Weak Interactions

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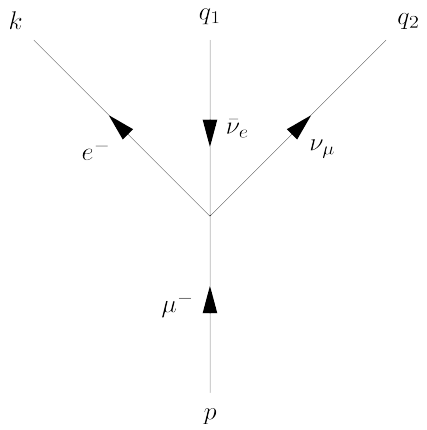
Muon decay

- main muon decay mode:

$$\mu^- \to e^- \, \bar{\nu}_e \, \nu_\mu$$

- relevant current product: $j^{(\mu)\dagger} \cdot j^{(e)}$



$$\mathcal{M}_{\rm fi} = -\frac{G}{\sqrt{2}} \langle e^-(k, s_e)\, \bar{\nu}_e(q_1)\, \nu_\mu(q_2) | \Big(\bar{\nu}_\mu(0) \mathcal{O}_L^\alpha \, \mu(0) \Big) \Big(\bar{e}(0) \mathcal{O}_{L\alpha} \, \nu_e(0) \Big) | \mu^-(p, s_\mu) \rangle$$

Use momentum space Feynman rules $\Longrightarrow$

ampl.: $$\mathcal{M}_{\rm fi} = -\frac{G}{\sqrt{2}} [\bar{u}^{(\nu_\mu)}(q_2) \mathcal{O}_L^\alpha u^{(\mu)}(p, s_\mu)] [\bar{u}^{(e)}(k, s_e) \mathcal{O}_{L\alpha} v^{(\nu_e)}(q_1)]$$

prob.: $$|\mathcal{M}_{\rm fi}|^2 = \frac{G^2}{2} [\bar{u}^{(\nu_\mu)} \mathcal{O}_L^\alpha \overbrace{u^{(\mu)}][\bar{u}^{(\mu)}}^{\Pi^{(\mu)}_{\alpha\beta}} \mathcal{O}_L^\beta u^{(\nu_\mu)}] [\bar{u}^{(e)} \mathcal{O}_{L\alpha} v^{(\nu_e)}] [\bar{v}^{(\nu_e)} \mathcal{O}_{L\beta} u^{(e)}]$$

$$= \frac{G^2}{2} \mathrm{tr}\, [u^{(\nu_\mu)} \bar{u}^{(\nu_\mu)} \mathcal{O}_L^\alpha u^{(\mu)} \bar{u}^{(\mu)} \mathcal{O}_L^\beta] \mathrm{tr}\, [v^{(\nu_e)} \bar{v}^{(\nu_e)} \mathcal{O}_{L\beta} u^{(e)} \bar{u}^{(e)} \mathcal{O}_{L\alpha}]$$

Unpolarised muons, electron spin not measured

Sum over electron spin and average over muon spin

$$\sum_s u_s(\vec{p}\,)\bar{u}_s(\vec{p}\,) = \not{p} + m \qquad \sum_s v_s(\vec{p}\,)\bar{v}_s(\vec{p}\,) = \not{p} - m$$

Massless, definite-handedness particles

$$u_\pm(\vec{p}\,)\bar{u}_\pm(\vec{p}\,) = \not{p}\frac{1\mp\gamma^5}{2} \qquad v_\pm(\vec{p}\,)\bar{v}_\pm(\vec{p}\,) = \not{p}\frac{1\pm\gamma^5}{2}$$

First factor (second factor analogous):

$$\sum_{s_\mu} \mathrm{tr}\,[u^{(\nu_\mu)}\bar{u}^{(\nu_\mu)}\mathcal{O}_L^\alpha u^{(\mu)}\bar{u}^{(\mu)}\mathcal{O}_L^\beta] = \mathrm{tr}\,\not{q}_2\frac{1+\gamma^5}{2}\mathcal{O}_L^\alpha(\not{p}+m_\mu)\mathcal{O}_L^\beta$$

$$\frac{1+\gamma^5}{2}\mathcal{O}_L^\alpha = \frac{1+\gamma^5}{2}\gamma^\alpha\frac{1-\gamma^5}{2} = \gamma^\alpha(\frac{1-\gamma^5}{2})^2 = \mathcal{O}_L^\alpha$$

$$\langle\langle|\mathcal{M}_{\mathrm{fi}}|^2\rangle\rangle \equiv \sum_{s_\mu,s_e}|\mathcal{M}_{\mathrm{fi}}|^2 = \frac{G^2}{2}\,\mathrm{tr}\,[\not{q}_2\mathcal{O}_L^\alpha(\not{p}+m_\mu)\mathcal{O}_L^\beta]\mathrm{tr}\,[\not{q}_1\mathcal{O}_{L\beta}(\not{k}+m_e)\mathcal{O}_{L\alpha}]$$

Unpolarised muons, electron spin not measured (contd.)

Terms proportional fermion masses drop ($\propto$ odd number of γs)

$$\begin{aligned}\langle\langle|\mathcal{M}_{\rm fi}|^2\rangle\rangle &= \frac{G^2}{2}\,\mathrm{tr}\,[\not{q}_2\mathcal{O}_L^\alpha\not{p}\mathcal{O}_L^\beta]\mathrm{tr}\,[\not{q}_1\mathcal{O}_{L\beta}\not{k}\mathcal{O}_{L\alpha}]\\ &= \frac{G^2}{2}\,\mathrm{tr}\,[\not{q}_2\gamma^\alpha(1-\gamma^5)\not{p}\gamma^\beta(1-\gamma^5)]\mathrm{tr}\,[\not{q}_1\gamma_\beta(1-\gamma^5)\not{k}\gamma_\alpha(1-\gamma^5)]\\ &= \frac{G^2}{2}\,4\,\mathrm{tr}\,[\not{q}_2\gamma^\alpha\not{p}\gamma^\beta(1-\gamma^5)]\mathrm{tr}\,[\not{q}_1\gamma_\beta\not{k}\gamma_\alpha(1-\gamma^5)]\end{aligned}$$

Trace identities:

$$\mathrm{tr}\,\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta = 4(\eta^{\mu\alpha}\eta^{\nu\beta}-\eta^{\mu\nu}\eta^{\alpha\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}) \equiv 4S^{\mu\alpha\nu\beta}$$

$$\mathrm{tr}\,\gamma^5\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta = -4i\varepsilon^{\mu\alpha\nu\beta}$$

$\varepsilon^{\mu\alpha\nu\beta}$ totally antisymmetric tensor, $\varepsilon^{0123}=1$

Crossed terms drop (symmetries of tensors)

$$\begin{aligned}\langle\langle|\mathcal{M}_{\rm fi}|^2\rangle\rangle &= \frac{G^2}{2}\,4^3(S^{\mu\alpha\nu\beta}-i\varepsilon^{\mu\alpha\nu\beta})(S_{\rho\alpha\sigma\beta}-i\varepsilon_{\rho\beta\sigma\alpha})q_{2\mu}p_\nu q_1^\rho k^\sigma\\ &= 32G^2(S^{\mu\alpha\nu\beta}S_{\rho\alpha\sigma\beta}+\varepsilon^{\mu\alpha\nu\beta}\varepsilon_{\rho\alpha\sigma\beta})q_{2\mu}p_\nu q_1^\rho k^\sigma\end{aligned}$$

More identities:

$$S^{\mu\alpha\nu\beta}S_{\rho\alpha\sigma\beta} = 2\left(\delta^\mu{}_\sigma\delta^\nu{}_\rho+\delta^\mu{}_\rho\delta^\nu{}_\sigma\right) \qquad \varepsilon^{\mu\alpha\nu\beta}\varepsilon_{\rho\alpha\sigma\beta} = 2\left(\delta^\mu{}_\sigma\delta^\nu{}_\rho-\delta^\mu{}_\rho\delta^\nu{}_\sigma\right)$$

Unpolarised muons, electron spin not measured (contd.)

Terms proportional fermion masses drop ($\propto$ odd number of γs)

$$\begin{aligned}\langle\langle|\mathcal{M}_{\rm fi}|^2\rangle\rangle &= \frac{G^2}{2}\,\mathrm{tr}\,[\not{q}_2\mathcal{O}_L^\alpha\not{p}\mathcal{O}_L^\beta]\mathrm{tr}\,[\not{q}_1\mathcal{O}_{L\beta}\not{k}\mathcal{O}_{L\alpha}]\\ &= \frac{G^2}{2}\,\mathrm{tr}\,[\not{q}_2\gamma^\alpha(1-\gamma^5)\not{p}\gamma^\beta(1-\gamma^5)]\mathrm{tr}\,[\not{q}_1\gamma_\beta(1-\gamma^5)\not{k}\gamma_\alpha(1-\gamma^5)]\\ &= \frac{G^2}{2}\,4\,\mathrm{tr}\,[\not{q}_2\gamma^\alpha\not{p}\gamma^\beta(1-\gamma^5)]\mathrm{tr}\,[\not{q}_1\gamma_\beta\not{k}\gamma_\alpha(1-\gamma^5)]\end{aligned}$$

Trace identities:

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Unpolarised muons, electron spin not measured (contd.)

Amplitude square *summed* over spins:

$$\langle\langle|\mathcal{M}_{\rm fi}|^2\rangle\rangle = 128G^2\delta^\mu{}_\sigma\delta^\nu{}_\rho q_{2\mu}p_\nu q_1^\rho k^\sigma = 128G^2(p\cdot q_1)(k\cdot q_2)$$

Averaging muon spin $\rightarrow$ factor $1/(2s+1) = 1/2$
Differential decay width in the muon rest frame

$$d\Gamma = \frac{1}{2m_\mu}\frac{\langle\langle|\mathcal{M}_{\rm fi}|^2\rangle\rangle}{2}d\Phi^{(3)} = \frac{32G^2}{m_\mu}(p\cdot q_1)(k\cdot q_2)d\Phi^{(3)}$$

Phase-space element:

$$d\Phi^{(3)} = (2\pi)^4\delta^{(4)}(p-k-q_1-q_2)\frac{d^3k}{(2\pi)^3 2E}\frac{d^3q_1}{(2\pi)^3 2\omega_1}\frac{d^3q_2}{(2\pi)^3 2\omega_2}$$

Electron energy $E = k^0 = \sqrt{\vec{k}^2 + m_e^2}$
Neutrino energies $\omega_i = q_i^0 = |\vec{q}_i|$

Neutrinos not detected, only electron $\rightarrow$ integrate over neutrino momenta

Unpolarised muons, electron spin not measured (contd.)

$$d\Gamma = \frac{G^2}{8m_\mu \pi^5}\frac{d^3k}{E}p^\alpha k^\beta \int \frac{d^3q_1}{\omega_1}\int \frac{d^3q_2}{\omega_2}\,\delta^{(4)}(p-k-q_1-q_2)q_{1\alpha}q_{2\beta}$$
$$= \frac{G^2}{8m_\mu \pi^5}\frac{d^3k}{E}p^\alpha k^\beta I_{\alpha\beta}(p-k)$$

$I_{\alpha\beta}(q)$: symmetric tensor, mass dimension 2, built out of q

$$I_{\alpha\beta}(q) = \int \frac{d^3q_1}{\omega_1}\int \frac{d^3q_2}{\omega_2}\,\delta^{(4)}(q-q_1-q_2)q_{1\alpha}q_{2\beta} = Aq^2\eta_{\alpha\beta} + Bq_\alpha q_\beta$$
$$\eta^{\alpha\beta}I_{\alpha\beta}(q) = q^2(4A+B) = \int \frac{d^3q_1}{\omega_1}\int \frac{d^3q_2}{\omega_2}\,\delta^{(4)}(q-q_1-q_2)\,q_1\cdot q_2 = \frac{q^2}{2}C$$
$$q^\alpha q^\beta I_{\alpha\beta}(q) = (q^2)^2(A+B) = \int \frac{d^3q_1}{\omega_1}\int \frac{d^3q_2}{\omega_2}\,\delta^{(4)}(q-q_1-q_2)\,(q_1\cdot q_2)^2 = \frac{(q^2)^2}{4}C$$
$$C = \int \frac{d^3q_1}{\omega_1}\int \frac{d^3q_2}{\omega_2}\,\delta^{(4)}(q-q_1-q_2) \qquad A = \frac{C}{12} \quad B = \frac{C}{6}$$

C L-invariant, $q = q_1 + q_2$ t-like $\rightarrow$ use frame where $\vec{q} = 0$ (CM of ν_μ-$\bar{\nu}_e$ system)

$$C = \int \frac{d^3q_1}{\omega_1}\int \frac{d^3q_2}{\omega_2}\,\delta(q^0-\omega_1-\omega_2)\delta^{(3)}(\vec{q}_1+\vec{q}_2) = \frac{1}{2}\int d\Omega \int \frac{d\omega_1\,\omega_1^2}{\omega_1^2}\,\delta(\tfrac{1}{2}q^0-\omega_1) = 2\pi$$

$$I_{\alpha\beta}(q) = \frac{\pi}{6}\left(q^2\eta_{\alpha\beta} + 2q_\alpha q_\beta\right)$$

Unpolarised muons, electron spin not measured (contd.)

$$
\begin{aligned}
d\Gamma &= \frac{G^2}{8m_\mu\pi^5}\frac{d^3k}{E}p^\alpha k^\beta\frac{\pi}{6}\left((p-k)^2\eta_{\alpha\beta}+2(p-k)_\alpha(p-k)_\beta\right)\\
&= \frac{G^2}{48m_\mu\pi^4}\frac{d^3k}{E}\left((p-k)^2 p\cdot k+2p\cdot(p-k)\,k\cdot(p-k)\right)\\
&= \frac{G^2}{48m_\mu\pi^4}\frac{d^3k}{E}\left((p^2+k^2-2p\cdot k)\,p\cdot k-2(p^2-p\cdot k)(k^2-p\cdot k)\right)\\
&= \frac{G^2}{48m_\mu\pi^4}\frac{d^3k}{E}\left(3(p^2+k^2)p\cdot k-4(p\cdot k)^2-2p^2k^2\right)\\
&= \frac{G^2}{48m_\mu\pi^4}\frac{d^3k}{E}\left(3(m_\mu^2+m_e^2)m_\mu E-4(m_\mu E)^2-2m_\mu^2m_e^2\right)
\end{aligned}
$$

Approximations:

- $m_e \ll m_\mu \Rightarrow m_e^2/(m_\mu E) < m_e/m_\mu \ll 1$
- electron ultrarelativistic $m_e/E \ll 1$

$\Longrightarrow$ neglect m_e^2 in the first term, last term altogether

Unpolarised muons, electron spin not measured (contd.)

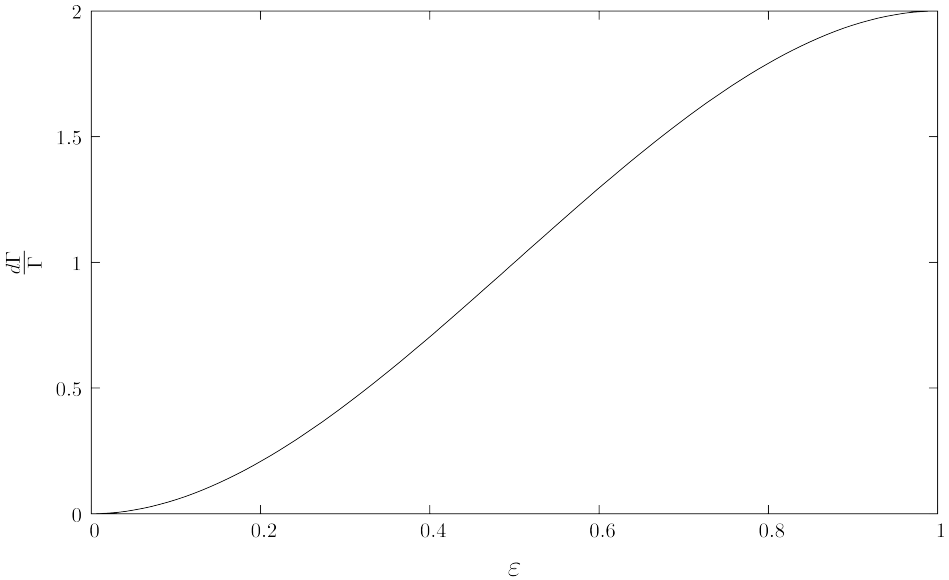
$$d\Gamma = \frac{G^2}{48 m_\mu \pi^4} \frac{d^3k}{E} (p \cdot k) \left(3p^2 - 4(p \cdot k)\right)$$
$$= \frac{G^2}{48 m_\mu \pi^4} \frac{d^3k}{E} m_\mu^2 E \left(3m_\mu - 4E\right)$$

Integrate over e direction: $d^3k \to 4\pi k^2 dk = 4\pi k E dE$

$$d\Gamma = \frac{G^2 m_\mu}{12\pi^3} \left(3m_\mu - 4E\right) E \sqrt{E^2 - m_e^2}\, dE$$
$$= \frac{G^2 m_\mu^2}{12\pi^3} \left(3 - \frac{2E}{m_\mu/2}\right) \left(\frac{m_\mu}{2}\right)^3 \frac{E}{m_\mu/2} \sqrt{\left(\frac{E}{m_\mu/2}\right)^2 - \left(\frac{m_e}{m_\mu/2}\right)^2} \frac{dE}{m_\mu/2}$$
$$= \frac{G^2 m_\mu^5}{96\pi^3} \left(3 - 2\varepsilon\right) \varepsilon^2 d\varepsilon$$

$\varepsilon = \frac{E}{E_{\max}} \in [0, 1]$, $E_{\max} = \frac{m_\mu}{2}$, term $\mathcal{O}((\frac{m_e}{m_\mu})^2)$ neglected in the square root

Unpolarised muons, electron spin not measured (contd.)



Total width:

$$\boxed{\Gamma = \frac{G^2 m_\mu^5}{96\pi^3} \int_0^1 d\varepsilon \, (3 - 2\varepsilon) \, \varepsilon^2 = \frac{G^2 m_\mu^5}{192\pi^3}}$$

$\Rightarrow$ extract G from muon lifetime (after including EM radiative corrections)

Energy distribution:

$$\frac{1}{\Gamma} \frac{d\Gamma}{d\varepsilon} = (6 - 4\varepsilon) \, \varepsilon^2$$

$\gamma_e > 20$ in more than 98% of cases $\rightarrow$ ultrarel. assumption justified

Polarised muons

Fermion of mass m, definite positive spin in direction $\vec{\eta}$ in its rest frame

$$u(p,s)\bar{u}(p,s) = (\not{p}+m)\frac{1+\gamma^5\not{s}}{2}$$

$$s|_{\text{rest frame}} = (0,\vec{\eta}) \qquad s = \left(\frac{\vec{\eta}\cdot\vec{p}}{m}, \vec{\eta} + \frac{\vec{p}(\vec{\eta}\cdot\vec{p})}{m(p^0+m)}\right)$$

Spacelike $s^2 = -\vec{\eta}^2 = -1$ and $s\cdot p = 0$

Square amplitude for fixed spins:

$$|\mathcal{M}_{\text{fi}}|^2 = \frac{G^2}{2}\frac{1}{2^2}\text{tr}\,[\not{q}_2\mathcal{O}_L^\alpha(\not{p}+m_\mu)(1+\gamma^5\not{s}_\mu)\mathcal{O}_L^\beta]\text{tr}\,[\not{q}_1\mathcal{O}_{L\beta}(\not{k}+m_e)(1+\gamma^5\not{s}_e)\mathcal{O}_{L\alpha}]$$

$$= \frac{G^2}{2}\text{tr}[\not{q}_2\gamma^\alpha(\not{p}+m_\mu)(1+\gamma^5\not{s}_\mu)\gamma^\beta(1-\gamma^5)]\text{tr}[\not{q}_1\gamma_\beta(\not{k}+m_e)(1+\gamma^5\not{s}_e)\gamma_\alpha(1-\gamma^5)]$$

$$s_\mu = \left(\frac{\vec{\eta}\cdot\vec{p}}{m_\mu}, \vec{\eta} + \frac{\vec{p}(\vec{\eta}\cdot\vec{p})}{m_\mu(p^0+m_\mu)}\right) \qquad s_e = \left(\frac{\vec{\zeta}\cdot\vec{k}}{m_e}, \vec{\zeta} + \frac{\vec{k}(\vec{\zeta}\cdot\vec{k})}{m_e(k^0+m_e)}\right)$$

- $\langle\langle|\mathcal{M}_{\text{fi}}|^2\rangle\rangle = 4\lim_{s_{\mu,e}\to 0}|\mathcal{M}_{\text{fi}}|^2$ (= summing over $s_{\mu,e}$)
- partially polarised muons: average over $\vec{\eta}$ with some prob. distr., s linear in $\vec{\eta} \Rightarrow s(\vec{\eta}) \to s(\langle\vec{\eta}\rangle) = \bar{s}$, with $\bar{s}\cdot p = 0$, $-1 \le \bar{s}^2 \le 0$

Polarised muons (contd.)

First factor (same applies to the second)

$$\mathrm{tr}\,[\not{q}_2\gamma^\alpha \underbrace{(\not{p}+m_\mu)(1+\gamma^5\not{s}_\mu)}_{\text{only odd n. of }\gamma\text{ contribute}} \gamma^\beta(1-\gamma^5)] \to \mathrm{tr}\,[\not{q}_2\gamma^\alpha(\not{p}+\gamma^5 m_\mu\not{s}_\mu)\gamma^\beta(1-\gamma^5)]$$

$\Rightarrow$ replace $(\not{p}+m_\mu)(1+\gamma^5\not{s}_\mu)\to\not{p}+\gamma^5 m_\mu\not{s}_\mu$

$$\mathrm{tr}[\not{q}_2\gamma^\alpha\gamma^5\not{s}_\mu\gamma^\beta(1-\gamma^5)] = \mathrm{tr}[\not{q}_2\gamma^\alpha\not{s}_\mu\gamma^\beta\gamma^5(1-\gamma^5)] = -\mathrm{tr}[\not{q}_2\gamma^\alpha\not{s}_\mu\gamma^\beta(1-\gamma^5)]$$

$\Rightarrow$ replace $p\to\frac{1}{2}(p-m_\mu s_\mu)$ and $k\to\frac{1}{2}(k-m_e s_e)$ in $\langle\langle|\mathcal{M}_{\mathrm{fi}}|^2\rangle\rangle$

$$\boxed{|\mathcal{M}_{\mathrm{fi}}|^2 = 32G^2[(p-m_\mu s_\mu)\cdot q_1][(k-m_e s_e)\cdot q_2]}$$

$$\boxed{d\Gamma = \frac{1}{2m_\mu}|\mathcal{M}_{\mathrm{fi}}|^2 d\Phi^{(3)} = \frac{16G^2}{m_\mu}[(p-m_\mu s_\mu)\cdot q_1][(k-m_e s_e)\cdot q_2]d\Phi^{(3)}}$$

Polarised muons (contd.)

Integrating over undetected neutrinos

$$d\Gamma = \frac{G^2}{16 m_\mu \pi^5} \frac{d^3k}{E} (p - m_\mu s_\mu)^\alpha (k - m_e s_e)^\beta I_{\alpha\beta}(p-k)$$
$$= \frac{G^2}{96 m_\mu \pi^4} \frac{d^3k}{E} \big[(p-k)^2 (p - m_\mu s_\mu) \cdot (k - m_e s_e) + 2(p-k)\cdot(p - m_\mu s_\mu)\,(p-k)\cdot(k - m_e s_e) \big]$$

Set $\tilde{p} = p - m_\mu s_\mu$, $\tilde{k} = k - m_e s_e$

$$[\ldots] = (p-k)^2 \tilde{p}\cdot\tilde{k} + 2(p-k)\cdot\tilde{p}\,(p-k)\cdot\tilde{k}$$
$$= (p-k)^2 (p\cdot\tilde{k} - m_\mu s_\mu\cdot\tilde{k}) + 2[(p-k)\cdot p + m_\mu s_\mu \cdot k](p\cdot\tilde{k} - k^2)$$
$$= (p\cdot\tilde{k})[(p-k)^2 + 2(p-k)\cdot p] - m_\mu[(p-k)^2 s_\mu\cdot\tilde{k} - 2(p\cdot\tilde{k} - k^2) s_\mu\cdot k] - 2k^2(p-k)\cdot p$$
$$= (p\cdot\tilde{k})(3p^2 - 4p\cdot k + k^2) - m_\mu[(p^2 - 2p\cdot k + k^2) s_\mu\cdot\tilde{k} - 2p\cdot\tilde{k} s_\mu\cdot k + 2k^2 s_\mu\cdot k] - 2k^2(p^2 - p\cdot k)$$

$m_\mu \gg m_e \Rightarrow$ neglect $k^2 = m_e^2$ vs. $p^2 = m_\mu^2$, $p\cdot k = E m_\mu$, $p\cdot\tilde{k} = m_\mu(E - \vec{\zeta}\cdot\vec{k})$

$$[\ldots] = (p\cdot\tilde{k})(3p^2 - 4p\cdot k) - m_\mu[(p^2 - 2p\cdot k) s_\mu\cdot\tilde{k} - 2p\cdot\tilde{k} s_\mu\cdot k]$$

Polarised muons (contd.)

1. If we *sum* over the electron spin states (unobserved e polarisation)

$$[\ldots] \xrightarrow[\text{sum over } s_e]{} 2\left[(p\cdot k)(3p^2-4p\cdot k)-m_\mu s_\mu\cdot k(p^2-4p\cdot k)\right] .$$

$$d\Gamma = \frac{G^2}{48 m_\mu \pi^4} d^3k \left[(3m_\mu^2-4m_\mu E)+m_\mu \vec{\eta}\cdot\vec{n}\frac{|\vec{k}|}{E}(m_\mu^2-4m_\mu E)\right]$$

Going over to ε and neglecting powers of m_e/m_μ we find

$$\begin{aligned} d\Gamma &= \frac{G^2 m_\mu^5}{96\pi^3}\left(3-2\varepsilon+\vec{\eta}\cdot\vec{n}\,(1-2\varepsilon)\right) d\varepsilon\varepsilon^2 \frac{d\Omega}{4\pi} \\ &= \frac{G^2 m_\mu^5}{192\pi^3}\left(3-2\varepsilon+\cos\theta\,(1-2\varepsilon)\right) d\varepsilon\varepsilon^2 d\cos\theta \\ &= \Gamma\left(3-2\varepsilon+(1-2\varepsilon)\cos\theta\right)\varepsilon^2 d\varepsilon d\cos\theta\,. \end{aligned}$$

Integrating over energies $\Rightarrow$ angular distribution of the electron

$$\boxed{\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta} = \int_0^1 d\varepsilon\,\varepsilon^2\left(3-2\varepsilon+(1-2\varepsilon)\cos\theta\right) = \tfrac{1}{2}\left(1-\tfrac{1}{3}\cos\theta\right)}$$

Experimentally observed [Garwin *et al.* (1957)]

Polarised muons (contd.)

1. If we do not sum over but instead observe the electron spin

$$m_e s_e = \left(\vec{\zeta}\cdot\vec{k}, m_e\vec{\zeta} + \vec{k}\frac{\vec{\zeta}\cdot\vec{k}}{E+m_e}\right) \simeq E(\vec{\zeta}\cdot\vec{n})\,(1,\vec{n}) \simeq (\vec{\zeta}\cdot\vec{n})k \Rightarrow \tilde{k} \simeq (1-\vec{\zeta}\cdot\vec{n})k$$

ultrarelativistic electron $E \gg m_e$

$$[\ldots] \xrightarrow[E\gg m_e]{} (1-\vec{\zeta}\cdot\vec{n})\left\{(p\cdot k)(3p^2-4p\cdot k) - m_\mu s_\mu\cdot k(p^2-4p\cdot k)\right\}$$

$\Rightarrow$ just multiply the spin-unobserved case by $\frac{1}{2}(1-\vec{\zeta}\cdot\vec{n})$

$$\boxed{d\Gamma = \Gamma(1-\vec{\zeta}\cdot\vec{n})\,(3-2\varepsilon+\vec{\eta}\cdot\vec{n}\,(1-2\varepsilon))\,d\varepsilon\varepsilon^2\frac{d\Omega}{4\pi}}$$

Antimuon decay

Main mode $\mu^+ \to e^+ \nu_e \bar{\nu}_\mu$

$$\mathcal{M}_{\rm fi} = \frac{G}{\sqrt{2}}[\bar{v}^{(\mu)}(p, s_\mu)\mathcal{O}^\alpha v^{(\nu_\mu)}(q_2)][\bar{u}^{(\nu_e)}(q_1)\mathcal{O}_{L\alpha} v^{(e)}(k, s_e)]$$

Decay widths obtain from muon widths via

$$(\not{p} + m_\mu)\frac{1+\gamma^5 \not{s}_\mu}{2} \to (\not{p} - m_\mu)\frac{1+\gamma^5 \not{s}_\mu}{2} \qquad (\not{k} + m_e)\frac{1+\gamma^5 \not{s}_e}{2} \to (\not{k} - m_e)\frac{1+\gamma^5 \not{s}_e}{2}$$

$$\Longrightarrow p - m_\mu s_\mu \to p + m_\mu s_\mu \qquad \Longrightarrow k - m_e s_e \to k + m_e s_e$$

$$\Longrightarrow \eta \to -\eta \qquad \Longrightarrow \zeta \to -\zeta$$

Phase-space element unchanged

$$d\Gamma_{\mu^-} = \Gamma(1 - \vec{\zeta} \cdot \vec{n})\,(3 - 2\varepsilon + \vec{\eta} \cdot \vec{n}\,(1 - 2\varepsilon))\, d\varepsilon \varepsilon^2 \frac{d\Omega}{4\pi}$$

$$d\Gamma_{\mu^+} = \Gamma(1 + \vec{\zeta} \cdot \vec{n})\,(3 - 2\varepsilon - \vec{\eta} \cdot \vec{n}\,(1 - 2\varepsilon))\, d\varepsilon \varepsilon^2 \frac{d\Omega}{4\pi}$$

Qualitative discussion

1. Muon mass dependence $\Gamma \propto m_\mu^5$ for dimensional reasons

$$\Gamma = m_\mu F(Gm_\mu^2, m_e/m_\mu)$$

   - F dimensionless
   - $F(Gm_\mu^2, m_e/m_\mu \ll 1) \simeq F(Gm_\mu^2, 0)$
   - $F = \mathcal{O}(G^2)$

$$\Longrightarrow \Gamma \simeq m_\mu F(Gm_\mu^2, 0) \simeq m_\mu (Gm_\mu^2)^2 F_0$$

2. $d\Gamma_{\mu^\pm}$ break P and C but not CP

$$d\Gamma_{\mu^\mp}(\vec{n}, \vec{\eta}, \vec{\zeta}) \underset{P}{\longrightarrow} d\Gamma_{\mu^\mp}(-\vec{n}, \vec{\eta}, \vec{\zeta}) = d\Gamma_{\mu^\pm}(\vec{n}, \vec{\eta}, \vec{\zeta})$$

$$d\Gamma_{\mu^\mp}(\vec{n}, \vec{\eta}, \vec{\zeta}) \underset{C}{\longrightarrow} d\Gamma_{\mu^\pm}(\vec{n}, \vec{\eta}, \vec{\zeta}) = d\Gamma_{\mu^\mp}(-\vec{n}, \vec{\eta}, \vec{\zeta})$$

$$d\Gamma_{\mu^\mp}(\vec{n}, \vec{\eta}, \vec{\zeta}) \underset{CP}{\longrightarrow} d\Gamma_{\mu^\mp}(\vec{n}, \vec{\eta}, \vec{\zeta})$$

$\Rightarrow$ different angular distributions of $e^\pm$ (final spins summed over)

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta}\Big|_{e^-} = \frac{1}{2}\left(1 - \frac{1}{3}\cos\theta\right), \qquad \frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta}\Big|_{e^+} = \frac{1}{2}\left(1 + \frac{1}{3}\cos\theta\right)$$

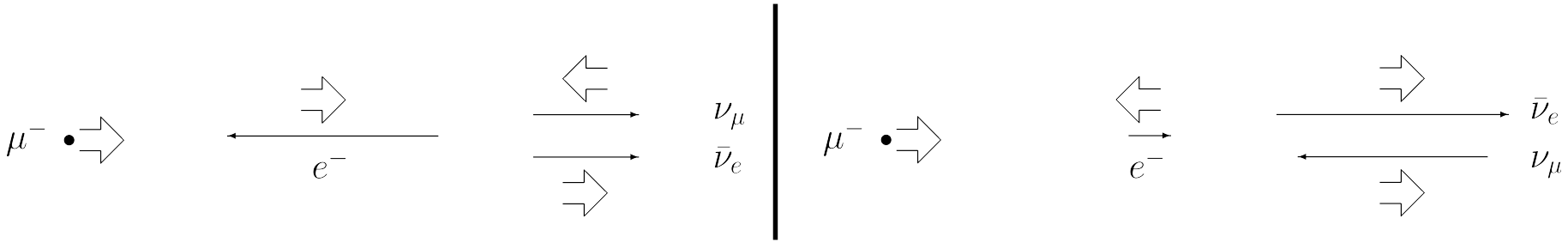
Qualitative discussion (contd.)

1. Breaking of parity can be inferred from the fact that unpolarised muon decay produces polarised electrons ($\vec{\zeta} \parallel \hat{k}$ as demanded by rotation invariance)

$$\left.\frac{d\Gamma}{\Gamma}\right|_{\substack{\mu \text{ unpol.} \\ s_e \text{ obs.}}} = \frac{1 - \vec{\zeta} \cdot \vec{n}}{2} (3 - 2\varepsilon)\, d\varepsilon \varepsilon^2 \frac{d\Omega}{4\pi}$$

2. Factor $(1 - \vec{\zeta} \cdot \vec{n})$ suppresses high-energy electrons with $\vec{\zeta} \parallel \vec{k}$ (same direction and orientation): consequence of chiral coupling of charged currents (suppresses massless particles with positive helicity) and of high–energy particle $\approx$ massless
3. e^- angular asymmetry consequence of angular momentum conservation and of fixed helicity of $\nu/\bar{\nu}$

Qualitative discussion (contd.)



- $\varepsilon \simeq 1$: $(p-k)^2 \simeq m_\mu^2 - 2m_\mu E \simeq 0$, $q_1 \cdot q_2 = \omega_1\omega_2(1-\cos\theta_\nu) \simeq 0$, phase space suppresses low $\omega_{1,2}$ $(\frac{dq_i}{\omega_i} = d\Omega_i d\omega_i \omega_i) \Rightarrow \cos\theta_\nu \simeq 1$
  - neutrino and the antineutrino momenta are parallel
  - their helicities add up to zero in the direction of their motion
  - electron momentum opposite to $\nu/\bar{\nu}$, negative helicity (high-energy limit + chirality of coupling)

  $\Rightarrow$ e emitted in the direction opposite to the muon spin
- low energy limit $\vec{k} \simeq 0$:
  - $\nu/\bar{\nu}$ travel in opposite directions
  - spins add up to 1 along momentum of $\bar{\nu}$
  - electron spin opposite to muon spin, parallel to ν momentum + negative helicity favoured by chiral coupling

  $\Rightarrow$ e emitted in the direction of muon spin

Strangeness-conserving semileptonic processes

Semileptonic processes with $\Delta S = 0$ governed by

$$\delta\mathcal{L} = -\frac{G}{\sqrt{2}}\cos\theta_C(\bar{u}\mathcal{O}_L^\alpha d)J_{l\alpha} + \text{h.c.}$$

We ignore heavy quarks b and t

Amplitude

$$\boxed{\mathcal{M}_{\text{fi}} = -\frac{G}{\sqrt{2}}\cos\theta_C(H^\alpha L_\alpha + \tilde{H}^\alpha \tilde{L}_\alpha)}$$

$$H^\alpha = \langle h_f|(\bar{u}\mathcal{O}_L^\alpha d)(0)|h_i\rangle \qquad L^\alpha = \langle \ell_f|J_l^\alpha(0)|\ell_i\rangle$$

$$\tilde{H}^\alpha = \langle h_f|(\bar{u}\mathcal{O}_L^\alpha d)(0)^\dagger|h_i\rangle \qquad \tilde{L}^\alpha = \langle \ell_f|J_l^\alpha(0)^\dagger|\ell_i\rangle$$

Initial/final hadronic states $h_{i,f}$, initial/final leptonic states $\ell_{i,f}$

Lowest-order perturbation theory

Only one of the terms in $\mathcal{M}_{\text{fi}}$ can be $\neq 0$, depending on $h_{i,f}$, $\ell_{i,f}$:

- $\bar{u}\mathcal{O}_L^\alpha d$: electric charge $Q = +1$, isospin $I_3 = +1$, hypercharge $Y = 0$
- $\bar{d}\mathcal{O}_L^\alpha u$: electric charge $Q = -1$, isospin $I_3 = -1$, hypercharge $Y = 0$

Drop the tilde in the following

Exact form H^α not known, use symmetry to constrain them $\Rightarrow$ depend on few phenomenological parameters (related by strong int. symmetries)

Isotopic spin (isospin) invariance

Up and down quark form an isospin doublet

$$q = \begin{pmatrix} u \\ d \end{pmatrix}$$

Exact $\mathrm{SU}(2)_I$ symmetry if $m_u = m_d$

Relevant charged weak currents $\bar{u}\mathcal{O}_L^\alpha d$, $\bar{d}\mathcal{O}_L^\alpha u$ part of the isovector triplet

$$\mathcal{J}_a^\mu = \bar{q}\mathcal{O}_L^\mu \frac{\tau_a}{2} q = \bar{q}\gamma^\mu \frac{\tau_a}{2} q - \bar{q}\gamma^\mu\gamma^5 \frac{\tau_a}{2} q = V_a^\mu - A_a^\mu$$

V_a^μ: (Lorentz) vector-isovector, A_a^μ: (Lorentz) axial vector-isovector
τ_a: Pauli matrices (generators of $\mathrm{SU}(2)_I$, standard notation)

$$\bar{u}\mathcal{O}_L^\mu d = \mathcal{J}_+^\mu = \bar{q}\mathcal{O}_L^\mu \frac{\tau_+}{2} q = V_+^\mu - A_+^\mu$$

$$\bar{d}\mathcal{O}_L^\mu u = \mathcal{J}_-^\mu = \bar{q}\mathcal{O}_L^\mu \frac{\tau_-}{2} q = V_-^\mu - A_-^\mu$$

$$\tau_\pm = \tau_1 \pm i\tau_2 \qquad \tau_+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

References

- R. L. Garwin, L. M. Lederman, M. Weinrich, Phys.Rev. 105 (1957) 1415