Weak Interactions

Matteo Giordano

Eötvös Loránd University (ELTE)
Budapest

November 5, 2020

Higgs mechanism: $G = \mathrm{SO}(2) \sim \mathrm{U}(1)$ (contd.)

In unitarity gauge the spectrum of the theory is manifest: the gauge-fixed Lagrangian can be quantised without the appearance of unphysical modes, the particle content is transparent

$\mathrm{SO}(2)$ point of view: doublet of real fields $\phi = \binom{\varphi_1}{\varphi_2}$

Vacuum/gauge choice: $\phi_0 = \begin{pmatrix} a \\ 0 \end{pmatrix}$, $\phi = \begin{pmatrix} a+\eta \\ 0 \end{pmatrix} \equiv \phi_0 + \tilde{\phi}$

Using SO(2) generator $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: $\tilde{\phi}^T T \phi_0 = \tilde{\phi}_i T_{ij} \phi_{0j} = 0$

Counting of degrees of freedom:

<u>before</u>		<u>after</u>	
2 real scalars	$\eta, \theta = 2$	1 real scalar	$\eta = 1$
1 massless vector	$A_\mu = 2$	1 massive vector	$B_\mu = 3$
	$= 4$		$= 4$

Nothing got lost!

Higgs mechanism: $G = \mathrm{SU}(2)$

Doublet of complex scalar fields ψ coupled to SU(2) gauge fields

$$\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + (D_\mu \psi)^\dagger (D^\mu \psi) - \lambda(\psi^\dagger \psi - a^2)^2 \qquad \lambda > 0$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon_{abc} A^b_\mu A^c_\nu \qquad D_\mu = \partial_\mu - igA^a_\mu \tfrac{\sigma^a}{2} \qquad a = 1,2,3$$

Ground state: $A_\mu = A^a_\mu \frac{\sigma^a}{2} = 0$, $\psi_0^\dagger \psi_0 = a^2$; choose $\psi_0 = \begin{pmatrix} 0 \\ a \end{pmatrix}$

Most general field configuration: $\psi(x) = e^{i \frac{\theta^a(x)}{a} \frac{\sigma^a}{2}} \begin{pmatrix} 0 \\ a + \frac{\eta(x)}{\sqrt{2}} \end{pmatrix}$

- real fluctuation η around the vacuum ψ_0
- SU(2) rotation parameterised by three real fields θ^a

Gauge fixing to unitarity gauge:
set $\psi_1 = 0$, $\psi_2 \in \mathbb{R}$ with gauge transformation $\Omega(x) \in \mathrm{SU}(2)$

$$\psi'(x) = \Omega(x)\psi(x)$$
$$A'_\mu(x) = \Omega(x) A_\mu(x) \Omega(x)^\dagger - \tfrac{i}{g}(\partial_\mu \Omega(x))\Omega(x)^\dagger$$
$$\Omega(x) = e^{-i \frac{\theta^a(x)}{a} \frac{\sigma^a}{2}}$$

Higgs mechanism: $G = \mathrm{SU}(2)$ (contd.)

$$
\begin{aligned}
&(D_\mu\psi')^\dagger(D^\mu\psi') \\
&= \left(\tfrac{1}{\sqrt{2}}(0,\partial_\mu\eta) + ig\left(0, a+\tfrac{\eta(x)}{\sqrt{2}}\right)A'_\mu\right)\left(\tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\ \partial^\mu\eta\end{pmatrix} - igA'^\mu\begin{pmatrix}0\\ a+\frac{\eta(x)}{\sqrt{2}}\end{pmatrix}\right) \\
&= \tfrac{1}{2}\partial_\mu\eta\partial^\mu\eta + \tfrac{ig}{\sqrt{2}}\left(\left(0, a+\tfrac{\eta(x)}{\sqrt{2}}\right)A'_\mu\begin{pmatrix}0\\ \partial^\mu\eta\end{pmatrix} - (0,\partial_\mu\eta)A'^\mu\begin{pmatrix}0\\ a+\frac{\eta(x)}{\sqrt{2}}\end{pmatrix}\right) \\
&\quad + g^2\left(0, a+\tfrac{\eta(x)}{\sqrt{2}}\right)A'_\mu A'^\mu\begin{pmatrix}0\\ a+\frac{\eta(x)}{\sqrt{2}}\end{pmatrix}
\end{aligned}
$$

Middle term vanishes: contributions of $A'^{1,2}_\mu\sigma^{1,2}$ identically zero, contributions of $A'^3_\mu\sigma^3$ cancel out

$$
A'_\mu A'^\mu = \tfrac{1}{4}A'^a_\mu A'^{b\mu}\sigma^a\sigma^b = \tfrac{1}{4}A'^a_\mu A'^{b\mu}(\delta^{ab} + i\varepsilon^{abc}\sigma^c) = \tfrac{1}{4}A'^a_\mu A'^{a\mu}
$$

$$
\Rightarrow (D_\mu\psi')^\dagger(D^\mu\psi') = \tfrac{1}{2}\partial_\mu\eta\partial^\mu\eta + \tfrac{g^2}{4}A'^a_\mu A'^{a\mu}\left(a + \tfrac{\eta(x)}{\sqrt{2}}\right)^2
$$

Higgs mechanism: $G = \mathrm{SU}(2)$ (contd.)

Gauge-fixed Lagrangian (dropping the primes)

$$\mathcal{L}_{\mathrm{UG}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{g^2}{4}A^a_\mu A^{a\mu}\left(a + \frac{\eta(x)}{\sqrt{2}}\right)^2 - \lambda\frac{\eta^2}{2}\left(2a + \frac{\eta}{\sqrt{2}}\right)^2$$

- read off degrees of freedom from the quadratic part, transparent in unitarity gauge
- one real massive scalar η with mass $m_\eta = 2a\sqrt{\lambda}$
- three massive vectors A^a_μ with mass $m_A = \frac{ga}{\sqrt{2}}$
- SU(2) symmetry completely broken, all gauge bosons acquire a mass
- none of the would-be massless modes θ^a in the physical spectrum, absorbed as longitudinal component of massive vector fields

Higgs mechanism: $G = \mathrm{SU}(2)$ (contd.)

In terms of a quartet of real scalars $\phi = (\phi_1, \phi_3, \phi_2, \phi_4)$, gauge group representation built out of generators $\frac{\Sigma^a}{2}$ with $i\Sigma^a = \begin{pmatrix} -\mathrm{Im}\sigma^a & -\mathrm{Re}\sigma^a \\ \mathrm{Re}\sigma^a & -\mathrm{Im}\sigma^a \end{pmatrix}$

Vacuum ϕ_0 and fluctuations $\tilde{\phi}$ around vacuum in unitarity gauge

$$\phi_0 = (0, a, 0, 0) \qquad \tilde{\phi} = \frac{1}{\sqrt{2}}(0, \eta, 0, 0)$$

Unitarity gauge condition: $\tilde{\phi}^T i\Sigma^a \phi_0 = 0 \quad a = 1, 2, 3$
Same as request that Goldstone modes be zero

Counting of degrees of freedom:

before		after	
4 real scalars	$\eta, \theta^a = 4$	1 real scalar	$\eta = 1$
3 massless vectors	$A^a_\mu = 6$	3 massive vectors	$A^a_\mu = 9$
	$= 10$		$= 10$

Match before and after symmetry breaking

Higgs mechanism: $G = \mathrm{SO}(3)$

SU(2) with triplet of adjoint scalars $\approx$ SO(3) with triplet of fundamental real scalars ϕ_i, $i = 1, 2, 3$

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + (D_\mu \phi)^T (D^\mu \phi) - \lambda(\phi^T \phi - \Lambda^2)^2 \qquad \lambda > 0$$

$$D_\mu = \partial_\mu - igA^a_\mu T^a \qquad a = 1, 2, 3 \qquad (T^a)_{bc} = -i\varepsilon_{abc}$$

Orthogonal representation $(T^a)^T = -T^a$

Vacuum configuration: $A^a_\mu = 0$, $\phi_i = \Lambda \delta_{i3}$

Most general configuration:

$$\phi(x) = e^{i(\theta^1(x)T^1 + \theta^2(x)T^2)} \begin{pmatrix} 0 \\ 0 \\ \Lambda + \eta(x) \end{pmatrix} = U(x) \begin{pmatrix} 0 \\ 0 \\ \Lambda + \eta(x) \end{pmatrix} = U(x)v(x)$$

- No term proportional to T^3 since rotations around $\hat{3}$ leave $v(x)$ invariant $\sim$ SO(2) stability group transformations
- Symmetry breaking pattern SO(3)$\to$ SO(2)

Higgs mechanism: $G = \mathrm{SO}(3)$ (contd.)

Unitarity gauge: $\phi \to \phi' = U^T \phi$, A^a_μ accordingly

Dropping primes, $\phi_a = \delta_{a3}\phi_3$

$$\begin{aligned}(D_\mu\phi)^T(D^\mu\phi) &= \partial_\mu\phi^T\partial^\mu\phi + igA^a_\mu(\phi^T T^a\partial^\mu\phi - \partial^\mu\phi^T T^a\phi) \\ &\quad + g^2 A^a_\mu A^{b\mu}\phi^T T^a T^b\phi\end{aligned}$$

$$\begin{aligned}(\phi^T T^a\partial^\mu\phi - \partial^\mu\phi^T T^a\phi) &= T^a_{bc}(\phi^b\partial^\mu\phi^c - \partial^\mu\phi^b\phi^c) \\ &= T^a_{33}(\phi^3\partial^\mu\phi^3 - \partial^\mu\phi^3\phi^3) = 0 \\ \phi^T T^a T^b\phi &= (\phi_3)^2(T^a T^b)_{33} = -(\phi_3)^2\varepsilon_{a3m}\varepsilon_{bm3} \\ &= (\phi_3)^2(\delta_{ab}\delta_{33} - \delta_{a3}\delta_{b3})\end{aligned}$$

$$(D_\mu\phi)^T(D^\mu\phi) = \partial_\mu\eta\partial^\mu\eta + g^2(\Lambda+\eta)^2(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu})$$

Higgs mechanism: $G = \mathrm{SO}(3)$ (contd.)

Lagrangian in unitarity gauge

$$\mathcal{L}_{\mathrm{UG}} = \tfrac{1}{2}\partial_\mu\eta\partial^\mu\eta - \lambda\eta^2(2\Lambda+\eta)^2 - \tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \tfrac{g^2}{2}(\Lambda+\eta)^2(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu})$$

- only two of the gauge fields acquire a mass $m_{1,2} = g\Lambda$
- A^3_μ remains massless, corresponding to T^3 not broken by the vacuum
- remaining scalar field also massive, $m_\eta = 2\Lambda\sqrt{2\lambda}$

Counting degrees of freedom:

before		after	
3 real scalars	$\eta, \theta^{1,2} = 3$	1 real scalar	$\eta = 1$
3 massless vectors	$A^{1,2,3}_\mu = 6$	2 massive vectors	$A^{1,2}_\mu = 6$
		1 massless vector	$A^3_\mu = 2$
	$= 9$		$= 9$

Match before and after symmetry breaking

Spontaneously broken gauge theories: Higgs mechanism

General case: gauge theory with gauge group G, $\mathrm{dimG} = n$, and a set of scalar fields + Mexican-hat type potential that break G down to H, $\dim H = n'$

- When G only global symmetry $\Rightarrow n - n'$ massless Goldstone bosons
- When G is local
  - the n' gauge bosons corresponding to the generators of H remain massless
  - the $n - n'$ gauge bosons corresponding to the broken generators acquire a mass
  - no massless scalars (Goldstone bosons) appear in the spectrum

Relevant part of most general Lagrangian of interest:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + (D_\mu \phi)^T (D^\mu \phi) - \mathscr{U}(\phi)$$

$\mathscr{U}(\phi) \geq 0$, ϕ real scalars
Real representation, $D_\mu = \partial_\mu - igA^a_\mu T^a$ with iT^a real antisymmetric

Higgs mechanism

Assume $\exists\phi_0 \neq 0$ s.t. $\mathscr{U}(\phi_0) = 0$

- build vacuum manifold $\mathcal{M} = \{G\phi_0\}$ (assume $\mathcal{M}$ is a single G-orbit)
- identify the stability group H, $H\phi_0 = \phi_0$

Set $\phi = \phi_0 + \tilde{\phi}$ and impose *unitarity gauge* condition on fluctuations $\tilde{\phi}$

$$\tilde{\phi}_i T^a_{ij}\phi_{0j} = 0 \qquad a = n'+1, \ldots, n$$

$\{T^a \mid a = 1, \ldots, n\}$ generators of algebra of G
$\{T^a \mid a = 1, \ldots, n'\}$ generators of algebra of H

Condition trivially satisfied for $a = 1, \ldots, n'$

Gauge condition amounts to setting would-be Goldstone modes to zero

One has to show that it is an admissible gauge condition: see [Weinberg (1995)]

Gauge fixed Lagrangian contains

$$(D_\mu\phi)^T(D^\mu\phi) = \partial_\mu\phi_i\partial^\mu\phi_i - igA^a_\mu(\partial^\mu\phi_i T^a_{ik}\phi_k - \phi_i T^a_{ik}\partial^\mu\phi_k)$$
$$+ g^2\phi_i(T^aT^b)_{ij}\phi_j A^a_\mu A^{b\mu}$$

Higgs mechanism (contd.)

Using unitarity gauge condition

$$\begin{aligned}
&\partial^\mu \phi_i T^a_{ik} \phi_k - \phi_i T^a_{ik} \partial^\mu \phi_k \\
&= \partial^\mu (\tilde{\phi}_i T^a_{ik} \phi_{0k} - \phi_{0i} T^a_{ik} \tilde{\phi}_k) + \partial^\mu \tilde{\phi}_i T^a_{ik} \tilde{\phi}_k - \tilde{\phi}_i T^a_{ik} \partial^\mu \tilde{\phi}_k \\
&= \partial^\mu \tilde{\phi}_i T^a_{ik} \tilde{\phi}_k - \tilde{\phi}_i T^a_{ik} \partial^\mu \tilde{\phi}_k
\end{aligned}$$

$\Rightarrow$ cubic interaction term $\tilde{\phi}\tilde{\phi}A$

Quadratic part:

$$(D_\mu \phi)^T (D^\mu \phi)|_{\text{quadratic part}} = \partial_\mu \tilde{\phi}_i \partial^\mu \tilde{\phi}_i + g^2 \phi_{0i} (T^a T^b)_{ij} \phi_{0j} A^a_\mu A^{b\mu}$$

Contains a mass term for the gauge fields, mass matrix

$$M^2_{ab} = g^2 \phi_{0i} (T^a T^b)_{ij} \phi_{0j} = -g^2 \langle T^a \phi_0, T^b \phi_0 \rangle$$

$\langle v, w \rangle = \sum_i v_i w_i$

M^2 is positive-definite: since $\forall v_a \in \mathbb{R}$ the matrix $igv_a T^a$ is real

$$v_a v_b M^2_{ab} = \langle igv_a T^a \phi_0, igv_b T^b \phi_0 \rangle \geq 0$$

Since $T^a \phi_0 = 0$ for $a = 1, \ldots, n'$, $M^2_{ab} = 0$ if $a \leq n'$ and/or $b \leq n'$

Higgs mechanism (contd.)

$$M^2 = \begin{pmatrix} \mathbf{0}_{n' \times n'} & \mathbf{0}_{n' \times (n-n')} \\ \mathbf{0}_{(n-n') \times n'} & \tilde{M}^2_{(n-n') \times (n-n')} \end{pmatrix}$$

- n' massless vector bosons corresponding to generators of $\mathfrak{h}$
- $(n - n') \times (n - n')$ block $\tilde{M}^2$ diagonalisable, encodes masses of gauge bosons corresponding to generators of $\mathfrak{g} - \mathfrak{h}$

Remark on gauge fixing:

- in unitarity gauge, the spectrum of the theory is transparent, but the fate of renormalisability is unclear in this gauge

  A gauge theory is renormalisable when the symmetries are intact
- 't Hooft and others have shown that $\exists$ gauges where renormalisability is apparent, at the cost of a less clear particle spectrum
- gauge invariance = physics is independent of gauge choice
  - if theory is renormalisabile in a gauge then it is just renormalisable
  - if Goldstone bosons are absent in a gauge, then they are just unphysical (gauge) modes

$\boxed{\text{Higgs mechanism: renormalisable way to give mass to gauge bosons}}$

References

- S. Weinberg, "The Quantum Theory of Fields", volume II, CUP (1995)