Particle physics: practice 1
Relativistic Kinematics

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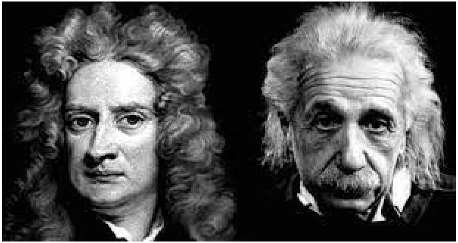
Special relativity

- Newton to Einstein: the problem of causality. Information can't be transferred instantaneously.



- "Special": no acceleration. "General": acceleration $\equiv$ gravity.
- HEP can be thought of as the ultimate testbed for SR, since it's bread and butter for us (all particles relativistic).
- https://pdg.lbl.gov/2019/reviews/rpp2019-rev-kinematics.pdf

Relativistic kinematics: Minkowski space

Relativistic theories are conveniently formulated in Minkowski space

$$\text{Minkowski space} = \mathbb{R}^4 + \text{Minkowski (pseudo)metric}$$

Euclidean space = $\mathbb{R}^3$ + Euclidean metric

Distance between points in E. space: $d(\vec{x},\vec{y}) = (\vec{x}-\vec{y})^2 = (\vec{x}-\vec{y})_i(\vec{x}-\vec{y})_j\delta_{ij}$

Latin indices $1,\ldots,3$, sum over repeated indices understood

Invariant under translations $\vec{x}\to\vec{x}+\vec{a}$ and rotations $\vec{x}\to R\vec{x}$

Point in Minkowski space (=*event*): X^μ, $\mu = 0,1,2,3$

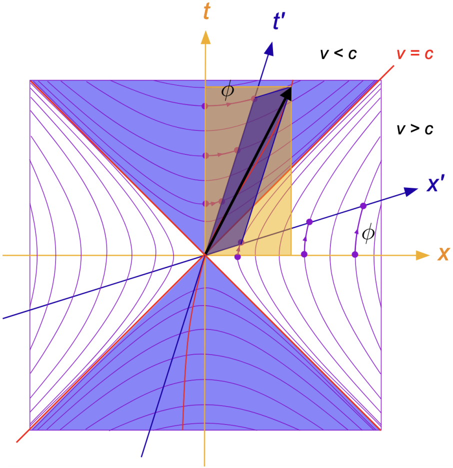
$$X^\mu = (ct,\vec{x}) = (t,\vec{x})$$

In Minkowski space distances replaced by *interval*

$$\begin{aligned}\Delta s^2 &\equiv (X-Y)^2 \equiv (X-Y)^\mu(X-Y)^\nu g_{\mu\nu} \equiv (X-Y)^\mu(X-Y)_\mu \\ &= (X^0-Y^0)^2 - (\vec{X}-\vec{Y})^2\end{aligned}$$

Minkowski metric tensor: $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

Lorentz transformation in Minkowski space



- Boost will shift the event along the hyperbolas in Minkowski space.

The lightcone

Contravariant vectors: $X^\mu = (X^0, \vec{X})$
Covariant vectors: $X_\mu = g_{\mu\nu} X^\nu = (X^0, -\vec{X})$
Indices lowered by $g_{\mu\nu}$ and raised by $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

$g^{\mu\nu}$ defined by $g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu$

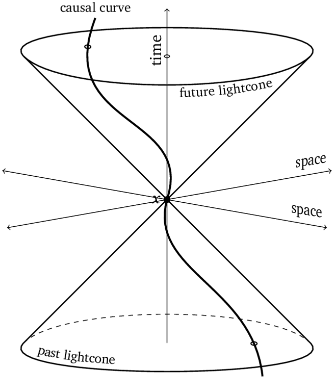
Minkowski scalar product $X \cdot Y \equiv X^\mu Y^\nu g_{\mu\nu} = X^\mu Y_\mu = X^0 Y^0 - \vec{X} \cdot \vec{Y}$

$\vec{X} \cdot \vec{Y}$: three-dimensional Euclidean scalar product

Interval is not a distance because it is not positive-definite:

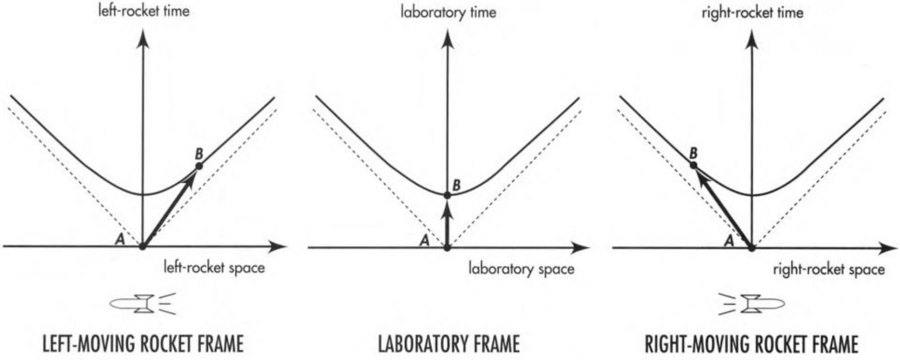
- $\Delta s^2 > 0$ *timelike* interval — $X^2 > 0$ timelike vector
- $\Delta s^2 < 0$ *spacelike* interval — $X^2 < 0$ spacelike vector
- $\Delta s^2 = 0$ *lightlike* or *null* interval — $X^2 = 0$ lightlike or null vector

The lightcone (contd.)



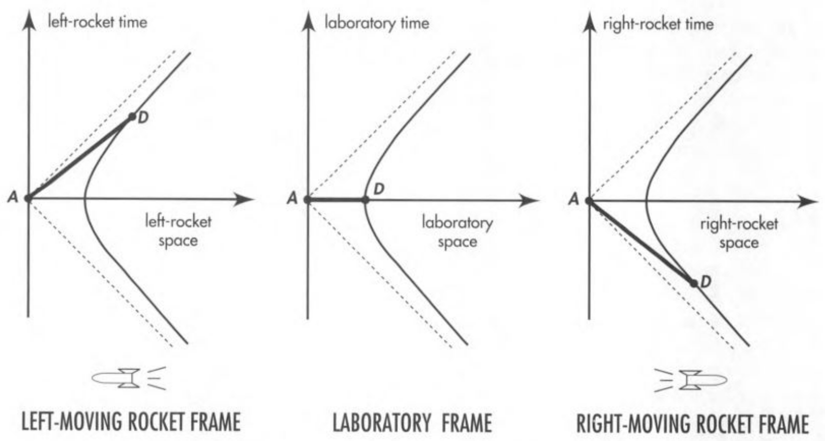
- For a fixed event X
  - $(Y-X)^2 = 0$, $Y^0 - X^0 > 0$: *forward (future) lightcone* of X
  - $(Y-X)^2 = 0$, $Y^0 - X^0 < 0$: *backward (past) lightcone* of X
  - $(Y-X)^2 > 0$, $Y^0 - X^0 > 0$: future of X (inside future lightcone)
  - $(Y-X)^2 > 0$, $Y^0 - X^0 < 0$: past of X (inside past lightcone)

TIMELIKE VECTORS, $X^2 > 0$



- Causally connected. Can go to a frame where the two events occur at the same point in space but at two times. Unique time ordering.
- This special frame Δt is called *proper time*. Eg. decay of an unstable particle in the particle's restframe.

Spacelike vectors, $X^2 < 0$



- Can go to a frame where two events at the same time occur at two places. But this simultaneity is frame-dependent. No unique time-ordering.
- Eg., virtual particles have spacelike 4-momentum

Lorentz transformations

Principles of special relativity:

- homogeneity and isotropy of space
- equivalence of all inertial reference frames

  = travelling at a relative constant speed
- constancy of speed of light

$\Rightarrow$ equivalent frames are related by a Lorentz transformation $X' = \Lambda X$: linear transformation that leaves every interval invariant

$$(X' - Y')^2 = (X - Y)^2 \qquad \forall X, Y$$
$$\Rightarrow X'^2 + Y'^2 - 2X' \cdot Y' = X^2 + Y^2 - 2X \cdot Y \qquad \forall X, Y$$
$$\Rightarrow X' \cdot Y' = X \cdot Y \qquad \forall X, Y$$

In components $X'^\mu = \Lambda^\mu{}_\alpha X^\alpha$

$$g_{\alpha\beta} X^\alpha Y^\beta = g_{\mu\nu} X'^\mu Y'^\nu = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta X^\alpha Y^\beta \qquad \forall X, Y$$
$$\Longrightarrow g_{\alpha\beta} = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta$$

Lorentz transformations (contd.)

Using matrix notation $\mathbf{\Lambda}_{\mu\alpha} = \Lambda^{\mu}{}_{\alpha}$, $\mathbf{g}_{\mu\nu} = g_{\mu\nu}$, $\mathbf{g}^{-1}_{\mu\nu} = g^{\mu\nu} = \mathbf{g}_{\mu\nu}$

$$\mathbf{g} = \mathbf{\Lambda}^T \mathbf{g} \mathbf{\Lambda}$$

$(\det \mathbf{\Lambda})^2 = 1 \Rightarrow \det \mathbf{\Lambda} = \pm 1$, $\mathbf{\Lambda}$ invertible

- $\det \mathbf{\Lambda} = 1$: *proper* transformations, leave orientation of space unchanged
- $\det \mathbf{\Lambda} = -1$: *improper* transformations invert the orientation of space

$\mathbf{\Lambda}^{-1} = \mathbf{g}^{-1}\mathbf{\Lambda}^T\mathbf{g}$ still a Lorentz transformation

- $\mathbf{g} = [\mathbf{\Lambda}\mathbf{\Lambda}^{-1}]^T \mathbf{g} [\mathbf{\Lambda}\mathbf{\Lambda}^{-1}] = \mathbf{\Lambda}^{-1\,T}[\mathbf{\Lambda}^T\mathbf{g}\mathbf{\Lambda}]\mathbf{\Lambda}^{-1} = \mathbf{\Lambda}^{-1\,T}\mathbf{g}\mathbf{\Lambda}^{-1}$
- $\mathbf{\Lambda}^{-1}_{\alpha\beta} = g^{\alpha\mu}\Lambda^{\nu}{}_{\mu}g_{\nu\beta} = \Lambda_{\beta}{}^{\alpha}$

From the $\alpha = 0$, $\beta = 0$ component of the defining relation

$$1 = \Lambda^0{}_0\Lambda^0{}_0 - \Lambda^i{}_0\Lambda^i{}_0 \Longrightarrow \Lambda^0{}_0\Lambda^0{}_0 = 1 + \Lambda^i{}_0\Lambda^i{}_0 \geq 1$$

- $\Lambda^0{}_0 \geq 1$: *orthochronous* (does not change the sign of time)
- $\Lambda^0{}_0 \leq -1$: *non-orthochronous* (changes the sign of time)

Proper orthochronous Lorentz group

Proper orthochronous Lorentz transformations = three-dimensional rotations (the SO(3) group) and boosts

Most general transformation: rotation × boost in x direction × rotation

Boost along x: $$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \beta = \frac{v}{c} = v < 1 \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Boost in general direction $\vec{n}$: rotate $\vec{n}$ to x, boost, rotate back

Coordinates in the new frame:

$$ct' = \gamma(ct - \beta x) \qquad x' = \gamma(x - \beta ct)$$
$$y' = y \qquad z' = z$$

$\Rightarrow$ relates R to R' moving with speed β in the *negative* x direction

Nonrelativistic limit $\beta = v/c \ll 1 \Rightarrow$ Galilei transformations

$$ct' = ct \qquad x' = x - vt$$

Full Lorentz group

Most general Lorentz transformation = proper orthochronous transformation times P (*parity*), T (*time reversal*), or PT

$$P^{\mu}{}_{\nu} = \mathrm{diag}(1,-1,-1,-1) \qquad T^{\mu}{}_{\nu} = \mathrm{diag}(-1,1,1,1)$$

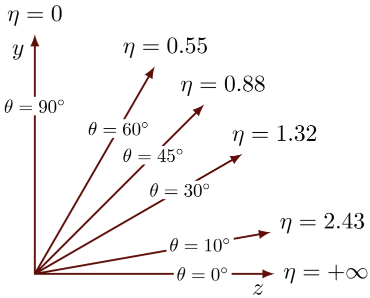
	$\det \Lambda = 1$		$\det \Lambda = -1$
$\Lambda^0{}_0 \geq 1$	$\boxed{\text{proper orthochronous}}$	$\underset{P}{\Rightarrow}$	improper orthochronous
	$\Downarrow_T$		$\Downarrow_T$
$\Lambda^0{}_0 \leq -1$	proper non-orthochronous	$\underset{P}{\Rightarrow}$	improper non-orthochronous

Rapidity and pseudorapidity

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{bmatrix} \cosh\zeta & -\sinh\zeta \\ -\sinh\zeta & \cosh\zeta \end{bmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

- $\tanh\zeta \equiv \beta$. $\zeta = \frac{1}{2}\ln\frac{E+|\mathbf{p}|c}{E-|\mathbf{p}|c}$ is called the rapidity for the boost.
- Show that the transformation is $e^{\mathcal{Z}\zeta}$ where $\mathcal{Z} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
- Use this to show that rapidities are additive for two subsequent boosts.
- At colliders, rapidity is $y = \frac{1}{2}\ln\frac{E+p_L}{E-p_L}$, while pseudorapidity $\eta = \frac{1}{2}\ln\frac{|\mathbf{p}|+p_L}{|\mathbf{p}|-p_L}$

Rapidity and pseudorapidity (contd.)



- .Pseudorapidity $\eta = \frac{1}{2} \ln \frac{|\mathbf{p}|+p_L}{|\mathbf{p}|-p_L} = -\ln\left[\tan(\theta/2)\right]$ is purely angular term. Agrees with the usual rapidity definition in the limit $p_T \gg m$.
- Colliders typically use $\Delta R \equiv \sqrt{(\Delta\eta)^2 + (\Delta\phi)^2}$ as 3-d angular separation between particles/jets.
- $p_L = p_T \sinh\eta$ and $|\mathbf{p}| = p_T \cosh\eta$

Point particles: kinematics

Trajectory $X^\mu(t)$ of point particle; over infinitesimal dt, $X^\mu \to X^\mu + dX^\mu$

$$X^\mu(t) = (ct, \vec{x}(t)) = (t, \vec{x}(t))$$

$$dX^\mu(t) = (dt, d\vec{x}(t)) = dt(1, \tfrac{d\vec{x}}{dt}(t)) = dt(1, \vec{v}(t))$$

Empirical fact: for massive particles $\vec{v}^{\,2} < 1$, for massless particles $\vec{v}^{\,2} = 1$

$$(dX)^2 = dX^\mu dX_\mu = dt^2(1 - \vec{v}^{\,2}) \geq 0 \qquad \text{(timelike)}$$

$$\tfrac{dX^\mu}{dt}(t) = (1, \vec{v}(t))$$

$\frac{dX^\mu}{dt}$ *not* a Lorentz vector: $dX^\mu =$ vector, $dt \neq$ scalar

Massive particle $\vec{v}^{\,2} < 1$: $\exists$ reference frame in which $\vec{v} = 0$ (*rest frame*)

$$X^\mu_{\text{rest}}(\tau) = (\tau, \vec{0})$$

τ: *proper time* (time measured in the particle's rest frame)

$$(dX_{\text{rest}})^2 = d\tau^2 = (dX)^2 = dt^2(1 - \vec{v}^{\,2}) = \tfrac{dt^2}{\gamma^2}$$

Point particles: kinematics (contd.)

Proper time:

- $d\tau^2 = \frac{dt^2}{\gamma^2} \Rightarrow |dt| > |d\tau|$ (time-dilation effect)
- determine the elapsed proper time by going over to the instantaneous rest frame of the particle $\Rightarrow$ twins' paradox

$$\tau = \int d\tau = \int_{t_0}^{t} dt' \sqrt{1-\vec{v}^{\,2}(t')} \leq t - t_0$$

- true scalar $\Rightarrow \frac{d^n X^\mu}{d\tau^n}$ are true vectors

Four-velocity

$$u^\mu \equiv \frac{dX^\mu}{d\tau} = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}\right) = \left(\gamma, \gamma\frac{d\vec{x}}{dt}\right) = (\gamma, \gamma\vec{v}) = (\gamma, \gamma\vec{\beta})$$

Four-momentum (vector u^μ times scalar m)

$$p^\mu \equiv m u^\mu = (\gamma m, \gamma m \vec{\beta})$$

$$p^0 = m\gamma = \frac{m}{\sqrt{1-\vec{v}^{\,2}}} = E \qquad p^i = m\gamma\vec{\beta}^{\,i} = \frac{m\vec{v}^{\,i}}{\sqrt{1-\vec{v}^{\,2}}} = \vec{p}^{\,i}$$

Point particles: four-momentum in the NR limit

Do $E, \vec{p}$ match their non-relativistic definition when $\frac{|\vec{v}|}{c} \ll 1$?

Needs reinstating powers of c

$$p^0 = mc\frac{1}{\sqrt{1-\left(\frac{\vec{v}}{c}\right)^2}} = mc\left(1 + \frac{1}{2}\left(\frac{\vec{v}}{c}\right)^2 + \mathcal{O}((\tfrac{v}{c})^4)\right)$$

$$\vec{p} = mc\frac{\frac{\vec{v}}{c}}{\sqrt{1-\left(\frac{\vec{v}}{c}\right)^2}} = m\vec{v}\left(1 + \mathcal{O}((\tfrac{v}{c})^2)\right)$$

Second line ok, first line times c

$$p^0 c = mc^2 + \frac{1}{2}m\vec{v}^2 + \ldots = E_0 + E_K^{\mathrm{NR}} + \ldots$$

$\Rightarrow$ NR kinetic energy E_K^{NR} of a particle plus *rest energy* $E_0 = mc^2$

Point particles: four-momentum for $m \neq 0$ and $m = 0$

Massive particles: $p^2 = m^2 > 0$

$$p^\mu = m\frac{dX^\mu}{d\tau} = \left(\frac{E}{c}, \vec{p}\right) \underset{c=1}{=} (E, \vec{p}) = (p^0, \vec{p})$$

Mass = relativistic invariant

$$p^2 = m^2\gamma^2(1 - \vec{\beta}^{\,2}) = m^2 > 0 \qquad u^2 = \gamma^2(1 - \vec{\beta}^{\,2}) = 1$$

Trajectory always inside the forward lightcone

Any constant would do, but m is *the* constant such that total momentum $\sum_i p_i = \sum_i m_i u_i$ of a system of particles is conserved

Also: correct NR limit of $p^\mu = m u^\mu$

Energy-momentum relation is called *dispersion relation*

$$E^2 = \vec{p}^{\,2} + m^2$$

Massless particles: $p^2 = 0$

$$p^\mu = (\omega, \vec{k})$$

Kinematics of 2-particle scattering

Two particle $\rightarrow$ two particle scattering process $a\,b \rightarrow c\,d$

Lab frame: one initial particle is at rest (= target)

$$p_a = (E_L, \vec{p}_L) \qquad p_b = (m_b, 0)$$
$$p_c = (E_c, \vec{p}_c) \qquad p_d = (E_d, \vec{p}_d)$$

Scattering angle θ_L in the lab: angle between trajectories of c and a

$$\cos\theta_L = \frac{\vec{p}_L \cdot \vec{p}_c}{|\vec{p}_L||\vec{p}_c|}$$

CM frame: vanishing total spatial momentum

$$p_a = (E_a^*, \vec{p}^{\,*}) \qquad p_b = (E_b^*, -\vec{p}^{\,*})$$
$$p_c = (E_c^*, \vec{p}^{\,\prime *}) \qquad p_d = (E_d^*, -\vec{p}^{\,\prime *})$$

Scattering angle θ^* in the CM: angle formed by the trajectories of a and c

$$\cos\theta^* = \frac{\vec{p}^{\,*} \cdot \vec{p}^{\,\prime *}}{|\vec{p}^{\,*}||\vec{p}^{\,\prime *}|}$$

Total center of mass energy $\sqrt{s}$ = Lorentz invariant

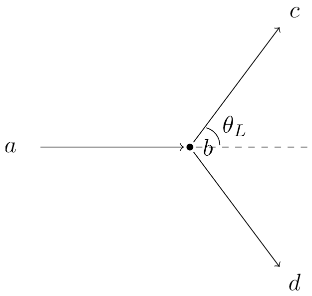
Kinematics of 2-particle scattering (contd.)

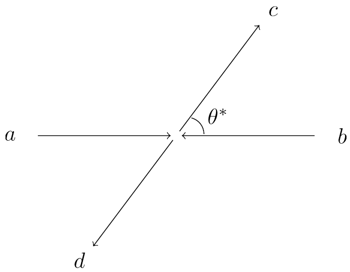
LAB

$$p_a = (E_L, \vec{p}_L) \qquad p_b = (m_b, 0)$$
$$p_c = (E_c, \vec{p}_c) \qquad p_d = (E_d, \vec{p}_d)$$

CM

$$p_a = (E_a^*, \vec{p}^{\,*}) \qquad p_b = (E_b^*, -\vec{p}^{\,*})$$
$$p_c = (E_c^*, \vec{p}^{\,\prime *}) \qquad p_d = (E_d^*, -\vec{p}^{\,\prime *})$$





Kinematics of 2-particle scattering (contd.)

$$\boxed{p_a + p_b = p_c + p_d}$$

- Four-momentum conservation implies $E^*_{c,d}$, $|\vec{p}^*_{c,d}| = |\vec{p}'^*|$ determined uniquely in the CM, independent of θ^*
- $E_{c,d}$, $|\vec{p}_{c,d}|$ and θ_L in the lab by Lorentz transf., depend on θ^*

$$\begin{aligned} p_b &= p_c + p_d - p_a \\ p_b^2 &= (p_c + p_d)^2 + p_a^2 - 2p_a \cdot (p_c + p_d) \\ m_b^2 &= s + m_a^2 - 2E_a^* \sqrt{s} \\ E_a^* &= \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} \quad \underset{a \leftrightarrow b}{\Rightarrow} \quad E_b^* = \frac{s + m_b^2 - m_a^2}{2\sqrt{s}} \end{aligned}$$

- CM energy squared s Lorentz invariant $\Rightarrow E_a^*$ from E_L in the lab:

$$s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2p_a \cdot p_b = m_a^2 + m_b^2 + 2E_L m_b \Rightarrow E_L = \frac{s - m_a^2 - m_b^2}{2m_b}$$

- Exchanging $a, b \leftrightarrow c, d$

$$E_c^* = \frac{s + m_c^2 - m_d^2}{2\sqrt{s}} \qquad E_d^* = \frac{s + m_d^2 - m_c^2}{2\sqrt{s}}$$

Kinematics of 2-particle scattering: CM

Center of mass energies:

$$E_a^* = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} \qquad E_b^* = \frac{s + m_b^2 - m_a^2}{2\sqrt{s}}$$

$$E_c^* = \frac{s + m_c^2 - m_d^2}{2\sqrt{s}} \qquad E_d^* = \frac{s + m_d^2 - m_c^2}{2\sqrt{s}}$$

Center of mass momentum magnitude $|\vec{p}^*|$:

$$\begin{aligned}
|\vec{p}^*|^2 &= E_a^{*2} - m_a^2 = \tfrac{(s+m_a^2-m_b^2)^2-4sm_a^2}{4s} = \tfrac{s^2+(m_a^2-m_b^2)^2-2s(m_a^2+m_b^2)}{4s} \\
&= \tfrac{(s-m_a^2-m_b^2)^2-4m_a^2m_b^2}{4s} = \tfrac{[s-(m_a+m_b)^2][s-(m_a-m_b)^2]}{4s} = \tfrac{\lambda(s,m_a^2,m_b^2)}{4s} \\
|\vec{p}'^*|^2 &= E_c^{*2} - m_c^2 = \tfrac{(s+m_c^2-m_d^2)^2-4sm_c^2}{4s} = \tfrac{s^2+(m_c^2-m_d^2)^2-2s(m_c^2+m_d^2)}{4s} \\
&= \tfrac{(s-m_c^2-m_d^2)^2-4m_c^2m_d^2}{4s} = \tfrac{[s-(m_c+m_d)^2][s-(m_c-m_d)^2]}{4s} = \tfrac{\lambda(s,m_c^2,m_d^2)}{4s}
\end{aligned}$$

Källén function: $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$

Kinematics of 2-particle scattering: lab

Lab kinematics recovered from CM kinematics

Given $\vec{p}_{\text{lab,CM}}$, $E_{\text{lab,CM}}$ total spatial momentum/total energy in lab/CM

$$|\vec{p}_{\text{CM}}| = 0 = \gamma_{\text{CM}}(|\vec{p}_{\text{lab}}| - \beta_{\text{CM}} E_{\text{lab}}) = \gamma_{\text{CM}}(|\vec{p}_L| - \beta_{\text{CM}}(m_b + E_L))$$
$$\implies \beta_{\text{CM}} = \frac{|\vec{p}_L|}{E_L + m_b}$$

Inverse Lorentz transformation from CM to lab

$$\begin{aligned} E_{c,\text{lab}} &= \gamma_{\text{CM}}(E_c^* + \beta_{\text{CM}}|\vec{p}'^*|\cos\theta^*)\,, \\ |\vec{p}_{c,\text{lab}}|\cos\theta_L &= \gamma_{\text{CM}}(|\vec{p}'^*|\cos\theta^* + \beta_{\text{CM}} E_c^*)\,, \\ |\vec{p}_{c,\text{lab}}|\sin\theta_L &= |\vec{p}'^*|\sin\theta^*\,, \end{aligned}$$

Transverse directions unaffected by Lorentz transformation, azimuthal angle transforms trivially

Example: proton-antiproton scattering

For $p\bar{p}$ scattering in circular collider, $E_p = E_{\bar{p}} = 270$ GeV

$$\Rightarrow \sqrt{s} = 540 \text{ GeV}$$

Let now p be at rest in the lab.

Q. What should be the energy E_L of $\bar{p}$ in the lab to obtain the same s?

A. CM energy square s is a relativistic invariant, can be evaluated in any reference frame; in the lab

$$s = (p_p + p_{\bar{p}})^2 = 2(m_p^2 + E_L m_p) = 2m_p(m_p + E_L)$$

Solve for E_L and impose $\sqrt{s} = 540$ GeV ($\gg m_p$)

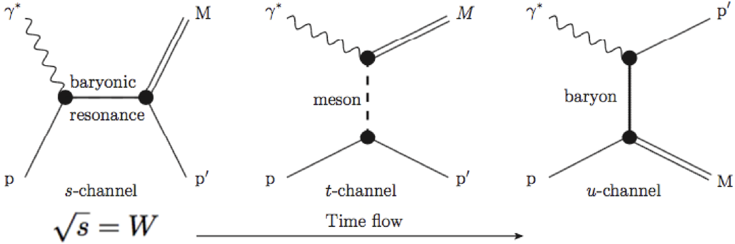
$$E_L = \frac{s - 2m_p^2}{2m_p} \simeq \frac{s}{2m_p} \simeq \frac{(540)^2}{2} \text{ GeV} \simeq \frac{30}{2} \cdot 10^4 \text{ GeV} = 150 \text{ TeV} \quad (!!!)$$

In general total CM energy $E_{\text{CM}} \simeq \sqrt{2m_p E_L}$

Mandelstam variables (cont.)

Convenient set of relativistic invariant variables for $2 \to 2$ scattering

$$s \equiv (p_a + p_b)^2 = (p_c + p_d)^2$$
$$t \equiv (p_a - p_c)^2 = (p_b - p_d)^2$$
$$u \equiv (p_a - p_d)^2 = (p_b - p_c)^2$$



$$\sqrt{s} = W$$

Mandelstam variables (cont.)

Convenient set of relativistic invariant variables for $2 \to 2$ scattering

$$
\begin{aligned}
s &\equiv (p_a + p_b)^2 = (p_c + p_d)^2 \\
t &\equiv (p_a - p_c)^2 = (p_b - p_d)^2 \\
u &\equiv (p_a - p_d)^2 = (p_b - p_c)^2
\end{aligned}
$$

- $s =$ total CM energy squared
- $t =$ square of four-momentum transfer from a to c

$$t = p_a^2 + p_c^2 - 2p_a \cdot p_c = m_a^2 + m_c^2 - 2(E_a^* E_c^* - |\vec{p}^*||\vec{p}'^*| \cos\theta^*)$$

- $u =$ square of four-momentum transfer from a to d

$$u = p_a^2 + p_d^2 - 2p_a \cdot p_d = m_a^2 + m_d^2 - 2(E_a^* E_d^* + |\vec{p}^*||\vec{p}'^*| \cos\theta^*)$$

u obtained from t after $m_c \to m_d$ and $\cos\theta^* \to -\cos\theta^*$

Energies and magnitudes of momenta entirely determined by s and particle masses $\Rightarrow t = t(s, \theta^*)$, or instead $\theta^* = \theta^*(s,t)$ and use s, t

Mandelstam variables (contd.)

Only two independent Mandelstam variables:

$$
\begin{aligned}
s+t+u &= (p_a+p_b)^2+(p_a-p_c)^2+(p_a-p_d)^2 \\
&= m_a^2+m_b^2+m_c^2+m_d^2+2p_a\cdot(p_a+p_b-p_c-p_d) \\
&= m_a^2+m_b^2+m_c^2+m_d^2
\end{aligned}
$$

Bounds on Mandelstam variables determine physical region for s, t, u

$$
s \geq \max((m_a+m_b)^2,(m_c+m_d)^2)
$$

$$
\begin{aligned}
t=(p_a-p_c)^2=m_a^2+m_c^2-2p_a\cdot p_c &= 2(m_a^2+m_c^2)-(p_a+p_c)^2 \\
&\leq 2(m_a^2+m_c^2)-(m_a+m_c)^2=(m_a-m_c)^2
\end{aligned}
$$

Similarly using p_b and p_d; same approach for u

$$
t \leq \min((m_a-m_c)^2,(m_b-m_d)^2) \qquad u \leq \min((m_a-m_d)^2,(m_b-m_c)^2)
$$

Lower bound from this and $t|u = m_a^2+m_b^2+m_c^2+m_d^2-s-u|t$

$$
\begin{aligned}
t &\geq \max(m_b^2+m_c^2+2m_am_d, m_a^2+m_d^2+2m_bm_c)-s \\
u &\geq \max(m_a^2+m_c^2+2m_bm_d, m_b^2+m_d^2+2m_am_c)-s
\end{aligned}
$$

Mandelstam variables (contd.)

Simplification if $m_a = m_b$, $m_c = m_d \Rightarrow E_a^* = E_b^* = E_c^* = E_d^* = \frac{\sqrt{s}}{2}$

$$t = m_a^2 + m_c^2 - \frac{s}{2}\left(1 - \cos\theta^* \sqrt{1 - \frac{4m_a^2}{s}}\sqrt{1 - \frac{4m_c^2}{s}}\right)$$

If also $m_a = m_c \equiv m$

$$t = 2m^2 - \frac{s}{2}\left(1 - \cos\theta^*\left(1 - \frac{4m^2}{s}\right)\right) = -\left(s - 4m^2\right)\sin^2\frac{\theta^*}{2}$$

$$\boxed{s \geq 4m^2 \qquad -\left(s - 4m^2\right) \leq t \leq 0}$$

- Upper limit: at threshold $s = 4m^2$ or when $\theta^* = 0$ (fwd scatter)
- Lower limit: when $\theta^* = \pi$ (backscattering)
- In this case $u(s, \theta^*) = t(s, \pi - \theta^*) \Rightarrow$ same bound applies to u; role of $\theta^* = 0$ and $\theta^* = \pi$ exchanged

Relevant for

- elastic processes involving only one type of particles/antiparticles
- very high energy limit (masses negligible, particles $\approx$ massless)

Example: proton-proton scattering

Elastic pp scattering, $\sqrt{s} = 53$ GeV

Differential cross section $\frac{d\sigma}{dt}(t)$ has a peak at $-t = t_0 = 1.81\ \text{GeV}^2$

E. Nagy *et al.*, Nucl. Phys. **B150** (1979) 221

Q. What is the corresponding scattering angle in the CM?

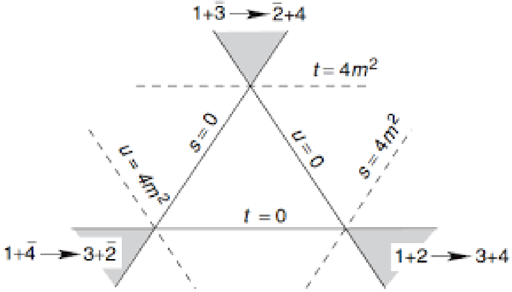
A. Elastic scattering of identical particles, $s/m_p^2 \gg 1$

$$-t = (s - 4m_p^2)\sin^2\tfrac{\theta^*}{2} \simeq s\sin^2\tfrac{\theta^*}{2}$$

$$\sin^2\tfrac{\theta^*}{2} = -\tfrac{t}{s-4m_p^2} = \tfrac{1.81}{53^2-4\cdot 0.938^2} = \tfrac{1.81}{2805} = 6.45\cdot 10^{-4}$$

$$\sin^2\tfrac{\theta^*}{2} \simeq \tfrac{(\theta^*)^2}{4} \quad \Longrightarrow \quad \theta^* \simeq 2\sqrt{5}\cdot 10^{-2} \simeq 5\cdot 10^{-2}$$

Mandelstam plane



- Sides of eq. triangle: $s=0$, $t=0$ and $u=0$ axes
- For appropriate side length $s+t+u = m_a^2 + m_b^2 + m_c^2 + m_d^2$
- Physical region for the $a+b \to c+d$ process (equal masses) = wedge defined by the prolongation of the u and t axes

Crossing symmetry

QFT result: scattering amplitudes for $a+b \to c+d$, $a+\bar{c} \to \bar{b}+d$, $a+\bar{d} \to c+\bar{b}$ are part of a single analytic function extending beyond physical momenta, and related to each other

$$A_{ab\to cd}(p_a, p_b; p_c, p_d) = A_{a\bar{c}\to\bar{b}d}(p_a, -p_c; -p_b, p_d) = A_{a\bar{d}\to c\bar{b}}(p_a, -p_d; p_c, -p_b)$$

Use Mandelstam variables

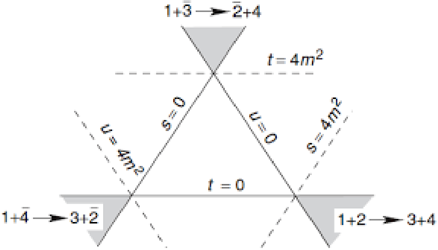
$$\begin{array}{lrl} a+b \to c+d & \mathcal{A}_s(s,t,u) = A_{ab\to cd}(p_a, p_b; p_c, p_d) & s\text{-channel} \\ a+\bar{c} \to \bar{b}+d & \mathcal{A}_t(s_t,t_t,u_t) = A_{a\bar{c}\to\bar{b}d}(p_a, p_{\bar{c}}; p_{\bar{b}}, p_d) & t\text{-channel} \\ a+\bar{d} \to c+\bar{b} & \mathcal{A}_u(s_u,t_u,u_u) = A_{a\bar{d}\to c\bar{b}}(p_a, p_{\bar{d}}; p_c, p_{\bar{b}}) & u\text{-channel} \end{array}$$

$$\begin{array}{lll} s = (p_a+p_b)^2 & t = (p_a-p_c)^2 & u = (p_a-p_d)^2 \\ s_t = (p_a+p_{\bar{c}})^2 & t_t = (p_a-p_{\bar{b}})^2 & u_t = (p_a-p_d)^2 \\ s_u = (p_a+p_{\bar{d}})^2 & t_u = (p_a-p_c)^2 & u_u = (p_a-p_{\bar{b}})^2 \end{array}$$

Crossing-symmetry relations

$$\mathcal{A}_s(s,t,u) = \mathcal{A}_t(t,s,u) = \mathcal{A}_u(u,t,s)$$

Crossing symmetry (contd.)



$$\mathcal{A}_s(s,t,u) = \mathcal{A}_t(t,s,u) = \mathcal{A}_u(u,t,s)$$

- If s, t, u take physical values for the s-channel process $a\,b \to c\,d$, crossing relations involve $\mathcal{A}_t$ and $\mathcal{A}_u$ at unphysical values of their arguments
- Relations fully meaningful if $\mathcal{A}_s$ can be analytically continued outside the physical domain
- For equal masses, physical regions of $\mathcal{A}_t$ and $\mathcal{A}_u$ are $s_t \geq 4m^2, t_t \leq 0$ and $s_u \geq 4m^2, t_u \leq 0$, but $t \leq 0$ and $s \geq 4m^2$

Physical regions = wedges outside Mandelstam triangle

Invariant phase space

States of spinless particle, mass m are characterised by four-momenta p^μ with $p^2 = m^2$ and positive energy $p^0 \geq m > 0$

One-particle phase space:

$$\{p \in \mathbb{R}^4 | p^2 - m^2 = 0\,, \quad p^0 > 0\} \subset \mathbb{R}^4$$

Measure of infinitesimal element of phase space

$$d\Phi^{(1)} = \frac{d^4 p}{(2\pi)^4} 2\pi \delta(p^2 - m^2)\theta(p^0)$$

- Manifestly invariant under orthochronous Lorentz transformations: p^2 invariant, $\mathrm{sign}(p^0)$ invariant under orthochronous transformations
- Overall scale appropriate for relativistic normalisation of one-particle states: $\langle \vec{p}\,' | \vec{p} \rangle = (2\pi)^3 2p^0 \delta^{(3)}(\vec{p}\,' - \vec{p})$

Invariant phase space (contd.)

Recast $d\Phi^{(1)}$ in more convenient form: for any f with simple zeros $\{x_n\}$

$$\delta(f(x)) = \sum_{x_n, f(x_n)=0} \frac{1}{|f'(x_n)|}\delta(x - x_n)$$

- multiply both sides by some function $h(x)$, integrate over $\mathbb{R}$, show that one gets the same result
- divide $\mathbb{R} = (-\infty, +\infty) = \cup_k I_k$ with $f(x)$ monotonic in I_k $\Rightarrow f$ invertible in I_k and vanishes at most once ($|f'| \neq 0$ there)
- set $y = f(x) \to x = f^{-1}(y)$ in each I_k

$$\begin{aligned}
\int_{-\infty}^{+\infty} dx\, \delta(f(x))h(x) &= \sum_k \int_{I_k} dx\, \delta(f(x))h(x) \\
&= \sum_k \int_{f(I_k)} dy\, \frac{1}{|f'(f^{-1}(y))|}\delta(y)h(f^{-1}(y)) \\
&= \sum_k \int_{0\in f(I_k)} dy\, \frac{1}{|f'(f^{-1}(0))|}\delta(y)h(f^{-1}(0)) = \sum_n \frac{1}{|f'(x_n)|}h(x_n)
\end{aligned}$$

Invariant phase space (contd.)

$$
\begin{aligned}
d\Phi^{(1)} &= \frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\theta(p^0) = \frac{d^4p}{(2\pi)^3}\delta(p^{0\,2}-\vec{p}^{\,2}-m^2)\theta(p^0) \\
&= \frac{d^4p}{(2\pi)^3}\frac{1}{2|p^0|}\left[\delta(p^0-\varepsilon(\vec{p}))+\delta(p^0+\varepsilon(\vec{p}))\right]\theta(p^0) \\
&= \frac{d^4p}{(2\pi)^3}\frac{1}{2\varepsilon(\vec{p})}\delta(p^0-\varepsilon(\vec{p}))\theta(p^0) = \frac{d^3p}{(2\pi)^3 2\varepsilon(\vec{p})} \equiv d\mathit{\Omega}_p \\
\varepsilon(\vec{p}) &\equiv \sqrt{\vec{p}^{\,2}+m^2}
\end{aligned}
$$

n-particle phase space $\subset \mathbb{R}^{4n}$ corresponding to four-momenta of n particles subjected to a constraint on the total four-momentum
Measure of infinitesimal element:

$$d\Phi^{(n)} = \prod_{j=1}^{n} d\mathit{\Omega}_{p_j}(2\pi)^4\delta^{(4)}\left(p_{\rm tot}-\sum_{j=1}^{n}p_j\right)$$

Lorentz invariant: $d\mathit{\Omega}_{p_j}$ Lorentz invariant

$$\delta^{(4)}(\mathit{\Lambda}P) = |\det\mathit{\Lambda}|^{-1}\delta^{(4)}(P) = \delta^{(4)}(P)$$

Invariant phase space: 2-particle case

Total momentum $p_{\rm tot} = (E_{\rm tot}, \vec{p}_{\rm tot})$, particle energies $\varepsilon_i(\vec{p}) = \sqrt{\vec{p}^2 + m_i^2}$

$$d\Phi^{(2)} = \frac{d^3p_1}{(2\pi)^3 2\varepsilon_1(\vec{p}_1\,)} \frac{d^3p_2}{(2\pi)^3 2\varepsilon_2(\vec{p}_2\,)} (2\pi)^4 \delta^{(4)}(p_{\rm tot} - p_1 - p_2)$$

$$= \frac{1}{(2\pi)^2} \frac{d^3p_1}{2\varepsilon_1(\vec{p}_1\,)} \frac{d^3p_2}{2\varepsilon_2(\vec{p}_2\,)} \delta^{(3)}(\vec{p}_{\rm tot} - \vec{p}_1 - \vec{p}_2) \delta(E_{\rm tot} - \varepsilon_1(\vec{p}_1\,) - \varepsilon_2(\vec{p}_2\,))$$

Integrate trivially over $\vec{p}_2$, setting it equal to $\vec{p}_2 = \vec{p}_{\rm tot} - \vec{p}_1$

$$d\Phi^{(2)} = \frac{1}{(2\pi)^2} \frac{d^3p_1}{2\varepsilon_1(\vec{p}_1\,)} \frac{1}{2\varepsilon_2(\vec{p}_{\rm tot} - \vec{p}_1\,)} \delta(E_{\rm tot} - \varepsilon_1(\vec{p}_1\,) - \varepsilon_2(\vec{p}_{\rm tot} - \vec{p}_1\,))$$

To further integrate over $|\vec{p}_1|$ requires changing variables, most easily done working in the CM

$$\vec{p}_{\rm tot,CM} = 0 \Rightarrow \vec{p}_{1\,\rm CM} = -\vec{p}_{2\,\rm CM}, \quad |\vec{p}_{1\,\rm CM}| = |\vec{p}_{2\,\rm CM}| = p$$

Dropping "CM" in the following

Invariant phase space: 2-particle case (contd.)

Delta function depends on $E_{\text{tot}} - \varepsilon_1(p) - \varepsilon_2(p)$

Dropped vector sign on $\pm\vec{p}$

$$\left|\frac{\Delta E}{\Delta E p}\left[E_{\text{tot}} - \varepsilon_1(p) - \varepsilon_2(p)\right]\right| = \left[\frac{p}{\varepsilon_1(p)} + \frac{p}{\varepsilon_2(p)}\right] = \frac{p}{\varepsilon_1(p)\varepsilon_2(p)}\left[\varepsilon_1(p) + \varepsilon_2(p)\right]$$

Changing variables to $d^3p_1 = dpp^2 d\cos\theta^* d\phi^* = dpp^2 d\Omega^*$

$$d\Phi^{(2)} = \frac{1}{(2\pi)^2}\frac{dpp^2 d\Omega^*}{2\varepsilon_1(p)}\frac{1}{2\varepsilon_2(p)}\underbrace{\frac{\varepsilon_1(p)\varepsilon_2(p)}{p}\left[\varepsilon_1(p) + \varepsilon_2(p)\right]^{-1}\delta(p - p^*)}_{\delta(E_{\text{tot}} - \varepsilon_1(p) - \varepsilon_2(p))}$$

$$= \frac{d\Omega^*}{(2\pi)^2}\frac{p^*}{4(\varepsilon_1(p^*) + \varepsilon_2(p^*))} = \frac{d\Omega^*}{(2\pi)^2}\frac{p^*}{4E^*_{\text{tot}}} = \frac{d\Omega^*}{16\pi^2}\frac{p^*}{\sqrt{s}}$$

$$= \frac{d\Omega^*}{32\pi^2}\frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{s}$$

For equal masses $\lambda(s, m^2, m^2) = s(s - 4m^2) \Rightarrow d\Phi^{(2)} = \frac{d\Omega^*}{32\pi^2}\sqrt{\frac{s - 4m^2}{s}}$