Weak Interactions

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Gauge theories: motivation

- Locality principle:
  - interactions are not instantaneous, take place *locally* between fields and then propagate at finite speed
  - No event can affect anything outside of its future lightcone
- Overall phase of state vector of a system is experimentally unobservable, can be chosen arbitrarily: e.g., changing phase of every electron state has no experimental effect
  - if all observers agree on phase redefinition: global U(1) transformation; leaves physics invariant $\Rightarrow$ *global* U(1) symmetry
  - if we redefine the phase in one way and another experimenter outside our lightcone in another way there should be no observable consequence $\Rightarrow$ invariance under U(1) *local* (*gauge*) symmetry
- Should be possible to choose phase of electron field (creates/destroy $e^{\mp}$ anywhere) *in different way in different places*
- We are assuming existence of a local symmetry: *gauge principle*

Gauge theories with scalar fields

Real or complex scalar fields, invariance under global symmetry group G

$$\mathcal{L}(\phi) = \tfrac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - (\phi) \qquad \mathcal{L}(\phi) = \partial_\mu\phi_i^*\partial^\mu\phi_i - (\phi)$$

$$\mathcal{L}(g\phi) = \mathcal{L}(\phi)\ \forall\, g \in G \qquad (g\phi)_i(x) = U_{ij}(g)\phi_j(x) \qquad U_{ij}(g) = e^{i\varepsilon_a(g)T^a}$$

$U(g)$: unitary representation of G
T^a: Hermitian (representatives of) group generators $[T^a, T^b] = if_{abc}T^c$

For real fields: orthogonal representation
$\Rightarrow$ T^a purely imaginary, antisymmetric

Real, x-independent parameters $\varepsilon_a \in \mathbb{R}$
(ϕ): potential, function of ϕ only but not of $\partial_\mu\phi$
Invariance under $G \leftrightarrow (g\phi) = (\phi)$
What happens if we promote global to local symmetry, $\varepsilon_a \to \varepsilon_a(x)$?

- $(g(x)\phi(x)) = (\phi(x))$ also for local transformation
- kinetic term depends on $\partial_\mu\phi$, not invariant under local transformation:

$$\partial_\mu\phi_i(x) \to \partial_\mu(U_{ij}(x)\phi_j(x)) = U_{ij}(x)\partial_\mu\phi_j(x) + \partial_\mu(U_{ij}(x))\phi_j(x)$$

Gauge theories with scalar fields (contd.)

To recover invariance introduce new set of fields to reabsorb the extra term: *gauge fields* $A^a_\mu(x)$, $a = 1, \ldots, \dim G$ (one for each generator of G)

- must be Lorentz vectors like ∂_μ under Lorentz transformations
- transform almost like adjoint objects under internal G transformation

Replace ordinary derivative ∂_μ with *covariant derivative* D_μ

$$(D_\mu\phi)_i \equiv \partial_\mu\phi_i - igT^a_{ij}A^a_\mu\phi_j = \partial_\mu\phi_i - ig(A_\mu)_{ij}\phi_j$$

g: dimensionless *coupling constant*

How should A^a_μ transform in the appropriate way to make Lagrangian invariant under $\phi(x) \to U(x)\phi(x)$, $A^a_\mu(x) \to A'^a_\mu(x)$?

$$\begin{aligned}(\partial_\mu - igA_\mu)\phi &\to U\partial_\mu\phi + (\partial_\mu U)\phi - igA'_\mu U\phi \\ &= U[\partial_\mu - ig(U^{-1}A'_\mu U + \tfrac{i}{g}U^{-1}\partial_\mu U)]\phi\end{aligned}$$

Invariance requires:

$$A_\mu = U^{-1}A'_\mu U + \tfrac{i}{g}U^{-1}\partial_\mu U \Longrightarrow A'_\mu = UA_\mu U^{-1} - \tfrac{i}{g}(\partial_\mu U)U^{-1}$$

$U^{-1} = U^\dagger$ or $U^{-1} = U^T$ for unitary or orthogonal matrices

Gauge theories with scalar fields (contd.)

Transformation rule of covariant derivative:

$$D_\mu \phi(x) \to U(x) D_\mu \phi(x)$$

Replacing $\partial_\mu \to D_\mu$, $\mathcal{L}$ becomes invariant under local transformations

$$\mathcal{L}(\phi) = \tfrac{1}{2} D_\mu \phi_i D^\mu \phi_i - (\phi) \qquad \mathcal{L}(\phi) = (D_\mu \phi)_i^* (D^\mu \phi)_i - (\phi)$$

- Gauge fields/covariant derivative analogous to connections/covariant derivative of general relativity, but important difference: gauge connections act in internal space, while spacetime connections act on tangent space of spacetime itself
- Non-homogenous transformation rule of gauge fields analogous to transformation rule of connections in general relativity
  - first term corresponds to transformation rule of adjoint multiplet
  - second term spoils it, A_μ^a not exactly adjoint object (cf. connections are not tensors)

Gauge field tranformations

Representation U/T^a of group/algebra generators irrelevant for transformation properties of A^a_μ
Infinitesimal $U(x) = \mathbf{1} + i\varepsilon_a(x)T^a$

$$\begin{aligned} A'^a_\mu T^a &= (\mathbf{1} + i\varepsilon_b T^b)A_\mu(\mathbf{1} - i\varepsilon_c T^c) - \tfrac{i}{g}(\partial_\mu i\varepsilon_a T^a)(\mathbf{1} - i\varepsilon_b T^b) \\ &= A^a_\mu T^a + i\varepsilon_b A^c_\mu [T^b, T^c] + \tfrac{1}{g}\partial_\mu \varepsilon_a T^a \\ &= A^a_\mu T^a - \varepsilon_b A^c_\mu f_{bca} T^a + \tfrac{1}{g}\partial_\mu \varepsilon_a T^a \end{aligned}$$

For semi-simple compact groups cyclic, totally antisymmetric f_{abc}

$$\delta A^a_\mu \equiv A'^a_\mu - A^a_\mu = -f_{abc}\varepsilon_b A^c_\mu + \tfrac{1}{g}\partial_\mu \varepsilon_a$$

- intrinsic transformation properties, no reference to the representation under which scalar fields transform
- both a g-independent homogenous term, and a g-dependent inhomogenous term

Gauge group

Local symmetry group = *gauge group*

Groups of interest: direct products $G = \times_i G_i$ of simple or Abelian G_i

Definitions:

- *Abelian* group: all elements commute, corresponding algebra generated by commuting elements ⇒ direct sum of 1-dim. commuting algebras
- *non-Abelian* group: if elements do not commute; corresponding algebra also non-commutative
- *simple/semi-simple* Lie group G: group with simple/semi-simple algebra $\mathfrak{g}$
- *simple* Lie algebra: non-Abelian Lie algebra with no nontrivial *ideal*
  Ideal $\mathfrak{a}$: subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$, $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a}$, left invariant by the whole algebra, $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$
  For a simple Lie algebra the only ideals are $\{0\}$ and the whole algebra
- *semi-simple* Lie algebra: no nontrivial Abelian ideal; equivalently, direct sum of simple algebras

Examples:

- $\mathrm{SU}(N)$ simple; $\mathrm{SO}(N)$ simple;
- $\mathrm{SU}(N) \times \mathrm{SU}(N)$ semisimple; $\mathrm{U}(N) = \mathrm{U}(1) \times \mathrm{SU}(N)$ semisimple;
- $\mathrm{U}(1) \sim \mathrm{SO}(2)$ Abelian

Universality of coupling

Important property of non-Abelian gauge theories: single, universal coupling constant for each simple factor in the gauge group

- coupling constant to matter fields enters transformation properties of gauge field to ensure gauge invariance, uniquely defined by these
- if several matter-field multiplets present, they are all coupled with the same coupling to non-Abelian gauge fields
- Abelian case: redefine transformation laws $\phi \to e^{ig\alpha}\phi$, $A_\mu \to A_\mu - i\partial_\mu\alpha$, can choose different g for different fields
- non-Abelian case: trick does not work, coupling constant reappears in homogenous term in gauge-field transformation law, still constrained to be unique
- each simple/Abelian subalgebra commute with others, independent transformation properties $\sim$ independent couplings

Gauge-field dynamics: field-strength tensor

Gauge symmetry allows dynamics for A_μ^a

$$D_\mu D_\nu \phi = \underbrace{\partial_\mu \partial_\nu \phi - igA_\mu \partial_\nu \phi - igA_\nu \partial_\mu \phi}_{\mu \leftrightarrow \nu \text{ symmetric}} - ig(\partial_\mu A_\nu)\phi + (-ig)^2 A_\mu A_\nu \phi$$

$$[D_\mu, D_\nu]\phi = -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])\phi \equiv -igF_{\mu\nu}\phi$$

Field strength tensor $F_{\mu\nu} = F_{\mu\nu}^a T^a$

Measures curvature of internal space with gauge connection A_μ^a
Similar to Riemann tensor for spacetime connections

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc} A_\mu^b A_\nu^c$$

- first term like in QED
- second term typical only for non-Abelian groups: self-interacting gauge fields, even in the absence of matter
- again same coupling constant

Gauge-field dynamics: Yang-Mills Lagrangian

Transformation properties of $F_{\mu\nu}$

$$F_{\mu\nu} \to F'_{\mu\nu} = U F_{\mu\nu} U^{-1}$$

$F^a_{\mu\nu}$ transform properly as an adjoint multiplet, no inhomogenous term

Infinitesimal transformations

$$\delta F^a_{\mu\nu} = -f_{abc}\varepsilon_b F^c_{\mu\nu}$$

Gauge-invariant kinetic term for gauge fields:

$$\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\tfrac{1}{2}\mathrm{tr}_F F_{\mu\nu} F^{\mu\nu}$$

tr_F: trace in fundamental (defining) representation
$\mathrm{tr}\, t^a_F t^b_F = \frac{1}{2}\delta^{ab}$

$\mathcal{L}_{\mathrm{YM}}$: Yang-Mills(-Shaw) Lagrangian

- mass term $m^2 A_\mu A^\mu$ forbidden by gauge invariance
- (θ-*term*) $\epsilon_{\mu\nu\rho\sigma} F^{a\mu\nu} F^{a\sigma\rho}$ allowed by gauge invariance but forbidden by parity; total derivative, does not affect equations of motion

Not totally irrelevant: plays important role in axial anomaly and mass of η' meson

Fermions

Realistic gauge theory requires fermions, promote Dirac Lagrangian from globally to locally invariant by replacing $\partial_\mu \to D_\mu$

$$\bar{\psi}(i\not{\partial} - m)\psi \to \bar{\psi}(i\not{D} - m)\psi \qquad D_\mu = \partial_\mu - igA^a_\mu t^a$$

$\not{\partial} = \partial_\mu \gamma^\mu$, $\not{D} = D_\mu \gamma^\mu$

Same g but not necessarily same representation t^a of group generators

Lagrangian of general gauge theory

$$\mathcal{L} = \underbrace{-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}}_{A-A\text{ int.}} \underbrace{+\frac{1}{2}(D_\mu\phi)_i(D^\mu\phi)_i}_{\phi-A\text{ int.}} -(\phi) \underbrace{+\bar{\psi}(i\not{D} - m)\psi}_{\psi-A\text{ int.}} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{matter}}$$

Real scalar fields, without loss of generality

FF term:

$$F^a_{\mu\nu}F^{a\mu\nu} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) + \overbrace{2g(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) f_{abc} A^{b\mu} A^{c\nu}}^{\text{cubic interactions}}$$
$$+ \underbrace{g^2 f_{abc} f_{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu}}_{\text{quartic interactions}}$$

Equations of motion

EOM for gauge fields

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\nu)} = -\frac{1}{2}\frac{\partial F^m_{\rho\sigma}}{\partial(\partial_\mu A^a_\nu)} F^{m\rho\sigma} = -F^a_{\mu\nu}$$

$$\frac{\partial \mathcal{L}_{\mathrm{YM}}}{\partial A^a_\nu} = -\frac{1}{2}\frac{\partial F^m_{\rho\sigma}}{\partial A^a_\nu} F^{m\rho\sigma} = -g f_{mab} A^b_\mu F^{m\nu\mu} = g f_{abm} A^b_\mu F^{m\mu\nu}$$

$$= -ig[A_\mu, F^{\mu\nu}]^a$$

$$\frac{\partial \mathcal{L}_{\mathrm{matter}}}{\partial A^a_\nu} \equiv -J^{a\nu}$$

Matrix notation

$$\partial_\mu F^{\mu\nu} - ig[A_\mu, F^{\mu\nu}] = J^\nu$$

In components

$$(D^{(A)}_\mu)^a_b F^{b\mu\nu} = J^{a\nu}$$

$D^{(A)}_\mu$: covariant derivative in the adjoint representation

$$(D^{(A)}_\mu)^a_b = \delta^a_b \partial_\mu + g f_{acb} A^c_\mu = \delta^a_b \partial_\mu - ig(-if_{cab}) A^c_\mu = \delta^a_b \partial_\mu - ig T^{(A)c}_{ab} A^c_\mu$$

Free gauge fields

Gauge invariance $\Rightarrow$ gauge fields are massless vectors, only 2 physical d.o.f.
Free case $g = 0$

$$\partial_\mu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) = 0 \Longrightarrow \Box A^{a\nu} - \partial^\nu \partial \cdot A^a = 0$$

Gauge invariance ($g = 0$): $A^a_\mu \to A^a_\mu + \partial_\mu \Lambda^a$ further conditions must be imposed to obtain a unique solution (besides initial conditions)
General solution in momentum space

$$A^a_\mu(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x} \tilde{A}^a_\mu(p)\,, \qquad \tilde{A}^a_\mu(p) = \int d^4x\, e^{ip\cdot x} A^a_\mu(x)$$

$$(p^2 \delta^\nu{}_\mu - p^\nu p_\mu)\tilde{A}^{a\mu}(p) = 0$$

Decompose $\tilde{A}^a_\mu(p)$ on a (p-dependent) complete basis of four-vectors

$$\tilde{A}^{a\mu}(p) = a^i(p)\varepsilon^\mu_i(p) + b(p)\tilde{p}^\mu + c(p)p^\mu$$

$$p^\mu = (E, \vec{p}\,) \qquad \tilde{p}^\mu = (-E, \vec{p}\,)$$

$$\varepsilon^\mu_i(p) = (0, \vec{\varepsilon}_i(\vec{p}\,)) \qquad \vec{\varepsilon}_i(\vec{p}\,) \cdot \vec{p} = 0 \qquad \vec{\varepsilon}_i(\vec{p}\,) \cdot \vec{\varepsilon}_j(\vec{p}\,) = \delta_{ij}$$

No relation assumed between E and $\vec{p}$

Free gauge fields (contd.)

$p \cdot \varepsilon_i = \tilde{p} \cdot \varepsilon_i = 0$ p, $\tilde{p}$ linearly independent (for $E, \vec{p} \neq 0$)

$$p^2 \left[a^i(p)\varepsilon_i^\nu(p) + b(p)\tilde{p}^\nu + c(p)p^\nu\right] - p^\nu \left[b(p)p \cdot \tilde{p} + c(p)p^2\right] = 0$$
$$p^2 \left[a^i(p)\varepsilon_i^\nu(p) + b(p)\tilde{p}^\nu\right] - p \cdot \tilde{p}p^\nu b(p) = 0$$

Term proportional to $c(p)$ drops out: completely arbitrary, unphysical
Contracting with $\tilde{p}$

$$[p^2\tilde{p}^2 - (p \cdot \tilde{p})^2]b(p) = 0 \qquad [(E^2 - \vec{p}^{\,2})^2 - (E^2 + \vec{p}^{\,2})^2]b(p) = 0$$
$$4E^2\vec{p}^{\,2}b(p) = 0 \qquad \Longrightarrow b(p) = 0$$

Contracting with ε_i

$$p^2 a^i(p) = 0 \Longrightarrow p^2 = 0$$

$$\tilde{A}^{a\mu}(p) = a^i(p)\varepsilon_i^\mu(p) + c(p)p^\mu \qquad p^2 = 0$$

for arbitrary $a^i(p)$, $c(p)$ with $p^2 = 0$, but c unphysical
$\Rightarrow$ two degrees of freedom corresponding to transverse polarisations $\varepsilon_i^\mu(\vec{p}\,)$

Higgs mechanism: $G = \mathrm{SO}(2) \sim \mathrm{U}(1)$

If the spontaneously broken symmetry is gauged there are no Goldstones

Use U(1) version, two real scalars combined into a single complex field
Abelian gauge theory of a complex field with charge $-e$

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^*(D_\mu\varphi) - \tfrac{\lambda}{2}(2\varphi^*\varphi - a^2)^2 \qquad \lambda > 0$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu \qquad D_\mu = \partial_\mu + ieA_\mu$$

Minimal energy:

- $F_{\mu\nu} = 0 \Rightarrow A_\mu = 0$ (up to gauge transformations)
- $\varphi(x) = \varphi_0$ with $\varphi_0^*\varphi_0 = \frac{a^2}{2}$

Choose the ground state $\varphi_0 = \frac{a}{\sqrt{2}}$, $A_\mu = 0$, parameterise

$$\varphi(x) = \frac{1}{\sqrt{2}}(a + \eta(x))e^{i\frac{\theta(x)}{a}}$$

Need to fix the gauge, whether to solve the classical Cauchy problem or to quantise the theory in QFT

Higgs mechanism: $G = \mathrm{SO}(2) \sim \mathrm{U}(1)$ (contd.)

By a U(1) gauge transformation we can set φ real (*unitarity gauge*)

$$\varphi(x) \to e^{-i\frac{\theta(x)}{a}}\varphi(x) = \frac{1}{\sqrt{2}}(a + \eta(x))$$

How does A_μ transform?

$$A_\mu(x) \to A_\mu(x) - \frac{i}{-e}(\partial_\mu e^{-i\frac{\theta(x)}{a}})e^{i\frac{\theta(x)}{a}} = A_\mu(x) + \frac{1}{ea}\partial_\mu\theta(x) \equiv B_\mu$$

Set $G_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu$, $\bar{D}_\mu = \partial_\mu + ieB_\mu$

$$\begin{aligned}
\mathcal{L}_{\mathrm{UG}} &= -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \left|\bar{D}_\mu \frac{a+\eta}{\sqrt{2}}\right|^2 - \frac{\lambda}{2}((a+\eta)^2 - a^2)^2 \\
&= -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}\left|\partial_\mu\eta + ieB_\mu(a+\eta)\right|^2 - \frac{\lambda}{2}(\eta^2 + 2a\eta)^2 \\
&= -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}e^2B_\mu B^\mu(a+\eta)^2 - \frac{\lambda}{2}(\eta^2 + 2a\eta)^2
\end{aligned}$$

- a mass term has appeared for the gauge boson B_μ, with $m_B = ea$
- the "would-be" Goldstone mode $\theta(x)$ has disappeared, becoming the longitudinal component of the massive vector boson B_μ
- the other scalar field η, the *Higgs field*, is massive with $m_\eta = 2a\sqrt{\lambda}$