Weak Interactions

Matteo Giordano

Eötvös Loránd University (ELTE)
Budapest

September 23, 2020

Axial current and chiral symmetry (contd.)

Spontaneous breaking of axial part of chiral symmetry $\Rightarrow$
Goldstone bosons π_a (one per broken generator), coupled to axial current

$$\langle 0|A_a^\mu(0)|\pi_b\rangle = ip^\mu f_{ab} \qquad f_{ab}: \text{ constants}$$

$$\langle 0|A_a^\mu(0)|\pi_b\rangle = ip^\mu f_\pi \delta_{ab} \qquad \text{if vector symmetry (isospin) unbroken}$$

f_π: *pion decay constant*, real (T invariance), $[f_\pi] = [m]$

$$\langle 0|\partial_\mu A_a^\mu(0)|\pi_b\rangle = (-i)p_\mu ip^\mu f_\pi \delta_{ab} = m_\pi^2 f_\pi \delta_{ab}$$

PCAC hypothesis: generalise to an operator relation

$$\partial_\mu A_a^\mu(x) = f_\pi m_\pi^2 \phi_a(x)$$

ϕ_a: pion fields (mass dimension 1), $\phi_\pm = \phi_1 \pm i\phi_2$

$$\langle \pi_a|\phi_b(0)|0\rangle = \delta_{ab} \qquad \langle \pi_3|\phi_3(0)|0\rangle = 1 \quad \langle \pi_\pm = \tfrac{\pi_1 \pm i\pi_2}{\sqrt{2}}|\tfrac{\phi_\pm(0)}{\sqrt{2}}|0\rangle = 1$$

QCD perspective: Ward identity for A_a^μ in terms of effective mesonic field

$$\partial_\mu A_a^\mu(x) = 2m_{ud} P_a(x) \qquad P_a = \bar{q}\gamma^5 \tfrac{\tau_a}{2} q$$

$m_{ud} = m_u = m_d$

Pion decays: leptonic decays of charged pions

Purely leptonic decays

$$\pi^+ \to \ell^+ \nu_\ell \,, \qquad \pi^- \to \ell^- \bar{\nu}_\ell$$

$J_\pi^P = 0^-$ (pseudoscalars) $\Rightarrow$ pion leptonic decay mediated by axial current

$$\langle 0|V_\mp^\mu|\pi^\pm\rangle = 0 \qquad \langle 0|A_\mp^\mu|\pi^\pm\rangle = i\sqrt{2}f_\pi p^\mu$$

Fixed by Lorentz invariance, f_π dim=1 real (T: 2-family approx.) const., same for $\pi^\pm$ (isospin, or CP away from iso limit)

Normalisation: $\langle 0|A_i^\mu|\pi_j\rangle = if_\pi\delta_{ij}p^\mu$ so $\langle 0|A_3^\mu|\pi^0\rangle = if_\pi p^\mu$ and

$$\langle 0|A_\mp^\mu|\pi_\pm\rangle = \langle 0|A_1^\mu \mp iA_2^\mu|\tfrac{\pi_1 \pm i\pi_2}{\sqrt{2}}\rangle = \tfrac{1}{\sqrt{2}}(\langle 0|A_1^\mu|\pi_1\rangle + \langle 0|A_2^\mu|\pi_2\rangle) = if_\pi\sqrt{2}p^\mu$$

Choice of signs from isospin conservation, $A_-^\mu = \bar{d}\mathcal{O}_L^\mu u$ and $A_+^\mu = \bar{u}\mathcal{O}_L^\mu d$ coupled to $\bar{\nu}_\ell\mathcal{O}_L^\mu\ell$ and $\bar{\ell}\mathcal{O}_L^\mu\nu_\ell$

Quark model perspective: $\pi^+ = \bar{d}u$, $\pi^- = \bar{u}d$, annihilated by $A_-^\mu = \bar{d}\mathcal{O}_L^\mu u$ and $A_+^\mu = \bar{u}\mathcal{O}_L^\mu d$; $|\pi^+\rangle = -|I=1,\, I_3=1\rangle$ in Condon-Shortley convention

Pion decays: leptonic decays of charged pions (contd.)

Focus on $\pi^+ \to \ell^+ \nu_\ell$ (same width for $\pi^- \to \ell^- \bar{\nu}_\ell$ from CP)

$$\begin{aligned}
\mathcal{M}_{\rm fi} &= -\frac{G}{\sqrt{2}} \cos\theta_C \, i\sqrt{2} f_\pi p^\mu_{(\pi)} \langle \ell^+ \nu_\ell | (\bar{\nu}_\ell \mathcal{O}_{L\mu} \ell)(0) | 0 \rangle \\
\text{(Feynman rules)} &= -iG \cos\theta_C \, f_\pi p^\mu_{(\pi)} \bar{u}_{(\nu)} \gamma_\mu (1-\gamma^5) v_{(\ell)} \\
\text{(momentum cons.)} &= -iG \cos\theta_C \, f_\pi \bar{u}_{(\nu)} (\not{p}_{(\nu)} + \not{p}_{(\ell)}) (1-\gamma^5) v_{(\ell)} \\
\text{(Dirac eq.)} &= -iG \cos\theta_C \, f_\pi \bar{u}_{(\nu)} (1+\gamma^5) \not{p}_{(\ell)} v_{(\ell)} \\
\text{(Dirac eq.)} &= \quad iG \cos\theta_C \, f_\pi m_\ell \bar{u}_{(\nu)} (1+\gamma^5) v_{(\ell)}
\end{aligned}$$

Squaring

$$|\mathcal{M}_{\rm fi}|^2 = G^2 \cos^2\theta_C \, f_\pi^2 m_\ell^2 \, \bar{u}_{(\nu)} (1+\gamma^5) v_{(\ell)} \bar{v}_{(\ell)} (1-\gamma^5) u_{(\nu)}$$

Summing over ℓ^+ spins

$$\begin{aligned}
\langle\langle |\mathcal{M}_{\rm fi}|^2 \rangle\rangle &= G^2 \cos^2\theta_C \, f_\pi^2 m_\ell^2 \, {\rm tr}\, (\not{p}_{(\ell)} - m_\ell)(1-\gamma^5) \not{p}_{(\nu)} \tfrac{1+\gamma^5}{2} (1+\gamma^5) \\
&= G^2 \cos^2\theta_C \, f_\pi^2 m_\ell^2 \, 2{\rm tr}\, (\not{p}_{(\ell)} - m_\ell) \not{p}_{(\nu)} (1+\gamma^5) \\
&= 2G^2 \cos^2\theta_C \, f_\pi^2 m_\ell^2 \, {\rm tr}\, \not{p}_{(\ell)} \not{p}_{(\nu)} = 8G^2 \cos^2\theta_C \, f_\pi^2 m_\ell^2 \, p_{(\ell)} \cdot p_{(\nu)}
\end{aligned}$$

Pion decays: leptonic decays of charged pions (contd.)

$$p_{(\pi)} = p_{(\ell)} + p_{(\nu)} \Longrightarrow m_\pi^2 = m_\ell^2 + 2p_{(\ell)} \cdot p_{(\nu)} \Longrightarrow m_\pi^2 - m_\ell^2 = 2p_{(\ell)} \cdot p_{(\nu)}$$

$$\langle\langle |\mathcal{M}_{\rm fi}|^2 \rangle\rangle = 4G^2 \cos^2\theta_C \, f_\pi^2 m_\ell^2 (m_\pi^2 - m_\ell^2)$$

Isotropic (expected from $J_\pi = 0$); energies fixed (2-body decay) $\Rightarrow$ constant

$$\Gamma = \int d\Gamma = \int d\Phi^{(2)} \frac{\langle\langle |\mathcal{M}_{\rm fi}|^2 \rangle\rangle}{2m_\pi} = \frac{\langle\langle |\mathcal{M}_{\rm fi}|^2 \rangle\rangle}{2m_\pi} \int d\Phi^{(2)} = \frac{\langle\langle |\mathcal{M}_{\rm fi}|^2 \rangle\rangle}{2m_\pi} \Phi^{(2)}$$

$$d\Phi^{(2)} = (2\pi)^4 \delta^{(4)}(p_{(\pi)} - p_{(\ell)} - p_{(\nu)}) \frac{d^3 p_{(\ell)}}{(2\pi)^3 2E_\ell} \frac{d^3 p_{(\nu)}}{(2\pi)^3 2E_\nu}$$

$\Phi^{(2)}$ Lorentz invariant, work in pion rest frame

$$\begin{aligned}
\Phi^{(2)} &= \frac{1}{(4\pi)^2} \int \frac{d^3 p_{(\ell)}}{E_\ell} \int \frac{d^3 p_{(\nu)}}{E_\nu} \delta(m_\pi - E_\ell - E_\nu) \delta^{(3)}(\vec{p}_{(\ell)} - \vec{p}_{(\nu)}) \\
&= \frac{1}{(4\pi)^2} \int \frac{d^3 p_{(\ell)}}{E_\ell E_\nu} \delta(m_\pi - E_\ell - E_\nu) \\
&= \frac{1}{4\pi} \int_0^\infty \frac{dp\, p^2}{\sqrt{m_\ell^2 + p^2}\sqrt{m_\nu^2 + p^2}} \delta(m_\pi - \sqrt{m_\ell^2 + p^2} - \sqrt{m_\nu^2 + p^2})
\end{aligned}$$

Pion decays: leptonic decays of charged pions (contd.)

$$\delta(m_\pi - \sqrt{m_\ell^2 + p^2} - \sqrt{m_\nu^2 + p^2}) = (\tfrac{p}{E_\ell} + \tfrac{p}{E_\nu})^{-1}\delta(p - p_*) = \tfrac{E_\ell E_\nu}{p m_\pi}\delta(p - p_*)$$

p_*: magnitude of spatial momentum of final particles

$$m_\pi - \sqrt{m_\nu^2 + p^2} = \sqrt{m_\ell^2 + p^2}$$

$$m_\pi^2 - 2m_\pi\sqrt{m_\nu^2 + p^2} + m_\nu^2 + p^2 = m_\ell^2 + p^2$$

$$m_\pi^2 + m_\nu^2 - m_\ell^2 = 2m_\pi\sqrt{m_\nu^2 + p^2}$$

$$(m_\pi^2 + m_\nu^2 - m_\ell^2)^2 = 4m_\pi^2(m_\nu^2 + p^2)$$

Always squaring positive quantities $\Rightarrow$ equivalent equations

$$p_*^2 = \frac{(m_\pi^2 + m_\nu^2 - m_\ell^2)^2}{4m_\pi^2} - m_\nu^2 = \frac{[m_\pi^2 - (m_\nu + m_\ell)^2][m_\pi^2 - (m_\nu - m_\ell)^2]}{4m_\pi^2}$$

Pion decays: leptonic decays of charged pions (contd.)

Phase space volume:

$$\Phi^{(2)} = \frac{1}{4\pi}\int_0^\infty \frac{dp\, p^2}{E_\ell E_\nu}\frac{E_\ell E_\nu}{p m_\pi}\delta(p-p_*) = \frac{p_*}{4\pi m_\pi}$$

Set $m_\nu = 0$

$$p_*^2 = \left(\frac{m_\pi^2 - m_\ell^2}{2m_\pi}\right)^2 \Longrightarrow \Phi^{(2)} = \frac{m_\pi^2 - m_\ell^2}{8m_\pi^2}$$

Total width:

$$\Gamma = \frac{1}{2m_\pi}4G^2\cos^2\theta_C\, f_\pi^2 m_\ell^2 (m_\pi^2 - m_\ell^2)\frac{m_\pi^2 - m_\ell^2}{8m_\pi^2}$$

$$= \frac{G^2\cos^2\theta_C\, f_\pi^2 m_\ell^2}{4\pi m_\pi^3}(m_\pi^2 - m_\ell^2)^2 = \frac{G^2\cos^2\theta_C\, f_\pi^2}{4\pi}\, m_\pi m_\ell^2\left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2$$

$\Gamma = \Gamma(m_\ell)$ suppressed

- $m_\ell \sim m_\pi$: threshold effect due to the limited available phase space
- $m_\ell \sim 0$: dynamical effect due to definite handedness of the current

Pion decays: leptonic decays of charged pions (contd.)

Suppression of decay into light charged leptons:

- for very light lepton helicity almost good quantum number, and almost only left-handed leptons and right-handed antileptons appear
- $J_\pi = 0, \vec{p}_\pi = 0 \Rightarrow$ opposite ℓ^+/ν_ℓ spins and momenta in final state $\Rightarrow$ same helicity but ν_ℓ L-handed, ℓ^+ R-handed $\Rightarrow$ suppression
- vanishes exactly if $m_\ell = 0$

Dominant decay mode $\pi^+ \to \mu^+ \nu_\mu$ instead of $\pi^+ \to e^+ \nu_e$ available despite limited phase space

$$\frac{\Phi_\mu^{(2)}}{\Phi_e^{(2)}} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 - m_e^2} \simeq 0.4$$

$$\frac{\Gamma_{\pi^+ \to e^+ \nu_e}}{\Gamma_{\pi^+ \to \mu^+ \nu_\mu}} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 \simeq 1.2 \cdot 10^{-4}$$

$$m_{\pi^\pm} = 140\,\mathrm{MeV},\ m_\mu = 106\,\mathrm{MeV},\ m_e = 0.5\,\mathrm{MeV}$$

Pion beta decay

Three-body decay

$$\pi^+ \to \pi^0 \quad e^+ \quad \nu_e$$

$$p_1 = p_2 + p_{(e)} + p_{(\nu)}$$

What about $\pi^+ \to \pi^0\, \mu^+\, \nu_\mu$?

$$\mathcal{M}_{\rm fi} = -\frac{G}{\sqrt{2}}\, \cos\theta_C \langle \pi^0 | (\bar{d}\mathcal{O}_L^\mu u)(0) | \pi^+ \rangle \langle e^+ \nu_e | (\bar{\nu}_e \mathcal{O}_{L\mu} e)(0) | 0 \rangle$$
$$= -\frac{G}{\sqrt{2}}\, \cos\theta_C \langle \pi^0 | V_-^\mu | \pi^+ \rangle \bar{u}_{(\nu)}(p_{(\nu)}) \gamma_\mu (1-\gamma^5) v_{(e)}(p_{(e)})$$

Only vector current contributes (no axial vector is available)

$$\langle \pi^0 | V_-^\mu | \pi^+ \rangle = f_+(q^2) p^\mu + f_-(q^2) q^\mu$$

$p = p_1 + p_2$, $q = p_1 - p_2$

form factors $f_\pm$: dim.less real (T inv.) functions of q^2 ($q \cdot p = m_{\pi^+}^2 - m_{\pi^0}^2$, $p^2 = q^2 + 4q \cdot p$)

In the isospin limit conservation of the vector current implies

$$0 = q_\mu \langle \pi^0 | V_-^\mu | \pi^+ \rangle |_{\rm iso} = q^2 f_-(q^2)|_{\rm iso} \Rightarrow f_-(q^2)|_{\rm iso} \equiv 0$$

Pion beta decay (contd.)

Expand in s. breaking parameter $\Delta \equiv m_{\pi^+} - m_{\pi^0} \neq 0$ (otherwise no decay!)

$$f_-(q^2) = f_-(q^2)|_{\rm iso} + f_-^{(1)}(q^2)|_{\rm iso}\,\Delta + \mathcal{O}(\Delta^2) = f_-^{(1)}(q^2)|_{\rm iso}\,\Delta + \mathcal{O}(\Delta^2)$$

Small momentum transfer and little phase space available due to small Δ

$$q^2 = (p_1 - p_2)^2 = 2(p_1^2 + p_2^2) - (p_1 + p_2)^2 = 2(m_{\pi^+}^2 + m_{\pi^0}^2) - (p_1 + p_2)^2$$
$$\leq 2(m_{\pi^+}^2 + m_{\pi^0}^2) - (m_{\pi^+} + m_{\pi^0})^2 = (m_{\pi^+} - m_{\pi^0})^2 = \Delta^2$$

To leading order: neglect f_- and take f_+ constant

$$f_- q^\mu = \mathcal{O}(\Delta^2), \quad f_+(q^2) = f_+(0) + q^2 f_+'(0) + \ldots = f_+(0) + \mathcal{O}(\Delta^2)$$

Transition within isomultiplet $\rightarrow$ governed by weak charge $Q_W(I, I_3')$

$$I = 1,\ I_3 = 1,\ I_3' = 0,\ Q_W(I, I_3) = \sqrt{I(I+1) - I_3(I_3+1)}$$

$$\langle \pi^0 | V_-^0(0) | \pi^+ \rangle|_{{\rm iso},\,\vec{q}=0} = 2p_1^0 Q_W(1,0) = 2p_1^0\sqrt{2}$$

$$f_+(q^2)|_{\rm iso}\, p^0|_{\vec{q}=0} = f_+(0)|_{\rm iso}\, 2p_1^0 \quad \Longrightarrow \quad f_+(0)|_{\rm iso} = \sqrt{2}$$

Also $f_+(0) = f_+(0)|_{\rm iso} + \mathcal{O}(\Delta^2)$

Pion beta decay (contd.)

$$\langle \pi^0 | V_-^\mu | \pi^+ \rangle = f_+(0)|_{\text{iso}}\, p^\mu + \mathcal{O}(\Delta^2) = \sqrt{2} p^\mu + \mathcal{O}(\Delta^2)$$

To next-to-leading order in Δ we thus have

$$\begin{aligned}
\mathcal{M}_{\text{fi}} &= -\tfrac{G}{\sqrt{2}}\, \cos\theta_C \sqrt{2} p^\mu \bar{u}_{(\nu)} \gamma_\mu (1-\gamma^5) v_{(e)} \\
&= -G \cos\theta_C p^\mu \bar{u}_{(\nu)} \gamma_\mu (1-\gamma^5) v_{(e)} \\
|\mathcal{M}_{\text{fi}}|^2 &= G^2 \cos^2\theta_C p^\mu p^\nu \bar{u}_{(\nu)} \gamma_\mu (1-\gamma^5) v_{(e)} \bar{v}_{(e)} \gamma_\nu (1-\gamma^5) u_{(\nu)} \\
\langle\langle |\mathcal{M}_{\text{fi}}|^2 \rangle\rangle &= G^2 \cos^2\theta_C p^\mu p^\nu \operatorname{tr} \gamma_\mu (1-\gamma^5)(\not{p}_{(e)} - m_e) \gamma_\nu (1-\gamma^5) \not{p}_{(\nu)} \tfrac{1+\gamma^5}{2} \\
&= 2G^2 \cos^2\theta_C p^\mu p^\nu \operatorname{tr} (1+\gamma^5) \gamma_\mu (\not{p}_{(e)} - m_e) \gamma_\nu \not{p}_{(\nu)} \\
&= 2G^2 \cos^2\theta_C p^\mu p^\nu \operatorname{tr} (1+\gamma^5) \gamma_\mu \not{p}_{(e)} \gamma_\nu \not{p}_{(\nu)} \\
&= 8G^2 \cos^2\theta_C p^\mu p^\nu \big[p_{(e)\mu} p_{(\nu)\nu} + p_{(e)\nu} p_{(\nu)\mu} - \eta_{\mu\nu} (p_{(e)} \cdot p_{(\nu)}) \\
&\qquad - i\varepsilon_{\mu\alpha\nu\beta} p_{(e)}^\alpha p_{(\nu)}^\beta \big] \\
&= 8G^2 \cos^2\theta_C \left[2(p \cdot p_{(e)})(p \cdot p_{(\nu)}) - p^2 (p_{(e)} \cdot p_{(\nu)}) \right]
\end{aligned}$$

Pion beta decay (contd.)

Differential decay width (with [...] exact up to $\mathcal{O}(\Delta^2)$)

$$d\Gamma = \frac{4G^2\cos^2\theta_C}{m_{\pi^+}}\left[2(p\cdot p_{(e)})(p\cdot p_{(\nu)}) - p^2(p_{(e)}\cdot p_{(\nu)})\right] d\Phi^{(3)}\,,$$

$$\Gamma = \frac{4G^2\cos^2\theta_C}{m_{\pi^+}}\int \frac{d^3p_2}{(2\pi)^3 2E_2}\, p^\mu p^\nu [2\mathcal{I}_{\mu\nu}(q) - \eta_{\mu\nu}\mathcal{I}^\alpha{}_\alpha(q)]$$

General two-body integral for particles $X_{1,2}$

$$\mathcal{I}^{X_1X_2}_{\alpha\beta}(q) \equiv \int d\Omega^{X_1}_{q_1}\int d\Omega^{X_2}_{q_2}\,(2\pi)^4\delta^{(4)}(q-q_1-q_2)\,q_{1\alpha}\,q_{2\beta} = A^{X_1X_2}q^2\eta_{\alpha\beta} + B^{X_1X_2}q_\alpha\,q_\beta$$

For $X_1 = e^+$ and $X_2 = \nu_e$, and $q^2 = (p_1-p_2)^2 = (p_{(e^+)}+p_{(\nu)})^2 \geq m_e^2$ timelike

$$\mathcal{I}_{\alpha\beta}(q) \equiv \mathcal{I}^{e^+\nu_e}_{\alpha\beta}(q) = \tfrac{1}{32\pi}\left(1-\tfrac{m_e^2}{q^2}\right)\left\{\tfrac{1}{3}\left(1-\tfrac{m_e^2}{q^2}\right)^2(q^2\eta_{\alpha\beta}-q_\alpha q_\beta) + \left[1-\left(\tfrac{m_e^2}{q^2}\right)^2\right]q_\alpha q_\beta\right\}$$

$$\mathcal{I}^\alpha{}_\alpha(q) = \tfrac{q^2}{32\pi}\left(1-\tfrac{m_e^2}{q^2}\right)\left\{\left(1-\tfrac{m_e^2}{q^2}\right)^2 + \left[1-\left(\tfrac{m_e^2}{q^2}\right)^2\right]\right\} = \tfrac{q^2}{16\pi}\left(1-\tfrac{m_e^2}{q^2}\right)^2$$

Contracting with $p^\mu p^\nu$

$$\mathcal{Q} = p^\mu p^\nu[2\mathcal{I}_{\mu\nu}(q) - \eta_{\mu\nu}\mathcal{I}^\alpha{}_\alpha(q)] = \tfrac{1}{24\pi}\left(1-\tfrac{m_e^2}{q^2}\right)^2\left\{(p\cdot q)^2\left(1+\tfrac{2m_e^2}{q^2}\right) - p^2q^2\left(1+\tfrac{m_e^2}{2q^2}\right)\right\}$$

Pion beta decay (contd.)

Lorentz invariants: setting $2m \equiv m_{\pi^+} + m_{\pi^0}$

$$q^2 = m_{\pi^+}^2 + m_{\pi^0}^2 - 2m_{\pi^+}E_2 \equiv \omega^2 \geq 0$$
$$p^2 = m_{\pi^+}^2 + m_{\pi^0}^2 + 2m_{\pi^+}E_2 = q^2 + 4m_{\pi^+}E_2$$
$$q \cdot p = m_{\pi^+}^2 - m_{\pi^0}^2 = 2m\Delta$$

$\mathcal{Q}$ depends only on E_2, not on the angular variables

$$\int \frac{d^3p_2}{(2\pi)^3 2E_2} \to \frac{4\pi}{2(2\pi)^3}\int_0^{p_{2\max}} \frac{dp_2 p_2^2}{E_2} = \frac{1}{(2\pi)^2}\int_{E_{2\min}}^{E_{2\max}} dE_2\sqrt{E_2^2 - m_{\pi^0}^2}$$

$$E_2 = \frac{m_{\pi^+}^2 + m_{\pi^0}^2 - q^2}{2m_{\pi^+}}$$

$$p_2^2 = E_2^2 - m_{\pi^0}^2 = \frac{m_{\pi^+}^2 + m_{\pi^0}^2 - q^2}{2m_{\pi^+}} - m_{\pi^0}^2 = \frac{[m_{\pi^+}^2 - (m_{\pi^0}+\omega)^2][m_{\pi^+}^2 - (m_{\pi^0}-\omega)^2]}{4m_{\pi^+}^2}$$

$$= \frac{[m_{\pi^+} - m_{\pi^0} - \omega][m_{\pi^+} + m_{\pi^0} + \omega][m_{\pi^+} - m_{\pi^0} + \omega][m_{\pi^+}^2 + m_{\pi^0} - \omega]}{4m_{\pi^+}^2}$$

$$= \frac{[\Delta - \omega][2m + \omega][\Delta + \omega][2m - \omega]}{4m_{\pi^+}^2} = \frac{[\Delta^2 - \omega^2][4m^2 - \omega^2]}{4m_{\pi^+}^2}$$

$$p^2 = 2(m_{\pi^+}^2 + m_{\pi^0}^2) - \omega^2 = 4\left(m^2 + \left(\frac{\Delta}{2}\right)^2\right) - \omega^2 = 4m^2 + \Delta^2 - \omega^2$$

Pion beta decay (contd.)

Change integration variable to more convenient ω

$$\omega_{\min} = \sqrt{q^2_{\min}} = m_e \qquad \omega_{\max} = \sqrt{q^2_{\max}} = \Delta \qquad 2\omega d\omega = -2m_{\pi^+} dE_2$$

$$\int \frac{d^3p_2}{(2\pi)^3 2E_2} \to \frac{1}{(2\pi)^2 2m^2_{\pi^+}} \int_{m_e}^{\Delta} d\omega\, \omega \sqrt{[\Delta^2 - \omega^2][4m^2 - \omega^2]}$$

$$\mathcal{Q} = \frac{1}{24\pi}\left(1 - \frac{m_e^2}{\omega^2}\right)^2 \left\{4m^2\Delta^2\left(1 + \frac{2m_e^2}{\omega^2}\right) - (4m^2 + \Delta^2 - \omega^2)\omega^2\left(1 + \frac{m_e^2}{2\omega^2}\right)\right\}$$

Rescale $\omega \to \Delta\omega$

$$\int \frac{d^3p_2}{(2\pi)^3 2E_2} \to \frac{\Delta^3 2m}{(2\pi)^2 2m^2_{\pi^+}} \int_{\frac{m_e}{\Delta}}^{1} d\omega\, \omega \sqrt{[1 - \omega^2][1 - \frac{\Delta}{4m^2}\omega^2]}\,,$$

$$\mathcal{Q} = \frac{\Delta^2 m^2}{6\pi}\left(1 - \frac{m_e^2}{\Delta^2\omega^2}\right)^2 \left\{\left(1 + \frac{2m_e^2}{\Delta^2\omega^2}\right) - \left(1 + \frac{\Delta^2}{4m^2}(1 - \omega^2)\right)\omega^2\left(1 + \frac{m_e^2}{2\Delta^2\omega^2}\right)\right\}$$

$$= \frac{\Delta^2 m^2}{6\pi}\tilde{\mathcal{Q}}$$

Pion beta decay (contd.)

$$\Gamma = \frac{G^2 \cos^2\theta_C \Delta^5}{6\pi^3}\left(\frac{m}{m_{\pi^+}}\right)^3 \int_{\frac{m_e}{\Delta}}^{1} d\omega\, \omega \sqrt{[1-\omega^2][1-\tfrac{\Delta}{4m^2}\omega^2]}\,\tilde{\mathcal{Q}}(\omega)$$

Set $\varepsilon \equiv m_e^2/\Delta^2$, drop orders of Δ/m higher than $\mathcal{O}(\Delta/m)$

$$\Gamma = \frac{G^2 \cos^2\theta_C \Delta^5}{6\pi^3}\left(1-\frac{3\Delta}{2m}\right) K(\varepsilon)$$

$$K(\varepsilon) = \int_{\sqrt{\varepsilon}}^{1} d\omega\, \omega \left(1-\tfrac{\varepsilon}{\omega^2}\right)^2 \left\{ \left(1+\tfrac{2\varepsilon}{\omega^2}\right) - \omega^2 \left(1+\tfrac{\varepsilon}{2\omega^2}\right) \right\} \simeq \tfrac{2}{5}(1-5\varepsilon)$$

Total width (correct to $\mathcal{O}(\frac{\Delta}{m})$ and $\mathcal{O}(\frac{m_e^2}{\Delta^2})$)

$$\boxed{\Gamma = \frac{G^2 \cos^2\theta_C \Delta^5}{6\pi^3}\left(1-\frac{3\Delta}{2m}\right)\left(\frac{1}{5}-\varepsilon\right) = \frac{G^2 \cos^2\theta_C \Delta^5}{30\pi^3}\left(1-\frac{3\Delta}{2m}-5\frac{m_e^2}{\Delta^2}\right)}$$

References

- R. L. Garwin, L. M. Lederman, M. Weinrich, Phys.Rev. 105 (1957) 1415
- S. Fubini, G. Furlan, Physics Physique Fizika 1 (1965) 229