Weak Interactions

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November 12, 2020

The Standard Model: Fermionic sector (contd.)

Parameters of the theory that we need to fix: g, θ_W, v, f_i

- one relation with phenomenology already available: $e = g\sin\theta_W$
- Yukawa couplings f_i known from v and fermion masses
- relate v to Fermi constant G_F, $\sin^2\theta_W$ to elastic $\nu_e - e$ cross section

If m_W large, low-energy process corresponding to single W-exchange

- from Feynman diagram in low-energy approximation

$$\left(\frac{ig}{\sqrt{2}}\right)^2 \frac{i}{m_W^2}\langle f|J^+_\mu J^{-\mu}|i\rangle = -i\frac{g^2}{2m_W^2}\langle f|J^+_\mu J^{-\mu}|i\rangle$$

- described equally well by effective four-fermion interaction

$$\mathcal{L}_{\rm eff} = -\frac{g^2}{2m_W^2}J^+_\mu J^{-\mu} = -\frac{g^2}{8m_W^2}j^W_\mu j^{W\mu\dagger} = -\frac{G_F}{\sqrt{2}}j^+_\mu j^{-\mu}$$

$$G_F = \frac{g^2}{4m_W^2\sqrt{2}} = \frac{g^2}{4\frac{g^2v^2}{4}\sqrt{2}} = (v^2\sqrt{2})^{-1} \Longrightarrow v = 2^{-\frac{1}{4}}G_F^{-\frac{1}{2}} \simeq 250\,\mathrm{GeV}$$

Scale much larger than $m_{u,d,e}$ $\Rightarrow$ corresponding Yukawa couplings small

The Standard Model: Fermionic sector (contd.)

Low-energy processes involving neutral weak current (single Z-exchange)

- relevant Feynman diagram (factor $\frac{1}{2}$ to avoid double counting)

$$\left(\frac{ig}{\cos\theta_W}\right)^2 \frac{i}{m_Z^2}\langle f|\tfrac{1}{2}J^0_\mu J^{0\mu}|i\rangle = -i\frac{g^2}{2\cos\theta_W^2 m_Z^2}\langle f|J^0_\mu J^{0\mu}|i\rangle$$

- equivalently obtained from the low-energy effective Lagrangian

$$\mathcal{L}^0_{\mathrm{eff}} = -\frac{g^2}{2\cos\theta_W^2 m_Z^2}J^0_\mu J^{0\mu} = -\frac{g^2}{2m_W^2}J^0_\mu J^{0\mu}$$

- same coupling appears in charged and neutral current interactions

Neutral current: $J^0_\mu = \sum_i g_L^{(i)}\bar{\psi}_L^{(i)}\gamma_\mu\psi_L^{(i)} + g_R^{(i)}\bar{\psi}_R^{(i)}\gamma_\mu\psi_R^{(i)}$

$$g^{(i)}_{L,R} = T^3(\psi^{(i)}_{L,R}) - \sin^2\theta_W Q(\psi^{(i)}_{L,R})$$

Contribution to elastic $\nu_e - e$ scattering $\propto \frac{1}{2}\bar{\nu}_{eL}\gamma_\mu\nu_{eL}\,\bar{e}\gamma_\mu(a+b\gamma^5)e$

$$a = g_R^{(e)} + g_L^{(e)} = -\tfrac{1}{2} + 2\sin^2\theta_W \quad b = g_R^{(e)} - g_L^{(e)} = \tfrac{1}{2}$$

Exp.: $\sin^2\theta_W \simeq 0.22\div 0.23$

Prediction $m_W = |\frac{gv}{2}| = |\frac{ev}{2\sin\theta_W}| = |2^{-\frac{5}{4}}eG_F^{-\frac{1}{2}}/\sin\theta_W| = \frac{37\,\mathrm{GeV}}{|\sin\theta_W|} \to 81.8\,\mathrm{GeV}$

The Standard Model: Fermionic sector (contd.)

Summary for one generation of fermions:

- 15 Weyl (2-component) fermion fields with definite chirality
- 4 vector bosons (3 massive and 1 massless)
- 1 Higgs field
- Lagrangian parameters: $g, g', \mu^2, \lambda, f_e, f_u, f_d$, corresponding to phenomenological parameters $e, \sin\theta_W, m_W, m_\eta, m_e, m_u, m_d$
- unification not complete like in the electromagnetic case: there are still two independent coupling constants
- by construction, baryon and lepton number are conserved
- results of the model depend heavily on having a single complex doublet of scalar fields in the unbroken theory

The Standard Model: Fermionic sector (contd.)

Generalisation to more generations of fermions almost straightforward: replicating families guarantees anomaly-free theory

- add four SU(2) doublets of LH fields, and four singlets of RH fields
- assign T_3, Y as with lightest fermion generation
- allow for mixing of the various fermion species

$$\tilde{\ell}_{A\,L} = \begin{pmatrix} \tilde{\nu}_A \\ \tilde{e}_A \end{pmatrix}_L \qquad \tilde{q}_{A\,L} = \begin{pmatrix} \tilde{p}_A \\ \tilde{n}_A \end{pmatrix}_L \qquad \tilde{e}_{A\,R} \qquad \tilde{p}_{A\,R} \qquad \tilde{n}_{A\,R}$$

$$\tilde{e}_A = \tilde{e}, \tilde{\mu}, \tilde{\tau} \qquad \tilde{p}_A = \tilde{u}, \tilde{c}, \tilde{t} \qquad \tilde{n}_A = \tilde{d}, \tilde{s}, \tilde{b}$$

Fields with definite gauge transformation properties, couple simply to gauge fields

$$\begin{aligned} \bar{\tilde{\Psi}} \not{D} \tilde{\Psi} = \;& \bar{\tilde{\ell}}_{A\,L}(\not{\partial} - \tfrac{i}{2} g \vec{\tau} \vec{\not{W}}_\mu + \tfrac{i}{2} g' \not{B}_\mu) \tilde{\ell}_{A\,L} \\ &+ \bar{\tilde{q}}_{A\,L}(\not{\partial} - \tfrac{i}{2} g \vec{\tau} \vec{\not{W}}_\mu - \tfrac{i}{6} g' \not{B}_\mu) \tilde{q}_{A\,L} \\ &+ \bar{\tilde{\ell}}_{A\,R}(\not{\partial} + i g' \not{B}_\mu) \tilde{\ell}_{A\,R} \\ &+ \bar{\tilde{p}}_{A\,R}(\not{\partial} - i \tfrac{2}{3} g' \not{B}_\mu) \tilde{p}_{A\,R} + \bar{\tilde{n}}_{A\,R}(\not{\partial} + i \tfrac{1}{3} g' \not{B}_\mu) \tilde{n}_{A\,R} \end{aligned}$$

The Standard Model: Fermionic sector (contd.)

Yukawa couplings: allow for mixing of fields with same quantum numbers

- quarks and leptons do not mix due to different colour charges
- quarks of types p and n do not mix due to different electric charges
- any other mixing is allowed

Most general Yukawa term

$$\mathcal{L}_{\text{Yukawa}} = f^{(e)}_{AB}(\bar{\tilde{\ell}}_{A\,L}\phi)\tilde{e}_{A\,R} + f^{(p)}_{AB}(\bar{\tilde{q}}_{A\,L}\tilde{\phi})\tilde{p}_{B\,R} + f^{(n)}_{AB}(\bar{\tilde{q}}_{A\,L}\phi)\tilde{n}_{B\,R} + \text{h.c.}$$

After symmetry breaking, mass matrices in unitarity gauge

$$M^{(i)}_{AB} = \frac{v}{\sqrt{2}} f^{(i)}_{AB} \qquad i = e, p, n$$

$M^{(i)}$: general complex 3×3 matrices without further structure, transform to real positive diagonal matrix by means of a pair of unitary matrices

$$S^{(i)} M^{(i)} T^{(i)\dagger} = M^{(i)}_{\text{diag}}$$

Polar decomposition theorem: $M = HU$, H Hermitian, U unitary;
$H = V^\dagger D V$, D real diagonal, V unitary; denote $\Sigma = \mathrm{diag}(\mathrm{sgn}\, D_i)$;
set $S = \Sigma V$, $T^\dagger = U^\dagger V^\dagger \Rightarrow SMT^\dagger = \Sigma V V^\dagger D V U U^\dagger V^\dagger = \Sigma D$,
ΣD real positive diagonal matrix

The Standard Model: Fermionic sector (contd.)

$$
\begin{aligned}
M^{(i)}_{AB}\bar{\tilde{\psi}}^{(i)}_{A\,L}\tilde{\psi}^{(i)}_{B\,R} &= \bar{\tilde{\psi}}^{(i)}_{L} M^{(i)} \tilde{\psi}^{(i)}_{R} = \bar{\tilde{\psi}}^{(i)}_{L} S^{(i)\dagger} M^{(i)}_{\mathrm{diag}} T^{(i)} \tilde{\psi}^{(i)}_{R} \\
&= \overline{S^{(i)}\tilde{\psi}}^{(i)}_{L} M^{(i)}_{\mathrm{diag}} (T^{(i)}\tilde{\psi}^{(i)}_{R}) = \bar{\psi}^{(i)}_{L} M^{(i)}_{\mathrm{diag}} \psi^{(i)}_{R}
\end{aligned}
$$

- fields $\tilde{\psi}^{(i)}$: definite gauge transformation properties, not definite mass
- fields $\psi^{(i)}$: definite mass, not definite gauge transformation properties

Quarks charged current:

$$
\begin{aligned}
J^{h+}_{\mu} &= \bar{\tilde{q}}_{A\,L}\tau^{+}\gamma_{\mu}\tilde{q}_{A\,L} = \bar{\tilde{p}}_{A\,L}\gamma_{\mu}\tilde{n}_{A\,L} \\
&= \bar{p}_{A\,L}\gamma_{\mu}[S^{(p)}S^{(n)\dagger}]_{AB}n_{B\,L} = \bar{p}_{A\,L}\gamma_{\mu}U_{AB}n_{B\,L}
\end{aligned}
$$

U_{AB}: unitary CKM matrix, set

$$
\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = U \begin{pmatrix} d \\ s \\ b \end{pmatrix}
$$

- recover phenomenological description of quark mixing
- choice of rotating only $Q = -\frac{1}{3}$ quarks is purely conventional

The Standard Model: Fermionic sector (contd.)

Leptonic current:

$$J_\mu^{\ell +} = \bar{\tilde{\ell}}_{A\,L} \tau^+ \gamma_\mu \tilde{\ell}_{A\,L} = \bar{\tilde{\nu}}_{A\,L} \gamma_\mu \tilde{e}_{A\,L} = \bar{\tilde{\nu}}_{A\,L} \gamma_\mu S^{(e)\dagger}_{AB} e_{B\,L} = \overline{S^{(e)} \tilde{\nu}}_{A\,L} \gamma_\mu e_{A\,L}$$

- taking neutrinos massless there is no matrix $S^{(\nu)}$
- define $(S^{(e)} \tilde{\nu})_A = \nu_A$: still massless fields, coupled to e_A in charged weak current $J_\mu^{\ell +} = \overline{\nu}_{A\,L} \gamma_\mu e_{A\,L}$

more generally: for massive, mass-degenerate neutrinos add ν_R,
set $S^{(\nu)\prime} M^{(\nu)} T^{(\nu)\prime\dagger} = m_\nu \mathbf{1} \Rightarrow M^{(\nu)} = m_\nu S^{(\nu)\prime\dagger} T^{(\nu)\prime} \equiv m_\nu T^{(\nu)}$,
rotate only RH part to diagonalise mass, $S^{(\nu)} = \mathbf{1}$, $T^{(\nu)} = \frac{1}{m_\nu} M^{(\nu)}$,
and only LH part to diagonalise flavour $(S^{(e)} \tilde{\nu})_A = \nu_A$, $M^{(\nu)}_{\rm diag} = m_\nu \mathbf{1}$

- doublet (ν_A, e_A) still has definite gauge transformation properties
- define fields ν_A and e_A as neutrinos/charged leptons with definite lepton flavour, which is then a conserved quantity
- non-mixing and exact lepton family number conservation would then be a consequence of mass-degeneracy of the neutrinos

The Standard Model: Fermionic sector (contd.)

If neutrinos massive, analogous situation as with quarks:

- $\nu_A = (S^{(e)} S^{(\nu)\dagger} \tilde{\nu})_A = (U^{\dagger}_{\mathrm{PMNS}} \tilde{\nu})_A$
- mass eigenstates $\neq$ lepton flavour eigenstates
- mixing allowed, neutrino oscillations

Neutral current:

$$
\begin{aligned}
J^0_\mu &= \sum_i g_L^{(i)} \bar{\tilde{\psi}}^{(i)}_{A\,L} \gamma_\mu \tilde{\psi}^{(i)}_{A\,L} + g_R^{(i)} \bar{\tilde{\psi}}^{(i)}_{A\,R} \gamma_\mu \tilde{\psi}^{(i)}_{A\,R} \\
&= \sum_i g_L^{(i)} \bar{\psi}^{(i)}_{A\,L} \gamma_\mu [S^{(i)} S^{(i)\dagger}]_{AB} \psi^{(i)}_{B\,L} + g_R^{(i)} \bar{\psi}^{(i)}_{A\,R} \gamma_\mu [T^{(i)} T^{(i)\dagger}]_{AB} \psi^{(i)}_{B\,R} \\
&= \sum_i g_L^{(i)} \bar{\psi}^{(i)}_{A\,L} \gamma_\mu \psi^{(i)}_{A\,L} + g_R^{(i)} \bar{\psi}^{(i)}_{A\,R} \gamma_\mu \psi^{(i)}_{A\,R}
\end{aligned}
$$

Same form in terms of gauge or mass eigenstates