Weak Interactions

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Conservation of the vector current

Electromagnetic current has isosinglet + isotriplet part

$$V^\mu_{\rm em} = \tfrac{2}{3}\bar{u}\gamma^\mu u - \tfrac{1}{3}\bar{d}\gamma^\mu d = \tfrac{1}{2}\left(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d\right) + \tfrac{1}{6}\left(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d\right) = V^\mu_3 + S^\mu$$

S^μ: (Lorentz) vector-isoscalar current

Vector charged current $V^\mu_\pm$ and isotriplet EM current V^μ_3: same isotriplet

Conservation of EM current implies in the exact isospin limit ($m_u = m_d$)

$$\partial_\mu V^\mu_{\rm em} = 0 \Longrightarrow \partial_\mu V^\mu_3 = 0,\ \partial_\mu S^\mu = 0 \Longrightarrow \partial_\mu V^\mu_a = 0$$

Matrix elements current divergence $\partial_\mu J^\mu$ between momentum eigenstates

$$\langle \vec{p}\,'|\partial_\mu J^\mu(x)|\vec{p}\,\rangle = \partial_\mu\langle \vec{p}\,'|J^\mu(x)|\vec{p}\,\rangle = \partial_\mu\langle \vec{p}\,'|e^{iP\cdot x}J^\mu(0)e^{-iP\cdot x}|\vec{p}\,\rangle$$

$$= \partial_\mu e^{i(p'-p)\cdot x}\langle \vec{p}\,'|J^\mu(0)|\vec{p}\,\rangle = \partial_\mu e^{-iq\cdot x}\langle \vec{p}\,'|J^\mu(0)|\vec{p}\,\rangle = -ie^{-iq\cdot x}q_\mu\langle \vec{p}\,'|J^\mu(0)|\vec{p}\,\rangle$$

$$\Longrightarrow \langle \vec{p}\,'|\partial_\mu J^\mu(0)|\vec{p}\,\rangle = -iq_\mu\langle \vec{p}\,'|J^\mu(0)|\vec{p}\,\rangle$$

For conserved current $\partial_\mu J^\mu(x) = 0$, matrix element transverse to $q = p - p'$

$$\boxed{q_\mu\langle \vec{p}\,'|J^\mu(0)|\vec{p}\,\rangle = 0}$$

Isospin selection rules – Wigner-Eckart theorem

Isospin invariance $\Rightarrow$ matrix elements of $\langle A(I_f)|\mathcal{J}^\mu_a|B(I_i)\rangle \neq 0$ (possibly) only if $\Delta I = I_f - I_i = 0, \pm 1$ (composition rules of SU(2) representations)

For A, B in the same multiplet, $I_f = I_i = I$

$$\langle A|V^\mu_a|B\rangle = C^\mu_{(I)}\,(T^{(I)}_a)_{AB}$$

$$\langle A|S^\mu|B\rangle = \tilde{C}^\mu_{(I)}\,\mathit{1}^{(I)}_{AB}$$

$T^{(I)}_a$: generators in representation I, $\vec{I}^{\,2} = I(I+1)$
$\mathit{1}^{(I)}$: (2I+1)-dimensional identity matrix

- Isovector current:
  - V^μ_a isovector $\Rightarrow$ transforms in the $I = 1$ representation, i.e.

    $$D^{(I)*}_{A'A}\langle A'|V^\mu_a|B'\rangle D^{(I)}_{B'B} = D^{(1)}_{a'a}\langle A|V^\mu_{a'}|B\rangle$$

  - LHS transforms in $\overline{(\mathbf{2I+1})} \otimes (\mathbf{2I+1}) = (\mathbf{2I+1}) \otimes (\mathbf{2I+1})$
  - Decompose $(\mathbf{2I+1}) \otimes (\mathbf{2I+1}) = \oplus_{d=1}^{2(2I+1)}\mathbf{d}$ and find $d = 3$ $(I = 1)$
  - d=3 appears only once $\Rightarrow$ single tensorial structure, provided by $T^{(I)}_a$
- Isoscalar current: identity = only scalar (invariant) structure out of $\bar{\mathbf{R}} \otimes \mathbf{R}$

$$D^{(I)*}_{A'A}\langle A'|S^\mu|B'\rangle D^{(I)}_{B'B} = \langle A|S^\mu|B\rangle$$

Matrix elements of charged and electromagnetic currents

Take A and B (I, I_3) eigenstates

$$\langle I\, I_3' | V_+^\mu | I\, I_3 \rangle = C^\mu_{(I)}\, (T_+^{(I)})_{I_3'\, I_3} = C^\mu_{(I)}\, \sqrt{I(I+1) - I_3(I_3+1)}\, \delta_{I_3'\, I_3+1}$$

$$\langle I\, I_3' | V_3^\mu | I\, I_3 \rangle = C^\mu_{(I)}\, (T_3^{(I)})_{I_3'\, I_3} = C^\mu_{(I)}\, I_3\, \delta_{I_3'\, I_3}$$

$$\langle I\, I_3' | S^\mu | I\, I_3 \rangle = \tilde{C}^\mu_{(I)}\, \delta_{I_3'\, I_3}$$

Condon-Shortley convention on isospin eigenstates

$$\begin{aligned} C^\mu_{(I)} &= \langle I\, I_3+1 | V_3^\mu | I\, I_3+1 \rangle - \langle I\, I_3 | V_3^\mu | I\, I_3 \rangle \\ &= \langle I\, I_3+1 | V_{\mathrm{em}}^\mu + S^\mu | I\, I_3+1 \rangle - \langle I\, I_3 | V_{\mathrm{em}}^\mu + S^\mu | I\, I_3 \rangle \\ &= \langle I\, I_3+1 | V_{\mathrm{em}}^\mu | I\, I_3+1 \rangle - \langle I\, I_3 | V_{\mathrm{em}}^\mu | I\, I_3 \rangle \end{aligned}$$

Relation between matrix elements of V_+^μ and V_{em}^μ

$$\begin{aligned} \langle I\, I_3+1 | V_+^\mu | I\, I_3 \rangle &= \sqrt{I(I+1) - I_3(I_3+1)}\, \Big[\langle I\, I_3+1 | V_3^\mu | I\, I_3+1 \rangle - \langle I\, I_3 | V_3^\mu | I\, I_3 \rangle \Big] \\ &= \sqrt{I(I+1) - I_3(I_3+1)}\, \Big[\langle I\, I_3+1 | V_{\mathrm{em}}^\mu | I\, I_3+1 \rangle - \langle I\, I_3 | V_{\mathrm{em}}^\mu | I\, I_3 \rangle \Big] \end{aligned}$$

Weak charge

Noether charges associated to vector-isovector current

$$T_a = \int d^3x\, V_a^0(x)$$

x^0 arbitrary due to conservation, take $x^0 = 0$

$A(I', I_3', \vec{p}\,')$ and $B(I, I_3, \vec{p})$ isospin and momentum eigenstates

$$\begin{aligned}\langle A|T_a|B\rangle &= \int d^3x\, \langle A|V_a^0(x)|B\rangle = \int d^3x e^{i(\vec{p}-\vec{p}\,')\cdot x}\langle A|V_a^0(0)|B\rangle \\ &= (2\pi)^3\delta^{(3)}(\vec{q}\,)\langle A|V_a^0(0)|B\rangle\end{aligned}$$

$\vec{q} = \vec{p} - \vec{p}\,'$

$$\begin{aligned}\int d^3x\, \langle A|V_+^0(x)|B\rangle &= \langle A|T_+|B\rangle = (2\pi)^3\delta^{(3)}(\vec{q}\,)\langle A|V_+^0(0)|B\rangle \\ &= \delta_{I'I}\delta_{I_3'\,I_3+1}\sqrt{I(I+1)-I_3(I_3+1)}(2\pi)^3 2p^0\delta^{(3)}(\vec{q}\,)\end{aligned}$$

relativistic normalisation of states + Condon-Shortley

Weak charge (contd.)

$$\begin{aligned}\langle A|V_+^0(0)|B\rangle|_{\vec{q}=0} &= 2p^0\delta_{I'I}\delta_{I'_3 I_3+1}\sqrt{I(I+1)-I_3(I_3+1)} \\ &= 2p^0\delta_{I'I}\delta_{I'_3\, I_3+1}Q_W(I,I_3)\end{aligned}$$

$$\begin{aligned}\langle A|V_-^0(0)|B\rangle|_{\vec{q}=0} &= 2p^0\delta_{I'I}\delta_{I'_3 I_3-1}\sqrt{I(I+1)-I_3(I_3-1)} \\ &= 2p^0\delta_{I'I}\delta_{I'_3\, I_3-1}Q_W(I,I'_3)\end{aligned}$$

For transitions between states in the same isomultiplet, amplitude in the static limit $\vec{q}=0$ determined entirely by *weak charge* $Q_W(I,I_3)$

Since m is the same, $\vec{q}=0 \Rightarrow q^0=0$

Same argument for EM current

$$\int d^3x\,\langle A|V^0_{\rm em}(x)|B\rangle = \langle A|Q_{\rm em}|B\rangle = \delta_{Q'Q}Q(2\pi)^3 2p^0\delta^{(3)}(\vec{q}\,)$$

$$\langle A|V^0_{\rm em}(0)|B\rangle|_{\vec{q}=0} = 2p^0\delta_{Q'Q}Q$$

From $V^0_{\rm em}=V^0_3+S^0$ follows $Q=I_3+\frac{1}{2}(\mathcal{B}+S)=I_3+\frac{Y}{2}$ (Gell-Mann–Nishijima relation)

Axial current and chiral symmetry

For massless u, d quarks

- extended symmetry $\mathrm{SU}(2)_I \to \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$

  $\mathrm{SU}(2)_{L,R}$ acting independently on L, R quark chiralities

- chiral currents conserved $\Rightarrow$ both vector and axial current conserved

Quark masses small $\neq 0 \Rightarrow$ partial axial current conservation $\partial_\mu A^\mu_a \propto m_{u,d}$
Even for $m_{u,d} = 0$, V and A part of chiral symmetry realised differently: vacuum invariant under vector isospin rotation, not under an axial one $\Rightarrow$

- chiral symmetry spontaneously broken
- massless bosons generated through Goldstone mechanism: pions
- small pion mass due to explicit but soft breaking of chiral symmetry due to $m_{u,d} \neq 0$, would vanish for $m_{u,d} = 0$

Partial conservation of axial current (PCAC) correctly guessed before discovery of quarks and QCD