Weak Interactions

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Limitations of the four-fermion theory (contd.)

When does the effective theory break down?
Problems with unitarity at high energy already at tree level

Can be cured only using higher orders of perturbation theory
$\Rightarrow$ back to problem of non-renormalisability

$e\,\nu_e$ elastic scattering:

- tree-level amplitude polynomial in four-momenta (no propagators) $\Rightarrow$ polynomial in $\cos\theta_{\mathrm{CM}}$

θ_{CM}: angle between incoming/outgoing trajectories in CM frame

- partial-wave expansion of the amplitude contains only a finite number of partial waves, with amplitude f_J
- from dimensional analysis: $\sigma_{\mathrm{tot}} \sim G^2 s$ at high energy
- from partial-wave expansion: $\sigma_{\mathrm{tot}} \propto p^{-1} \sum (2J+1)|f_J|^2$
- from unitarity of the S-matrix: $|f_J|^2 \leq \mathrm{Im} f_J$

p: magnitude of the initial momenta in the CM frame

$\boxed{\Rightarrow \text{one of partial wave amplitudes will eventually violate unitarity bound}}$

Limitations of the four-fermion theory (contd.)

$e\,\nu_e$ elastic scattering: considering only charged current for simplicity

- $J = 0$ partial wave: $f_0 = \frac{Gs}{2\sqrt{2}\pi}$
- unitarity bound: $|\mathrm{Re} f_0| \le \frac{1}{2}$
- tree-level amplitude is real $\Rightarrow \frac{Gs}{\sqrt{2}\pi} \le 1$

$\Rightarrow$ unitarity violated for $s > \frac{\sqrt{2}\pi}{G} \simeq (600\,\mathrm{GeV})^2$, breakdown of the theory

How to improve the situation? Replace four-fermion interaction with exchange of massive intermediate vector boson (Yukawa, 1930s)

Differential cross section (for $|t| \ll m_W$, m_W: mass of IVB):

$$\frac{d\sigma}{d|t|}\Big|_{\mathrm{FF}} = \frac{G^2}{\pi} \longrightarrow \frac{d\sigma}{d|t|}\Big|_{\mathrm{IVB}} = \frac{G^2}{\pi}\frac{m_W^4}{(m_W^2+|t|)^2}$$

Boson exchange $\rightarrow$ propagator $\frac{1}{m_W^2+|t|^2}$:

- all partial waves are present
- cuts off the contribution of large tranferred momentum

Limitations of the four-fermion theory (contd.)

For the total cross section, integrate over $0 \le |t| \le s$ (ignore e mass)

$$\sigma|_{\rm FF} = \int_0^s d|t| \, \frac{d\sigma}{d|t|}\Big|_{\rm FF} = \frac{G^2}{\pi} s \longrightarrow \sigma|_{\rm IVB} = \int_0^s d|t| \, \frac{d\sigma}{d|t|}\Big|_{\rm IVB} = \frac{G^2 m_W^2}{\pi} \frac{s}{s+m_W^2}$$

In the IVB case, total cross section σ

- rises linearly with s at low energy
- approaches the constant $\frac{G^2 m_W^2}{\pi}$ at high energy
- unitarity respected, and unitarity bound becomes a bound on m_W

Coupling $g_W^2 = G m_W^2$ in IVB theory is dimensionless: renormalisable?

- No: propagator contains $\frac{p_\mu p_\nu}{p^2 m_W^2} \Rightarrow$ problematic high-energy behaviour
- No problem if IVB coupled to conserved current (would give no contrib.), but V^μ $[m_e - m_{\nu_e} \neq 0]$, A^μ $[m_e + m_{\nu_e} \neq 0]$ not conserved

Even if $m_{e,\nu_e} = 0$, so weak current conserved:

- massive vector bosons must be electrically charged $\Rightarrow$ non-renormalisable EM interaction
- problems with unitarity in the boson-boson cross section, can be cured if a further, neutral boson is introduced

Way out: start with massless bosons (renormalisable) and generate mass dynamically (renormalisability remains)

Massive vector bosons

Proca Langrangian for free massive vector particles:

$$\mathcal{L}_{\mathrm{Proca}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 W_\mu W^\mu \qquad F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$$

Equations of motion:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}_{\mathrm{Proca}}}{\partial W_\mu} - \partial_\nu \frac{\partial \mathcal{L}_{\mathrm{Proca}}}{\partial(\partial_\nu W_\mu)} \\ \Rightarrow 0 &= m^2 W^\mu - \partial_\nu(-F^{\nu\mu}) \\ \Rightarrow 0 &= (\Box + m^2) W^\mu - \partial^\mu \partial_\nu W^\nu \end{aligned}$$

Taking the divergence

$$(\Box + m^2)\partial_\mu W^\mu - \Box \partial_\nu W^\nu = m^2 \partial_\mu W^\mu = 0 \Rightarrow \partial_\mu W^\mu = 0$$

Plug back

$$\begin{aligned} (\Box + m^2) W^\mu &= 0 \\ \partial_\mu W^\mu &= 0 \end{aligned}$$

Massive vector bosons (contd.)

EOM most easily solved in momentum space

$$W^\mu(x) = \int d\Omega_p \sum_{j=1}^3 \left\{\varepsilon_j^\mu(\vec{p}\,)e^{-ip\cdot x}a_j(\vec{p}\,) + \varepsilon_j^{\mu *}(\vec{p}\,)e^{ip\cdot x}b_j^\dagger(\vec{p}\,)\right\}$$

$d\Omega_p$: invariant phase-space measure, $p^0 = \sqrt{\vec{p}^{\,2} + m^2}$

Polarisation vectors $\varepsilon_j^\mu(\vec{p})$

- satisfy $p \cdot \varepsilon_j(\vec{p}\,) = 0$, three independent solutions, e.g.

$$\varepsilon_{1,2}^\mu = (0, \vec{s}_{1,2}\,) \qquad \vec{p}\cdot\vec{s}_{1,2} = 0 \qquad \vec{s}_i\cdot\vec{s}_j = \delta_{ij} \;\; i,j = 1,2$$

$$\varepsilon_3^\mu = \tfrac{1}{m}(|\vec{p}\,|, p^0\hat{p}) \qquad \hat{p} = \tfrac{\vec{p}}{|\vec{p}|}$$

- longitudinal polarisation $\varepsilon_3^\mu = \frac{p^\mu}{m} + \frac{m}{p^0+|\vec{p}\,|}(-1,\hat{p})$
- for our choice $\varepsilon_i^\mu = \varepsilon_i^{\mu *}$; in general one imposes

$$\varepsilon_i \cdot \varepsilon_j^* = -\delta_{ij} \qquad i,j = 1,2,3$$

- orthonormal basis of 3d space transverse to four-momentum p

$$\sum_{j=1}^3 \varepsilon_j^\mu(\vec{p}\,)\varepsilon_j^{\nu *}(\vec{p}\,) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}$$

Massive vector bosons (contd.)

Class of interacting theories: massive vector boson coupled to current j^μ

$$\mathcal{L} = \mathcal{L}_{\mathrm{Proca}} - W_\mu j^\mu$$

EOM: replace $m^2 W^\mu \to m^2 W^\mu - j^\mu$

$$(\Box + m^2) W^\mu - \partial^\mu \partial_\nu W^\nu = j^\mu$$

Taking the divergence

$$m^2 \partial_\mu W^\mu = \partial_\mu j^\mu$$

Plug back

$$(\Box + m^2) W^\mu = \left(\eta^{\mu\nu} + \tfrac{\partial^\mu \partial^\nu}{m^2}\right) j_\nu$$

Green's function (propagator) $D^{\mu\nu}(x)$ connects solution to the current

$$W^\mu(x) = \int d^4y \, D^{\mu\nu}(x-y) j_\nu(y)$$

$$D^{\mu\nu} = \left(\eta^{\mu\nu} + \tfrac{\partial^\mu \partial^\nu}{m^2}\right) \frac{1}{\Box + m^2}$$

Massive vector bosons (contd.)

Momentum space propagator

$$\tilde{D}^{\mu\nu} = \frac{-\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}}{p^2 - m^2}$$

We are ignoring the choice of prescription to deal with the pole at $p^2 = m^2$

Second term can lead to bad high energy behaviour: in momentum space $\tilde{D}^{\mu\nu}$ couples to Fourier transform $\tilde{\jmath}^\mu$ of $j^\mu \Rightarrow p_\mu \tilde{\jmath}^\mu$ from the second term

- conserved current $\partial_\mu j^\mu = 0 \Rightarrow p_\mu \tilde{\jmath}^\mu = 0$, potentially dangerous term has no effect
- only second term of longitudinal polarisation vector ε_3 contributes to Feynman diagrams, no troublesome high-energy behaviour

Can give the photon a mass without spoiling renormalisability despite loss of gauge invariance

- if $p_\mu \tilde{\jmath}^\mu \neq 0 \Rightarrow$ cannot drop $p^\mu p^\nu$ term, theory non renormalisable due to its bad high-energy behaviour

$\Rightarrow$ cannot give mass to IVB "by hand" for use in weak interactions

Spontaneous symmetry breaking: Goldstone's theorem

Trick used to give mass to IVB combines gauge invariance and massless scalar particles appearing in a theory with spontaneously broken symmetry

- What is spontaneous breaking of a symmetry?
- Why do massless scalars (*Goldstone bosons*) appear in the spectrum?

Massless modes from breaking of a global continuous symmetry due to the non-invariance of the vacuum (ground state)

- EOM of a system being symmetric does not mean that every solution should be symmetric
- if minimal-energy solution not symmetric then not unique: selecting one solution the system breaks symmetry (*spontaneous breaking*)
- if symmetry is continuous, moving from one minimal-energy solution to another costs no energy $\Rightarrow$ massless modes
- *Goldstone theorem*: one massless mode for every generator of the symmetry broken by the vacuum

Spontaneous breaking of a gauge symmetry

If "broken" symmetry is a gauge (local) symmetry, "would-be" Goldstone modes "absorbed" by gauge bosons corresponding to the broken generators as their longitudinal (zero helicity) modes, making them massive

A gauge symmetry cannot be broken (Elitzur's theorem): what can be broken is the corresponding global symmetry

Q. Why is Goldstone theorem ineffective when the symmetry is local?

A. Quantisation of a gauge theory requires gauge fixing (which breaks the local symmetry explicitly)

- using "physical" gauges (e.g., Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, axial gauge $A_3 = 0$): if we ask for only physical states in Hilbert space, then gauge choice cannot respect Lorentz covariance (required by theorem)
- using covariant gauges (e.g., Lorenz gauge $\partial_\mu A^\mu = 0$): theorem applies, but Hilbert space contains unphysical, negative-norm states corresponding to remaining gauge modes (gauge fixing only partial); Goldstone mode is such a gauge mode, decoupled from physical states

Goldstone bosons

N real scalar fields $\phi_i(x)$, treat classically

Path-integral quantisation in mind

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i - (\phi)$$

Potential (ϕ): polynomial, includes mass (quadratic) terms, at most 4th-order for renormalisability

Assume $\{\phi_i\}$ basis of rep. space of N-dim unitary rep. of Lie group G

$$\phi_i(x) \to (g\phi)_i(x) = D_{ij}(g)\phi_j(x) \qquad D(g) = e^{\epsilon_a(g)T^a} \quad \epsilon_a(g) \in \mathbb{R}$$

$D(g)$: $N \times N$ unitary matrices, $D(g_1)D(g_2) = D(g_1 g_2)\ \forall\, g_{1,2} \in G$

T^a: $N \times N$ real antisymmetric matrices, $a = 1, \dots n = \dim G$, provide representation of group algebra $[T^a, T^b] = -f^{abc}T^c$

Real fields $\Rightarrow$ real representation $\Rightarrow$ orthogonal representation

N complex scalar fields $\varphi_i = \varphi_i^R + i\varphi_i^I \Rightarrow 2N$ real fields
unitary $N \times N$ representation $\delta\varphi = i\epsilon_a t^a \varphi \Rightarrow 2N \times 2N$ orthogonal representation $\delta\phi = \epsilon_a T^a \phi$

$$T^a = \begin{pmatrix} -\mathrm{Im}\,t^a & -\mathrm{Re}\,t^a \\ \mathrm{Re}\,t^a & -\mathrm{Im}\,t^a \end{pmatrix}$$

Goldstone bosons (contd.)

Assume $\mathcal{L}$ invariant under G, $(g\phi) = (\phi)$ (G is an internal symmetry)
Energy functional: bounded from below if is, set minimum to 0

$$E[\phi] = \int d^3x \left[\tfrac{1}{2}\partial_0\phi_i\partial_0\phi_i + \tfrac{1}{2}\vec{\nabla}\phi_i \cdot \vec{\nabla}\phi_i + (\phi)\right] \geq 0$$

Ground state (*vacuum state* in relativistic QFT): state of minimal (zero) energy = constant field configuration $\phi_i(x) = \phi_{0\,i}$ with $(\phi_0) = 0$

Minimal E since no contribution from derivative terms, and minimal potential since min = 0

If $\exists g$, $g\phi_0 \neq \phi_0 \Rightarrow$ more than one ground state ($(g\phi_0) = (\phi_0) = 0$)

$\mathcal{M}$: manifold of ground states $\mathcal{M} = \{\phi_0 \mid (\phi_0) = 0\}$

G-orbit of $\phi_0 \in \mathcal{M}$: $G\phi_0 = \{g\phi_0 \mid g \in G\}$

By construction $G\mathcal{M} = \mathcal{M}$; we further assume $\mathcal{M} = G\phi_0$ for any ϕ_0 i.e., any ϕ_0' can be reached from any ϕ_0 by some g

If $\mathcal{M}$ contains more than one state, then we say that the symmetry G is broken: any $\phi_0 \in \mathcal{M}$ will not be left invariant by all g

Goldstone bosons (contd.)

Stability group H: subgroup of G that leaves ground state invariant

$$H = \{h \in G \mid h\phi_0 = \phi_0\}$$

If $h_{1,2} \in H$, $h_1 h_2 \phi_0 = \phi_0 \Rightarrow h_1 h_2 \in H$; $e \in H$;
if $h \in H$, $h^{-1}\phi_0 = h^{-1}h\phi_0 = \phi_0 \Rightarrow h^{-1} \in H$

Stability groups defined using different ϕ_0 are isomorphic since $\mathcal{M} = G\phi_0$

Unbroken part of the symmetry group

(Right) cosets: $gH = \{gh \,|\, h \in H\}$

- correspond uniquely to equivalence class wrt relation $g_1 \sim g_2$ if $g_1 = g_2 h$ for some $h \in H$ (elements of G modulo elements of H);
- invariant under right multiplication by $h \in H$

(Right) coset space G/H: set of cosets/equivalence classes gH

Choosing some ϕ_0

- for any $\phi_0' \in \mathcal{M}$ can write $\phi_0' = g\phi_0$ non-uniquely ($gh\phi_0 = g\phi_0$, $h \in H$)
- $g\phi_0$ corresponds *uniquely* to a coset $gH \Rightarrow \mathcal{M} = G/H$