alpha} P_{-}=f^{\dagger} \gamma^{0} P_{+} \gamma^{\alpha}=f^{\dagger} P_{-} \gamma^{0} \gamma^{\alpha}=\left(P_{-} f\right)^{\dagger} \gamma^{0} \gamma^{\alpha}=\bar{f}_{-} \gamma^{0} \gamma^{\alpha}$.

[^9]:    ${ }^{15}$ One can of course include them as non-interacting particles, the so-called sterile neutrinos. They would be, however, almost entirely unobservable, coupling only to gravity.
    ${ }^{16}$ If $i \not \not \emptyset \psi=0$, then also $i \not \partial \gamma \gamma^{5} \psi=0$, and we can form linear combinations with definite chirality.

[^10]:    ${ }^{17}$ The identities $V^{\alpha} V_{\alpha}^{\dagger}=V_{\alpha}^{\dagger} V^{\alpha}$ and likewise follow from the bosonic nature of the bilinears, and from the understanding that the Lagrangian is normal ordered, if we work in the operator formalism, or is just a $c$-number, if we work in the functional-integral formalism.
    ${ }^{18}$ One is tempted to say that they do not exist, but it is now known that neutrinos have masses, so that a "wrong"-handedness component exists. Nevertheless, it is not coupled in the same way as the "right"-handedness component.

[^11]:    ${ }^{19}$ This process would be allowed if neutrinos were massive, truly neutral particles, coinciding with their antiparticle (Majorana fermions), in which case of course they could not be assigned any nonzero conserved charge. (If massless, the two helicity states can still be interpreted as different particles independently of lepton family number conservation.)

[^12]:    ${ }^{20}$ In the case of lightlike $q, q^{2}=0$, the integral vanishes, since it receives contributions only from the sets of zero measure where $q_{1} \cdot q_{2}=0$, corresponding to collinear neutrinos, or where $\omega_{1}=\left|\vec{q}_{1}\right|=0$ or $\omega_{2}=\left|\vec{q}_{2}\right|=0$, corresponding to neutrinos of vanishing energy.

[^13]:    ${ }^{21}$ To prove Eqs. (2.34) and 2.35, check first that they are correct in the rest frame of a massive particle using the explicit expressions Eq. 1.30. Notice that if $\vec{\eta} \cdot \vec{\sigma} \varphi=\varphi$, then $\varphi \varphi^{\dagger}=\frac{1+\vec{\eta} \cdot \vec{\sigma}}{2}$. The validity of Eq. 2.34) in a general reference frame follows from Lorentz invariance, after showing that Eq. 2.35 provides the Lorentz transformed of $s$ under a pure boost in direction $\frac{\vec{p}}{|\vec{p}|}$ (without any further rotation around $\frac{\vec{p}}{|\vec{p}|}$ ) with $\beta \gamma=\frac{|\vec{p}|}{m}$ starting from the rest frame.

[^14]:    ${ }^{22}$ The neutrino-antineutrino system in this case is a zero-mass system with zero helicity, so a system with vanishing total angular momentum.

[^15]:    ${ }^{23}$ If solutions exist for $q_{R}^{0} \geq m_{1}+m_{2}$, then both sides of each equation in Eq. 2.89 are positive. Replacing them with their squares leads therefore to an entirely equivalent equation under the restriction on $q_{R}^{0}$, i.e., their solutions are in one-one correspondence.
    ${ }^{24}$ Notice that $\left(m_{1}+m_{2}\right)^{2} \geq\left(m_{1}-m_{2}\right)^{2}$ if $m_{1,2} \geq 0$.

[^16]:    ${ }^{25}$ The axial-vector current matrix elements have the same form, of course with a different coefficient, but this does not turn out to be as useful.

[^17]:    ${ }^{26}$ We know that the light quark masses are also different from each other, so that the vector current is also not exactly conserved, but this has a smaller effect on physical quantities.
    ${ }^{27}$ Exploiting invariance under the antiunitary transformation $\Theta=C P T$ and the transformation property $\Theta^{\dagger} A^{\mu}(x) \Theta=-A^{\mu}(-x)$ of an axial-vector fermion bilinear, one finds

    $$
    \langle 0| \Theta^{\dagger} A_{a}^{\mu}(0) \Theta\left|\pi_{b}(\vec{p})\right\rangle=-\langle 0| A_{a}^{\mu}(0)\left|\pi_{b}(\vec{p})\right\rangle=-i f_{\pi} \delta_{a b} p^{\mu}=\langle 0| A_{a}^{\mu}(0)\left|\pi_{b}(-\vec{p})\right\rangle^{*}=\left(i f_{\pi} \delta_{a b} p^{\mu}\right)^{*}
    $$

[^18]:    ${ }^{29}$ The transformation laws for fermion bilinears under $\Theta=C P T$ and $C P$ read

    $$
    \Theta^{\dagger} \bar{\psi}(x) \Gamma \psi(x) \Theta=\bar{\psi}(-x) \gamma^{5} \gamma^{0} \Gamma^{\dagger} \gamma^{0} \gamma^{5} \psi(-x) \quad(C P)^{\dagger} \bar{\psi}(x) \Gamma \psi(x) C P=\bar{\psi}\left(x_{P}\right) \gamma^{0} \mathcal{C}^{\dagger} \Gamma^{T} \mathcal{C} \gamma^{0} \psi\left(x_{P}\right)
    $$

[^19]:    ${ }^{30}$ Notice that with this definition of $\pi^{+}$the corresponding isospin state is $-|11\rangle$, if we adopt the usual CondonShortley convention.

[^20]:    ${ }^{33}$ This is so if we choose phases as follows: $K^{+}=\bar{s} u, K^{0}=\bar{s} d=I_{-} K^{+},\left|\pi^{+}\right\rangle=\bar{d} u=-W_{-} K^{+}$, and $\sqrt{2}\left|\pi^{0}\right\rangle=I_{-} \pi^{+}, \sqrt{2}\left|\pi^{-}\right\rangle=I_{-} \pi^{0}$, i.e., $\pi^{0}=\frac{\bar{d} d-\bar{u} u}{\sqrt{2}}, \pi^{-}=-\bar{u} d$.

[^21]:    ${ }^{34}$ This result differs from the one reported in Okun.
    ${ }^{35}$ The argument is that the matrix element $M^{\mu}\left(\left\{\vec{p}_{j}\right\},\left\{\vec{p}_{j}^{\prime}\right\}\right)=\left\langle f\left(\left\{\vec{p}_{j}^{\prime}\right\}\right)\right| V^{\mu}\left|i\left(\left\{\vec{p}_{j}\right\}\right)\right\rangle$ of a vector current satisfies $\left\langle f\left(\left\{\vec{p}_{j}^{\prime}\right\}\right)\right| P^{\dagger} V^{\mu} P\left|i\left(\left\{\vec{p}_{j}\right\}\right)\right\rangle=\eta_{f}^{\prime} \eta_{i}\left\langle f\left(\left\{-\vec{p}_{j}^{\prime}\right\}\right)\right| V^{\mu}\left|i\left(\left\{-\vec{p}_{j}\right\}\right)\right\rangle=\left\langle f\left(\left\{\vec{p}_{j}^{\prime}\right\}\right)\right| \mathcal{P}_{\nu}^{\mu} V^{\nu}\left|i\left(\left\{\vec{p}_{j}\right\}\right)\right\rangle$, i.e., $M^{\mu}\left(\left\{-\vec{p}_{j}\right\},\left\{-\vec{p}_{j}^{\prime}\right\}\right)=$ $\eta_{f}^{\prime} \eta_{i} \mathcal{P}_{\nu}^{\mu} M^{\nu}\left(\left\{\vec{p}_{j}\right\},\left\{\vec{p}_{j}^{\prime}\right\}\right)$, thus transforming as a vector if $\eta_{f}^{\prime} \eta_{i}=1$ and as pseudovector if $\eta_{f}^{\prime} \eta_{i}=-1$. As similar argument holds for an axial-vector current.

[^22]:    ${ }^{36}$ Minus signs are introduced to make the phases of the various baryon states match those required by the Condon-Shortley convention for the matrix elements of the lowering operators of $\mathrm{SU}(2)_{I}$. One could use different phase conventions, but this does not concern us here since superselection rules do not allow one to consider linear combinations of different baryons in physical matrix elements, and so such phases are physically irrelevant.

[^23]:    ${ }^{37} \mathrm{~A}$ second term, $\mathcal{I}_{L}$, is also present, obtained by replacing $u_{R}, d_{R} \rightarrow u_{L}, d_{L}$ in Eq. 5.7 , but since its overall coupling is small it can be neglected compared to $I_{3}$.

[^24]:    ${ }^{38}$ One can see this condition as eliminating one component of $\varepsilon$, since only three polarisations are available. In particular, $p \cdot \varepsilon$ is a Lorentz scalar, and so unwanted in the description of a vector particle.

[^25]:    ${ }^{39}$ We note in passing that the existence of three and only three families of leptons with light neutrinos has been extablished experimentally at LEP.

[^26]:    ${ }^{40}$ The cross section has dimensions of $m^{-2}$, and since a factor $G^{2}$ of dimension $m^{-4}$ is present, and $\sqrt{s}$ is the only relevant energy scale at high energy, the result follows.

[^27]:    ${ }^{41} \mathrm{We}$ ignore here the term $q_{\mu} q_{\nu} / m_{W}^{2}$ in the propagator, considering $|t| \ll m_{W}$.

[^28]:    ${ }^{42}$ A gauge symmetry cannot be broken due to Elitzur's theorem. What can be broken is the remaining global symmetry after gauge fixing, which is required to carry out the quantisation procedure.
    ${ }^{43}$ Alternatively, one can write $T^{a}=i \tilde{T}^{a}$ with $\tilde{T}^{a}$ purely imaginary and Hermitian, and satisfying $\left[\tilde{T}^{a}, \tilde{T}^{b}\right]=$ $i f^{a b c} \tilde{T}^{c}$.

[^29]:    ${ }^{44}$ In mathematical terms this is asking that the action of $G$ on $\mathcal{M}$ be transitive.
    ${ }^{45}$ If $h_{1,2} \in H$ then $h_{1} h_{2} \phi_{0}=\phi_{0}$ and so $h_{1} h_{2} \in H$. Clearly the neutral element belongs to $H$, and for any $h \in H$ one has $h^{-1} \phi_{0}=h^{-1} h \phi_{0}=\phi_{0}$, i.e., $h^{-1} \in H$.
    ${ }^{46}$ The equality sign stands here for "diffeomorphic to".

[^30]:    ${ }^{47}$ The space $G / H$ is in general not a Lie group, unless $H$ is a normal subgroup, i.e., $g H g^{-1}=H$ for any $g \in G$. It is however a manifold of dimension $n-n^{\prime}$, and $\left\{T^{n^{\prime}+1}, \ldots, T^{n}\right\}$ are a basis of its tangent space at the point corresponding to the neutral element $e H=H \in G / H$.

[^31]:    ${ }^{48}$ This does not mean that it is totally irrelevant: this term is known as the $\theta$-term in QCD, and plays an important role in such things as, e.g., the axial anomaly and the mass of the $\eta^{\prime}$ meson.

[^32]:    ${ }^{49}$ Alternatively, one can choose arbitrary $b(p)$ if either $E=0$ or $\vec{p}=0$, or both. Zero energy and nonzero momentum would however yield a negative mass squared, so a tachyonic mode which leads to instabilities. In general, the constraints on $E$ and $\vec{p}$ are incompatible with the Lorentz transformation properties of the field, so these solutions must be excluded.

[^33]:    ${ }^{50}$ The proof is given in Weinberg's "The Quantum Theory of Fields", volume II, CUP.

[^34]:    ${ }^{51}$ An alternative would have been to include the left-handed field $\bar{e}_{R}$ in the same multiplet, in which case on would find $T_{3} \propto Q$. This is the basis of the $\mathrm{SO}(3)$ model of Georgi and Glashow, which was however disproven by experiments.

[^35]:    ${ }^{52}$ The full set of global symmetries of a doublet of left-handed fields ( $\nu_{e L}, e_{L}$ ) and a singlet $e_{R}$ is $\mathrm{SU}(2)_{L} \times$ $\mathrm{U}(1)_{L} \times \mathrm{U}(1)_{R}$, where $\mathrm{U}(1)_{L, R}$ are chiral phase transformations generated by $t_{L}=T^{3}-\frac{1+\gamma^{5}}{2} Q$ and $t_{R}=-\frac{1-\gamma^{5}}{2} Q$, one acting in the same way on $\nu_{e L}$ and $e_{L}$ and one acting on $e_{R}$. Only the combination $\mathrm{U}(1)_{Y}$ happens to be gauged in nature, generated by $Y=-2\left(t_{R}+t_{L}\right)$. The other independent combination $2 t_{L}+t_{R}$, corresponding to lepton family number, happens to be only global.
    ${ }^{53}$ The alternative would be to choose $Y=-1$ and have the neutral field in the upper component and a negatively charged field in the lower component. This is nothing but the charge conjugate of $\phi$ in Eq. 9.8, so we are not losing generality.

[^36]:    ${ }^{54}$ As discussed in Weinberg, op. cit., it suffices to use the polar decomposition theorem to write $M=H U$ with $H$ Hermitian and $U$ unitary. In turn, $H=V^{\dagger} D V$ with $D$ real diagonal and $V$ unitary. If we denote with $\Sigma$ the diagonal matrix of the signs of the entries of $D$, then taking $S=\Sigma V$ and $T^{\dagger}=U^{\dagger} V^{\dagger}$ one finds $S M T^{\dagger}=\Sigma V V^{\dagger} D V U U^{\dagger} V^{\dagger}=\Sigma D$, which is real positive and diagonal.

[^37]:    ${ }^{55}$ Since there are only two families, any extra phase factor can be reabsorbed by redefining the phases of the eigenstates.

[^38]:    ${ }^{56}$ More generally, one can introduce mass terms $\overline{\nu_{\ell}} M_{\ell \ell^{\prime}}\left(\nu_{\ell^{\prime} R}\right)$, which can be diagonalised yielding Majorana terms.
    ${ }^{57}$ The sign of $m_{-}$is not problematic, as it can be changed by redefining $\psi \rightarrow \gamma^{5} \psi$.

[^39]:    ${ }^{58}$ Notice that $X^{c}, Y^{c} \sim(\mathbf{3}, \overline{\mathbf{2}})$.

