Weak Interactions

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Goldstone bosons (contd.)

1. $H = G$: ground state invariant under whole symmetry group $\Rightarrow \phi_0$ unique, $\mathcal{M} = \{\phi_0\}$, symmetry group G *unbroken*
2. $H = \{e\}$ (the neutral element): the ground state is not invariant under any transformation, and G is *completely broken*
3. $H \subset G$ is a proper (nontrivial) subgroup: G is *broken down to H*

Choose generators $\{T^1, \dots T^n\}$ of G to contain generators of H:

- first $n' = \dim H$: $\{T^1, \dots, T^{n'}\}$ generators of H

  Span the Lie algebra of H which is a subalgebra of the Lie algebra of G

- last $n - n'$: $\{T^{n'+1}, \dots, T^n\}$ span rest of G-algebra, "generate" G/H

  Coset space G/H in general is not a Lie group (unless H is a normal subgroup $H = gHg^{-1}$)

  It is though a manifold of dimension $n - n'$, with $\{T^{n'+1}, \dots, T^n\}$ a basis of its tangent space at the "identity" point $eH = H \in G/H$

- $H\phi_0 = \phi_0 \Rightarrow T^a\phi_0 = 0$ for $a = 1, \dots, n'$
- $T^a\phi_0 \neq 0$ for $a = n'+1, \dots, n$, and $\sum_{a=n'+1}^{n} c_a T^a \phi_0 = 0 \Rightarrow c_a = 0$ (otherwise $c_a T^a$ would belong to algebra of H against hypothesis)

$G = \mathrm{SO}(2)$

Doublet of scalar fields $\phi = \binom{\phi_1}{\phi_2}$ in the defining representation ($N = 2$)

$n = \dim \mathrm{SO}(N) = \frac{N(N-1)}{2} \underset{N=2}{\Rightarrow} n = 1$ generators

Group generator: $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$(\phi) = \frac{\lambda}{2}(\phi_1^2 + \phi_2^2 - a^2)^2$$

Mexican-hat potential

$$\lambda, a \in \mathbb{R} \quad \lambda > 0$$

Ground state manifold $(\phi) = 0$

$$\mathcal{M} = \{\phi \mid \phi_1^2 + \phi_2^2 = a^2\} \sim S^1$$

$$\forall \phi^{(0)} \in \mathcal{M},\ \mathcal{M} = G\phi^{(0)}$$

- no $\phi \in \mathcal{M}$ left invariant by any rotation
  $\Rightarrow H = \{e\}$, symmetry completely broken
- G has no proper subgroups, symmetry unbroken or completely broken
- only solution to $T\phi = 0$ (invariant under SO(2)): $\phi_1 = \phi_2 = 0 \notin \mathcal{M}$

$G = \mathrm{SU}(2)$

Doublet of complex fields in defining representation ($N = 4,\ n = 3$)

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \begin{aligned} \psi_1 &= \phi_1 + i\phi_2 \\ \psi_2 &= \phi_3 + i\phi_4 \end{aligned}$$

$$(\Psi) = \lambda \left(\Psi^\dagger \Psi - a^2 \right)^2 = \lambda \left(\psi_1^* \psi_1 + \psi_2^* \psi_2 - a^2 \right)^2 = \lambda \left(\textstyle\sum_{i=1}^4 \phi_i^2 - a^2 \right)^2$$

$\lambda, a \in \mathbb{R},\ \lambda > 0$

is $\mathrm{SU}(2)$ invariant: $g \in \mathrm{SU}(2)$ if $(g\Psi)^\dagger (g\Psi) = \Psi^\dagger \Psi$ by definition

$$g = \begin{pmatrix} c & d \\ -d^* & c^* \end{pmatrix} \quad |c|^2 + |d|^2 = 1 \Longrightarrow \mathrm{SU}(2) \sim S^3$$

$$\mathcal{M} = \left\{ \phi \mid \textstyle\sum_{i=1}^4 \phi_i^2 = a^2 \right\} \sim S^3 \sim \mathrm{SU}(2)$$

$\mathcal{M}$ diffeomorphic to G and to G/H: $\Psi \sim V = \frac{1}{a} \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \end{pmatrix}$, $g\Psi \sim UV$

- $G = \mathcal{M} = G/H \Rightarrow H = \{e\} \Rightarrow$ symmetry completely broken
- no $\phi \in \mathcal{M}$ left invariant by any subgroup of G: $UV = V \Rightarrow U = \mathbf{1}$

$G = \mathrm{SO}(3)$

Triplet of real fields $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$ in defining rep. (= adjoint rep. of SU(2))

$$(\phi) = \lambda \left(\sum_{i=1}^{3} \phi_i^2 - a^2 \right)^2$$

Ground-state manifold is S^2

$$\mathcal{M} = \left\{ \phi \mid \sum_{i=1}^{3} \phi_i^2 = a^2 \right\} \sim S^2$$

Choose $\phi_0 = (0, 0, a)$, stability group $H = \{ h(\alpha)\,, \alpha \in [0, 2\pi) \}$

$$h(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Nontrivial $\mathrm{SO}(2) \sim U(1)$ stability group, coset space = g-state manifold

$$G/H = \mathrm{SO}(3)/\mathrm{SO}(2) = S^2 = \mathcal{M}$$

Symmetry breaking pattern depends on group and choice of representation

Goldstone theorem

Theorem: If G with $\dim\, G = n$ is broken down to H with $\dim\, H = n'$, then there are $n - n'$ massless bosons (*Goldstone bosons*) in the spectrum, one per broken symmetry generator ("generator" of the coset space)

Proof: Since $(\phi) = (g\phi)$, under infinitesimal $g = e^{\epsilon\cdot T} \simeq \mathbf{1} + \epsilon \cdot T$

$$(\phi) = (g\phi) = (\phi + \epsilon \cdot T\phi) \simeq (\phi) + \frac{\partial}{\partial \phi_i}(\phi)\,(\epsilon \cdot T)_{ij}\,\phi_j$$

ϵ_a small but arbitrary $\Rightarrow$ for any field configuration ϕ

$$\frac{\partial}{\partial \phi_i}(\phi)\,T^a_{ij}\phi_j = 0$$

Take one more derivative wrt ϕ_k

$$\frac{\partial^2}{\partial \phi_k \partial \phi_i}(\phi)\,T^a_{ij}\phi_j + \frac{\partial}{\partial \phi_i}(\phi)\,T^a_{ik} = 0$$

Set now $\phi = \phi_0 \in \mathcal{M}$, $\frac{\partial}{\partial \phi_i}(\phi_0) = 0$ since minimum of potential $\Rightarrow$

$$\frac{\partial^2}{\partial \phi_k \partial \phi_i}(\phi_0)\,T^a_{ij}\phi_{0\,j} = 0$$

Goldstone theorem (contd.)

Coefficients of the quadratic part of the potential

$$M^2_{ki} \equiv \frac{\partial^2}{\partial\phi_k \partial\phi_i}(\phi_0)$$

Mass matrix of fluctuations $\tilde{\phi}$ of $\phi = \phi_0 + \tilde{\phi}$ around ground state ϕ_0

$$(\phi) = (\phi_0 + \tilde{\phi}) = \tfrac{1}{2}\tilde{\phi}_k M^2_{ki}\tilde{\phi}_i + \text{higher orders}$$

Eigenvalue equation for eigenvalue zero

$$M^2_{ki} T^a_{ij} \phi_{0\,j} = 0$$

If $T^a\phi_0 \neq 0 \Rightarrow$ zero-eigenmode of $M^2 \sim$ massless fluctuation

- $T^a\phi_0 = 0$ for $a = 1, \ldots, n'$, uninteresting
- $T^a\phi_0 \neq 0$ for $a = n'+1, \ldots, n$ and linearly independent

$\Rightarrow n - n'$ independent massless modes, one per broken generator of G $\square$

Goldstone theorem (contd.)

Goldstone modes can be taken as

$$\tilde{\phi}_i T^a_{ij} \phi_{0\,j} \qquad a = n' + 1, \ldots, n$$

- scalar product with any $T^a \phi_0$ removes from $\tilde{\phi}_i$ contributions of massive modes (eigenvectors of symmetric matrix M^2 with nonzero eigenvalue, orthogonal to massless modes)
- linearly independent combinations: if $\sum_{a=n'+1}^{n} c_a \tilde{\phi}_i T^a_{ij} \phi_{0\,j} = 0$ for all $\tilde{\phi} \Rightarrow c_a T^a \phi_0 = 0$, contradicts hypotheses

If for a given configuration, $\tilde{\phi}$, one has $\tilde{\phi}_i T^a_{ij} \phi_{0\,j} = 0 \ \forall\, a$
$\Rightarrow$ no contribution to $\tilde{\phi}$ from Goldstone modes

Goldstone theorem: example

$G = \mathrm{SO}(2)$, doublet of real scalars

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{\lambda}{2}(\phi_1^2 + \phi_2^2 - a^2)^2$$

Set $\phi = \phi_0 + \tilde{\phi}$, choose ground state $\phi_0 = \binom{a}{0}$

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu\tilde{\phi}_1\partial^\mu\tilde{\phi}_1 + \frac{1}{2}\partial_\mu\tilde{\phi}_2\partial^\mu\tilde{\phi}_2 - \frac{\lambda}{2}\left((a+\tilde{\phi}_1)^2 + \tilde{\phi}_2^2 - a^2\right)^2 \\
&= \frac{1}{2}\partial_\mu\tilde{\phi}_1\partial^\mu\tilde{\phi}_1 + \frac{1}{2}\partial_\mu\tilde{\phi}_2\partial^\mu\tilde{\phi}_2 - \frac{\lambda}{2}\left(\tilde{\phi}_1(2a+\tilde{\phi}_1) + \tilde{\phi}_2^2\right)^2 \\
&= \frac{1}{2}\partial_\mu\tilde{\phi}_1\partial^\mu\tilde{\phi}_1 + \frac{1}{2}\partial_\mu\tilde{\phi}_2\partial^\mu\tilde{\phi}_2 - \frac{\lambda}{2}\left(4a^2\tilde{\phi}_1^2 + 4a\tilde{\phi}_1(\tilde{\phi}_1^2 + \tilde{\phi}_2^2) + (\tilde{\phi}_1^2 + \tilde{\phi}_2^2)^2\right)
\end{aligned}$$

$\tilde{\phi}_1 \to m_1^2 = 4\lambda a^2$
$\tilde{\phi}_2 \to m_2^2 = 0$
$\tilde{\phi}_i T_{ij}\phi_{0\,j} = -a\tilde{\phi}_2$ is the Goldstone mode

Goldstone theorem: example (contd.)

Alternative: $\mathcal{L}$ as Lagrangian of complex field with U(1) internal symmetry

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - \frac{\lambda}{2}\left(2\varphi^*\varphi - a^2\right)^2 \qquad \varphi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$$

Ground state $\varphi_0 = \frac{a}{\sqrt{2}}$, parameterise fluctuations as

$$\varphi(x) = \frac{1}{\sqrt{2}}\rho(x)e^{i\frac{\theta(x)}{a}} = \frac{1}{\sqrt{2}}(a + \eta(x))e^{i\frac{\theta(x)}{a}}$$

$$\mathcal{L} = \overbrace{\frac{1}{2}\partial_\mu\eta\partial^\mu\eta - 2a^2\lambda\eta^2 + \frac{1}{2}\partial_\mu\theta\partial^\mu\theta}^{\text{free part}} \overbrace{-\frac{\lambda}{2}\left(4a\eta^3 + \eta^4\right) + \left(\frac{\eta}{a} + \frac{\eta^2}{2a^2}\right)\partial_\mu\theta\partial^\mu\theta}^{\text{interaction part}}$$

η: massive field, $m_\eta^2 = 4\lambda a^2$ $\qquad$ θ: massless Goldstone mode

Symmetry $\theta \to \theta + c$ for constant c guarantees mass term not generated in higher orders of perturbation theory

Ground state manifold: $|\varphi| = \rho = a$

- change of phase ($\sim$ fluctuation in θ) costs no energy (system moves along valley of minima)
- change of amplitude ($\sim$ fluctuation in η) displaces system from minimum, encounters inertia corresponding to nonzero mass

Gauge theories

Gauge theories have *local* symmetry rather than just a global one

Given a system of fields $\phi_i(x)$, internal transformations

- *global* transformation: independent of x, $\phi_i(x) \to S_{ij}\phi_j(x)$
- *local* transformation: x-dependent, $\phi_i(x) \to S_{ij}(x)\phi_j(x)$

Ex.: EM interactions have local U(1) symmetry under x-dependent change of electron field phase (so of electron and positron states)

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \qquad \bar{\psi}(x) \to e^{-i\alpha(x)}\bar{\psi}(x)$$

- What is the motivation for local symmetries?
- What are the consequences?