Weak Interactions

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The Standard Model: Introduction

Describe weak interactions as exchange of massive vector bosons:

- vector bosons = gauge bosons of some gauge theory
- mass acquired via spontaneous symmetry breaking

Guarantees renormalisability

Build the appropriate model:

1. find right gauge group G and unbroken subgroup H
2. find set of scalar fields realising symmetry breaking pattern $G \to H$
3. choose representation multiplets of physical fields

Phenomenologically successful model: Glashow-Salam-Weinberg model

- minimal model unifying electromagnetism and weak interactions using a spontaneously broken gauge theory
- gauge group $G = \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ broken to $H = \mathrm{U}(1)_{\mathrm{EM}}$
    - gauge group is chiral, treating L and R fermions differently
    - U(1) groups in G and H are different

Model world with only e, ν_e, EM and weak interactions
Phenomenologically known currents:

$$j_\mu^W = \bar{\nu}_e \mathcal{O}_{L\mu} e = \bar{\nu}_e \gamma_\mu (1-\gamma^5) e \qquad j_\mu^{EM} = -\bar{e}\gamma_\mu e$$

Matter currents couple to gauge bosons in gauge theories

$$\mathcal{L}_W = g(j_\mu^W W^\mu + j_\mu^{W\dagger} W^{\mu\dagger}) \qquad \mathcal{L}_{EM} = e j_\mu^{EM} A^\mu$$

g, e: coupling constants, W^μ: W-boson field, A^μ: photon field

At least 3 gauge bosons, associated vector currents/charges conserved due to global symmetry

$$\begin{aligned} T_+(t) &= \tfrac{1}{2}\int d^3x\, j_0^W(t,\vec{x}) = \tfrac{1}{2}\int d^3x\, (\nu_e^\dagger(1-\gamma^5)e)(t,\vec{x}) \\ &= \int d^3x\, ({\nu_{eL}}^\dagger e_L)(t,\vec{x}) \\ T_-(t) &= \tfrac{1}{2}\int d^3x\, j_0^W(t,\vec{x})^\dagger = T_+(t)^\dagger \\ Q(t) &= -\int d^3x\, j_0^{EM}(t,\vec{x}) = \int d^3x\, (e^\dagger e)(t,\vec{x}) \end{aligned}$$

Placement of factors $\frac{1}{2}$ conventional

The Standard Model: Finding the gauge group (contd.)

Conserved charges = generators of symmetry group, part of a Lie algebra
$\Rightarrow [T_+, T_-]$ another element of symmetry algebra

Using CAR for fermion fields

$$\{\psi_{i\alpha}{}^\dagger(t,\vec{x}), \psi_{j\beta}(t,\vec{y})\} = \delta_{ij}\delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y})$$

$$T_3 \equiv \tfrac{1}{2}[T_+, T_-] = \tfrac{1}{2}\int d^3x \int d^3y\, [\nu_{eL}{}^\dagger e_L(t,\vec{x}), e_L{}^\dagger \nu_{eL}(t,\vec{y})]$$
$$= \tfrac{1}{2}\int d^3x\, [\nu_{eL}{}^\dagger \nu_{eL} - e_L{}^\dagger e_L](t,\vec{x})$$

- $[T_3, Q] = 0$, independent of $T_\pm$ and $Q \Rightarrow$ requires introduction of third gauge boson for weak interactions
- further commutators do not yield new charges $\Rightarrow$ four-dimensional gauge group

Instead of Q more convenient to use

$$Y = 2(Q - T_3) \qquad T_\pm = T_1 \pm iT_2 \Rightarrow [Y, T_a] = 0 \quad a = 1,2,3$$

The Standard Model: Finding the gauge group (contd.)

Gauge group: $G = \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$

$$\mathrm{SU}(2)_L \Rightarrow \{ T_a \mid a = 1,2,3 \} \qquad \mathrm{U}(1)_Y \Rightarrow Y = 2(Q - T_3)$$

- $\mathrm{SU}(2)_L$: *weak isospin* symmetry group, acts only on left-handed component of matter fields
- $\mathrm{U}(1)_Y$: *weak hypercharge* symmetry group acts on both left-handed and right-handed parts

Unrelated to isospin and hypercharge symmetry of strong interactions in the SU(3) quark model

Full global symmetries of LH doublet (ν_{eL}, e_L) and RH singlet e_R: $\mathrm{SU}(2)_L \times \mathrm{U}(1)_L \times \mathrm{U}(1)_R$

- $\mathrm{U}(1)_{L,R}$: chiral phase transformations of LH doublet and RH singlet, generated by $t_L = T^3 - \frac{1+\gamma^5}{2}Q$, $t_R = -\frac{1-\gamma^5}{2}Q$
- only combination $\mathrm{U}(1)_Y$ generated by $Y = -2(t_R + t_L)$ happens to be gauged in nature
- other independent combination $2t_L + t_R =$ lepton family number and is only global

Four gauge bosons: $W^a_\mu \leftrightarrow T_a$, $B_\mu \leftrightarrow Y$, but only one long range force associated to electric charge $Q \Rightarrow G$ must break down to $\mathrm{U}(1)_Q$

The Standard Model: Higgs mechanism

Need scalar fields + potential to break symmetry

$$G = \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to H = \mathrm{U}(1)_{\mathrm{EM}}$$

- doublet of complex scalar fields + Mexican-hat potential break SU(2) completely when vacuum expectation value (vev) is nonzero
- need to preserve U(1) subgroup associated to Q when vev appears $\Rightarrow$ only electrically neutral field allowed to develop vev

$Q = T_3 + \frac{Y}{2}$, $T_3 = \pm\frac{1}{2}$ for doublet upper/lower component: choose $Y = 1$

$$\phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \qquad [Q, \varphi^+] = \varphi^+ \qquad [Q, \varphi^0] = 0$$

To preserve $\mathrm{U}(1)_{\mathrm{EM}}$ only φ^0 can develop nonzero vev

Alternative: choose $Y = -1 \Rightarrow$ neutral upper component, negatively charged lower component = charge conjugate of ϕ, no loss of generality

The Standard Model: Higgs mechanism (contd.)

Covariant derivative:

$$D_\mu = \partial_\mu - igt^a W^a_\mu - \tfrac{i}{2}g' Y B_\mu$$

- g, g': dimensionless coupling constants
- t^a: generators of SU(2) in the appropriate representation

$\frac{1}{2}$ factor: conventional

Covariant derivative of ϕ: weak isospin doublet, $t^a = \tau^a/2$ (Pauli matrices)

$$D_\mu \phi = (\partial_\mu - ig\tfrac{\tau^a}{2} W^a_\mu - \tfrac{i}{2}g' B_\mu)\phi$$

Potential (up to irrelevant additive constant)

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger\phi)^2 \,, \qquad \lambda, \mu^2 > 0 \,.$$

Ground state choice:

$$\langle \phi \rangle_0 \equiv \langle 0|\phi|0\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} = \phi_0 \qquad v^2 = \frac{\mu^2}{\lambda} \qquad V(v) = V_{\rm min} = -\frac{\mu^4}{4\lambda}$$

The Standard Model: Higgs mechanism (contd.)

Parameterise most general field configuration

$$\begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} = \mathscr{U}^{-1}(\vec{\xi}) \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix} \qquad \mathscr{U}(\vec{\xi}) = e^{i \frac{\vec{\xi}(x)}{v} \cdot \frac{\vec{\tau}}{2}} \in \mathrm{SU}(2)$$

Vev $\langle \eta \rangle_0 = 0$, $\langle \vec{\xi} \rangle_0 = 0$, corresponding to the choice ϕ_0 for the vacuum

Impose unitarity gauge condition:

$$\phi(x) \to \mathscr{U}(\vec{\xi}(x))\phi(x) = \frac{v+\eta(x)}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{v+\eta(x)}{\sqrt{2}} \chi$$

Potential in unitarity gauge:

$$V = \mu^2 \eta^2 + \lambda v \eta^3 + \frac{\lambda}{4} \eta^4 - \frac{\mu^4}{4\lambda}$$

- quadratic (mass) term $\frac{1}{2}(\sqrt{2}\mu)^2 \eta^2$ for Higgs field η
- cubic and quartic self interactions (renormalisable)
- irrelevant constant

The Standard Model: Higgs mechanism (contd.)

Kinetic term:

$$(D_\mu\phi)^\dagger(D^\mu\phi) = \tfrac{1}{2}(D_\mu(v+\eta)\chi)^\dagger(D^\mu(v+\eta)\chi)$$

$$= \tfrac{1}{2}\chi^\dagger\chi\partial_\mu\eta\partial^\mu\eta$$

$$+ \tfrac{1}{2}(v+\eta)\chi^\dagger(ig\tfrac{\tau^a}{2}W^a_\mu + \tfrac{i}{2}g'B_\mu)\chi\partial^\mu\eta - \tfrac{1}{2}\partial^\mu\eta(ig\tfrac{\tau^a}{2}W^a_\mu + \tfrac{i}{2}g'B_\mu)\chi(v+\eta)$$

$$+ \tfrac{1}{2}(v+\eta)^2\chi^\dagger(ig\tfrac{\tau^a}{2}W^a_\mu + \tfrac{i}{2}g'B_\mu)(-ig\tfrac{\tau^a}{2}W^a_\mu - \tfrac{i}{2}g'B_\mu)\chi$$

$$= \tfrac{1}{2}\chi^\dagger\chi\partial_\mu\eta\partial^\mu\eta + \tfrac{1}{2}(v+\eta)^2\chi^\dagger(g\tfrac{\tau^a}{2}W^a_\mu + \tfrac{1}{2}g'B_\mu)(g\tfrac{\tau^a}{2}W^a_\mu + \tfrac{1}{2}g'B_\mu)\chi$$

- Second term $= 0$ due to unitarity gauge choice
- Third term $= (2,2)$ element of matrix sandwiched between $\chi^\dagger$ and χ

$$\text{mass term} = \chi^\dagger(g\tfrac{\tau^a}{2}W^a_\mu + \tfrac{1}{2}g'B_\mu)(g\tfrac{\tau^a}{2}W^a_\mu + \tfrac{1}{2}g'B_\mu)\chi$$

$$= \tfrac{g^2}{4}(W^1_\mu W^{1\mu} + W^2_\mu W^{2\mu}) + \tfrac{1}{4}(gW^3_\mu - g'B_\mu)(gW^{3\mu} - g'B^\mu)$$

$$= \tfrac{g^2}{2}\left(\tfrac{W^1_\mu - iW^2_\mu}{\sqrt{2}}\right)\left(\tfrac{W^{1\mu}+iW^{2\mu}}{\sqrt{2}}\right) +$$

$$+ \tfrac{g^2+g'^2}{4}\left(\tfrac{gW^3_\mu}{\sqrt{g^2+g'^2}} - \tfrac{g'B_\mu}{\sqrt{g^2+g'^2}}\right)\left(\tfrac{gW^{3\mu}}{\sqrt{g^2+g'^2}} - \tfrac{g'B^\mu}{\sqrt{g^2+g'^2}}\right)$$

The Standard Model: Higgs mechanism (contd.)

Vector bosons fields

$$W_\mu^\pm = \frac{W_\mu^1 \pm i W_\mu^2}{\sqrt{2}} \qquad Z_\mu = \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu \qquad A_\mu = \cos\theta_W B_\mu + \sin\theta_W W_\mu^3$$

Mass terms from covariant derivative and scalar potential

$$\text{Mass terms} = \mu^2\eta^2 + \left(\frac{gv}{2}\right)^2 W_\mu^- W^{+\mu} + \frac{1}{2}\left(\frac{gv}{2\cos\theta_W}\right)^2 Z_\mu Z^\mu$$

$$m_\eta = \sqrt{2}\mu \qquad m_W = \frac{gv}{2} \qquad m_Z = \frac{gv}{2\cos\theta_W} = \frac{m_W}{\cos\theta_W} \geq m_W$$

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}},\ \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}$$

All gauge bosons have $Y = 0 \Rightarrow Q = T_3$

W_μ^a not coupled to B_μ; B_μ Abelian, not coupled to itself

- $W_\mu^\pm$ electrically charged $Q(W^\pm) = \pm 1$
- Z_μ, A_μ electrically neutral $Q(W^3) = Q(B) = Q(Z) = Q(A) = 0$
- $W_\mu^\pm$, Z_μ: massive; charged and neutral IVB $\sim$ weak interactions
- A_μ: massless; photon $\sim$ EM

The Standard Model: Fermionic sector

Model world: only e, ν_e leptons, u and d quarks

- Chiral gauge interactions, LH and RH components are independent and count as different dof
- No. of chiral (Weyl) fermions $= 15$
  - leptons: e_L, ν_{eL}, e_R
  - quarks: u_L, d_L, u_R, d_R (in three colours each)
- $\mathrm{SU}(2)_L$ gauge bosons couple to LH fields
- $\mathrm{U}(1)_Y$ couples to both LH and RH but with different couplings

Choice of multiplets guided by phenomenology and known leptonic and hadronic charged weak currents

The Standard Model: Fermionic sector (contd.)

Conserved weak isospin charges:

$$T_+ = \int d^3x \, ({\nu_{eL}}^\dagger e_L + {u_L}^\dagger d_L) \qquad T_- = \int d^3x \, ({e_L}^\dagger \nu_{eL} + {d_L}^\dagger u_L)$$

$$2T_3 = \int d^3x \, ({\nu_{eL}}^\dagger \nu_{eL} - {e_L}^\dagger e_L + {u_L}^\dagger u_L - {d_L}^\dagger d_L)$$

LH leptons/quarks form weak isodoublets, RH e, u, d are isosinglets

$$\ell_L = \binom{\nu_{eL}}{e_L} \qquad q_L = \binom{u_L}{d_L} \qquad e_R \qquad u_R \qquad d_R$$

Electric charge/ weak hypercharge:

$$Q = \int d^3x \, (-{e_L}^\dagger e_L - {e_R}^\dagger e_R + \tfrac{2}{3}{u_L}^\dagger u_L + \tfrac{2}{3}{u_R}^\dagger u_R - \tfrac{1}{3}{d_L}^\dagger d_L - \tfrac{1}{3}{d_R}^\dagger d_R)$$

$$\begin{aligned} Y &= 2(Q - T_3) \\ &= \int d^3x \, (-{\nu_{eL}}^\dagger \nu_{eL} - {e_L}^\dagger e_L - 2{e_R}^\dagger e_R + \tfrac{1}{3}{u_L}^\dagger u_L + \tfrac{4}{3}{u_R}^\dagger u_R + \tfrac{1}{3}{d_L}^\dagger d_L - \tfrac{2}{3}{d_R}^\dagger d_R) \end{aligned}$$

Y must be the same in each multiplet since $[T_a, Y] = 0$

- LH doublets: $Y_L = 2(Q_+ - \frac{1}{2}) = 2(Q_- + \frac{1}{2}) = Q_+ + Q_-$
- RH singlets: $Y_R = 2Q$
- $Y = 2 \times$ average charge of the multiplet

The Standard Model: Fermionic sector (contd.)

Weak hypercharge:

$$Y(\ell_L) = -1 \qquad Y(q_L) = \tfrac{1}{3}$$
$$Y(e_R) = -2 \qquad Y(u_R) = \tfrac{4}{3} \qquad Y(d_R) = -\tfrac{2}{3}$$

Important theoretical reason to treat e, ν, u, d together

- in QFT certain symmetries of the classical action do not survive quantisation: *anomalous* symmetries
- typical example: chiral symmetries, like the one used here!
- in the presence of an anomaly, Noether current J^μ associated to global symmetry is no more conserved $\Rightarrow$
  - breaks gauge symmetry as well
  - nice properties of a gauge theory are lost
- with the right matter content in the theory, contributions to $\partial_\mu J^\mu$ can cancel out: this is the case with our choice of fields, representations, and charges
- $e_L, \nu_L, u_L, d_L, e_R, u_R, d_R$ = one *generation* of fermions (anomaly-free)

The Standard Model: Fermionic sector (contd.)

$$\mathcal{L}_{\rm fermion} = \bar{\psi} i \not{D} \psi - m\bar{\psi}\psi - \mathcal{L}_{\rm Yukawa}(\phi, \psi, \bar{\psi})$$

$$D_\mu \psi = (\partial_\mu - igT^a W^\mu - \tfrac{i}{2} g' Y B_\mu)\psi$$

$$T^a = \begin{cases} \frac{\tau^a}{2} & \psi = \psi_L \\ 0 & \psi = \psi_R \end{cases}$$

$$Y = \begin{cases} -1 & \psi = \ell_L \\ +\frac{1}{3} & \psi = q_L \\ -2 & \psi = e_R \\ +\frac{4}{3} & \psi = u_R \\ -\frac{2}{3} & \psi = d_R \end{cases}$$

Explicit mass term forbidden by chiral nature of the symmetry (already at global level)

$$\bar{\psi}\psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L \Rightarrow m = 0$$

Different mechanism to provide masses to fermions: Yukawa terms $\phi\bar{\psi}\psi$

$$\mathcal{L}_{\rm Yukawa}(\phi, \psi, \bar{\psi}) = f_\ell\,(\bar{\ell}_L \phi) e_R + f_d\,(\bar{q}_L \phi) d_R + f_u\,(\bar{q}_L \tilde{\phi}) u_R + \text{h.c.}$$

$$\phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix},\ \tilde{\phi} = i\tau^2 \phi^* = \begin{pmatrix} \varphi^0 \\ -\varphi^+ \end{pmatrix}:\ \mathrm{SU}(2)_L \text{ doublets},\ Y(\phi) = 1,\ Y(\tilde{\phi}) = -1$$

$\phi \to U\phi$ doublet, $\phi^* \to U^*\phi^*$ antidoublet; $U^* = \tau^2 U \tau^2$ so $\tilde{\phi} \to U\tilde{\phi}$

$\tilde{\phi}$ needed to give mass to u quark while respecting symmetry

Dimensionless quantities $f_{\ell,d,u}$: *Yukawa couplings*

The Standard Model: Fermionic sector (contd.)

Going over to unitarity gauge:

$$\phi = \begin{pmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{pmatrix} \qquad \tilde{\phi} = \begin{pmatrix} \frac{v+\eta}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\mathcal{L}_{\text{Yukawa, UG}} = \frac{v+\eta}{\sqrt{2}} f_\ell \, \bar{e}_L e_R + \frac{v+\eta}{\sqrt{2}} f_d \, \bar{d}_L d_R + \frac{v+\eta}{\sqrt{2}} f_u \, \bar{u}_L u_R + \text{h.c.}$$

Masses of fermions and couplings to Higgs field η:

$$m_i = \frac{v f_i}{\sqrt{2}}\,, \qquad f_i = \frac{\sqrt{2} m_i}{v}$$

- the larger the mass, the stronger the coupling with the Higgs field
- if v is large, couplings are small
- no mass given to the neutrino

The Standard Model: Fermionic sector (contd.)

Coupling to gauge fields:

$$
\begin{aligned}
i\mathcal{L}_{\rm int} &= g\vec{J}_\mu \vec{W}^\mu + \tfrac{1}{2}g' J^y_\mu B^\mu \\
\vec{J}_\mu &= \bar{\ell}_L \tfrac{\vec{\tau}}{2}\ell_L + \bar{q}_L \tfrac{\vec{\tau}}{2} q_L \\
J^y_\mu &= -\bar{\ell}_L \ell_L + \tfrac{1}{3}\bar{q}_L q_L - 2\bar{e}_R e_R + \tfrac{4}{3}\bar{u}_R u_R - \tfrac{2}{3}\bar{d}_R d_R
\end{aligned}
$$

In terms of physical (mass eigenstates) gauge fields $W^\pm_\mu$, Z_μ, and A_μ

$$
\begin{aligned}
i\mathcal{L}_{\rm int} = \tfrac{g}{\sqrt{2}} \Big[(J^1_\mu + iJ^2_\mu)\tfrac{W^{1\mu} - iW^{2\mu}}{\sqrt{2}} + (J^1_\mu - iJ^2_\mu)\tfrac{W^{1\mu} + iW^{2\mu}}{\sqrt{2}} \Big] \\
+ \left(gJ^3_\mu W^{3\mu} + \tfrac{1}{2}g' J^y_\mu B^\mu \right)
\end{aligned}
$$

Invert relation between W^3_μ, B_μ and Z_μ, A_μ:

$$
\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W^3 \\ B \end{pmatrix} \Longrightarrow \begin{pmatrix} W^3 \\ B \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix}
$$

$$
\begin{aligned}
gJ^3_\mu W^{3\mu} + \tfrac{1}{2}g' J^y_\mu B^\mu = \left(g\cos\theta_W J^3_\mu - \tfrac{g'}{2}\sin\theta_W J^y_\mu \right) Z^\mu \\
+ \left(g\sin\theta_W J^3_\mu + \tfrac{g'}{2}\cos\theta_W J^y_\mu \right) A^\mu
\end{aligned}
$$

The Standard Model: Fermionic sector (contd.)

Relation between charges/currents: $Y = 2(Q - T_3) \leftrightarrow J^y_\mu = 2(J^{EM}_\mu - J^3_\mu)$

$$\begin{aligned}
gJ^3_\mu W^{3\mu} + \tfrac{1}{2}g' J^y_\mu B^\mu &= \left[(g\cos\theta_W + g'\sin\theta_W)J^3_\mu - g'\sin\theta_W J^{EM}_\mu\right] Z^\mu \\
&\quad + \left[(g\sin\theta_W - g'\cos\theta_W)J^3_\mu + g'\cos\theta_W J^{EM}_\mu\right] A^\mu \\
&= \tfrac{g}{\cos\theta_W}\left(J^3_\mu - \sin^2\theta_W J^{EM}_\mu\right) Z^\mu + g\sin\theta_W J^{EM}_\mu A^\mu \\
&= \tfrac{g}{\cos\theta_W} J^0_\mu Z^\mu + g\sin\theta_W J^{EM}_\mu A^\mu
\end{aligned}$$

Used the relations

$$\begin{aligned}
&g\sin\theta_W - g'\cos\theta_W = 0 \\
&g\cos\theta_W + g'\sin\theta_W = g(\cos\theta_W + \tan\theta\sin\theta_W) = \tfrac{g}{\cos\theta_W}
\end{aligned}$$

A_μ couples to $J^{EM}_\mu \Rightarrow$ identify with photon field, and $e = g\sin\theta_W$ with electromagnetic coupling constant

$$i\mathcal{L}_{\rm int} = \tfrac{g}{\sqrt{2}}\left(J^+_\mu W^{-\mu} + J^-_\mu W^{+\mu}\right) + \tfrac{g}{\cos\theta_W} J^0_\mu Z^\mu + eJ^{EM}_\mu A^\mu$$