Particle physics

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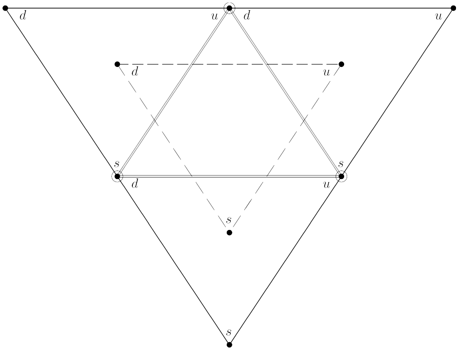
Composition of irreps

I_3, Y additive $\rightarrow$ draw triangles on triangles



Similarly, composition of lowest-dim irreps:

$$6 \otimes 3 = 10 \oplus 8 \qquad 3 \otimes 3 = 6 \oplus \bar{3}$$

$$\bar{3} \otimes 3 = 8 \oplus 1 \qquad 3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

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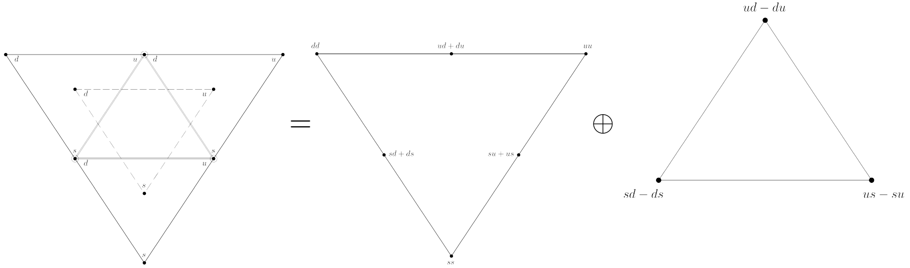

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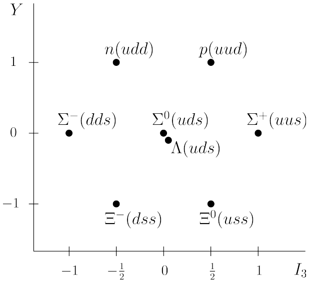
Quark content of hadrons

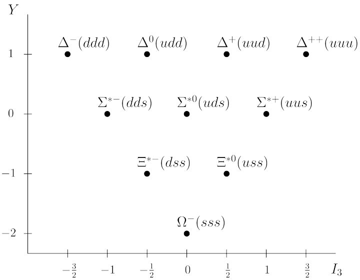
Quark content of hadrons follows from corresponding values of I_3 and Y (independently of how irreps are obtained from $3 \otimes 3 \otimes 3$ or $3 \otimes \bar{3}$)

$$n_u + n_d + n_s = 3B \qquad I_3 = \frac{1}{2}(n_u - n_d) \qquad Y = \frac{1}{3}(n_u + n_d - 2n_s)$$

Baryons $B = 1$:

$$n_u = I_3 + \frac{1}{2}Y + 1 \qquad n_d = -I_3 + \frac{1}{2}Y + 1 \qquad n_s = 1 - Y$$





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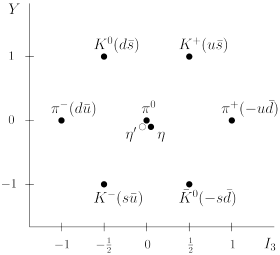
$$n_u + n_d + n_s = 3B \qquad I_3 = \tfrac{1}{2}(n_u - n_d) \qquad Y = \tfrac{1}{3}(n_u + n_d - 2n_s)$$

Mesons $B = 0$:

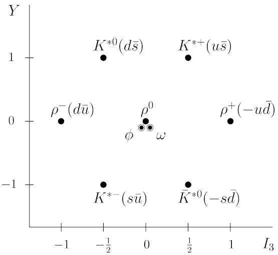
$$n_u = I_3 + \tfrac{1}{2}Y \qquad n_d = -I_3 + \tfrac{1}{2}Y \qquad n_s = -Y$$



$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$

$\eta = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$

$\eta' = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$



$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$

$\omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$

$\phi = s\bar{s}$

Quark masses: qualitative treatement

Assume baryon masses come mostly from constituents' masses (here $n_{u,d,s} \geq 0$)

$$m_B = n_u m_u + n_d m_d + n_s m_s$$

Mass splitting within isomultiplets: since S constant

$$\Delta m_B = m_1 - m_2 = (n_{u1} - n_{u2}) m_u + (n_{d1} - n_{d2}) m_d = (n_{u1} - n_{u2})(m_u - m_d)$$

Small, in a first approximation $m_u = m_d$

$$m_B = m_u(n_u + n_d) + m_s n_s = 3m_u + (m_s - m_u)|S|$$

- Nucleon mass $m_{p,n} \approx 3m_u \approx 940\ \mathrm{MeV}$
- Baryon masses linear in $|S|$, splittings $\approx m_s - m_u \approx 150\ \mathrm{MeV}$

$$m_u \simeq m_d \approx 300\ \mathrm{MeV} \qquad m_s \approx 450\ \mathrm{MeV}$$

Constituent masses very different from *current masses* discussed before: in fact, most of a hadron mass **not** from quark masses but from interaction

Estimate would not work with light pseudoscalar mesons: linearity of masses in $|S|$ does not hold there

Basics of *quark model* (Gell-Mann, Zweig, 1964) completed. . .

. . . but there are serious problems

Wave functions and the problem with statistics

Quarks could explain why only certain representations appear in nature, but can baryon wave functions be built consistently with Fermi statistics?

Different quark flavours $\sim$ different states of the same spin-$\frac{1}{2}$ particle

Baryons=fermions, wave functions antisymmetric under quark exchange

$$\psi = \psi_{\text{space}}\psi_{\text{spin}}\psi_{\text{flavour}}$$

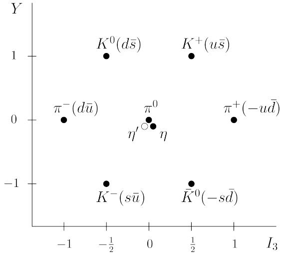
- lowest-lying states have usually $\ell_{1,2} = 0 \Rightarrow \psi_{\text{space}}$ symmetric
- antisymmetry from spin-flavour part

Decuplet:

- $s = \frac{3}{2} \Rightarrow$ symmetric spin wf
- flavour content: $\Delta^{++} = (uuu)$, ladder operators do not change symmetry properties $\Rightarrow$ symmetric flavour wf
- not acceptable for fermions

$$\begin{aligned}
\Delta^{++} &= |uuu\rangle \\
\Delta^{+} &\propto I_- \Delta^{++} \\
&\propto |uud\rangle + |udu\rangle + |duu\rangle \\
\Sigma^{*+} &\propto V_- \Delta^{++} \\
&\propto |uus\rangle + |usu\rangle + |suu\rangle \\
\Xi^{*0} &\propto V_-^2 \Delta^{++} \\
&\propto |uss\rangle + |sus\rangle + |ssu\rangle \\
\Omega^{-} &\propto V_-^3 \Delta^{++} \propto |sss\rangle
\end{aligned}$$

Meson wave functions



$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$

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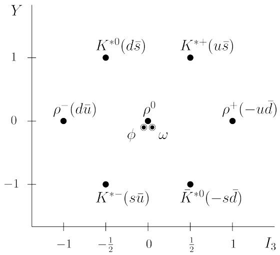

$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$

$\omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$

$\phi = s\bar{s}$

No restriction on the symmetry of the wave function ($q \neq \bar{q}$)

- total $q\bar{q}$ spin $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$, ground states ($\ell = 0$) are $J = 0$ or $J = 1$
- intrinsic parity $\eta_q \eta_{\bar{q}} = -1 \Rightarrow$ lightest mesons: pseudoscalars or vectors
- flavour $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1} \Rightarrow$ an octet and a singlet
- for exact SU(3) pseudoscalars and vectors have identical $q\bar{q}$ content
- SU(3) broken to $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$, $I = 0$ states from SU(3) octet and SU(3) singlet can mix $\Rightarrow$ meson nonets
- mixing small for pseudoscalars but almost maximal for vectors

Colour

How can the antisymmetrisation problem be fixed? Greenberg, 1964:

- add extra degree of freedom (*colour*) $q, \bar{q} \to q_i, \bar{q}_i$, $i = 1, \ldots, N_c$
- require $\psi_{\rm colour}$ antisymmetric
- associated internal SU(N_c) symmetry, but no further degeneracies among hadrons masses $\Rightarrow$ hadrons must be SU(N_c) singlets

Lowest-dimensional singlets:

- $\delta_{i_1 i_2}$: symmetric, singlet in $N_c \otimes \bar{N}_c \Rightarrow q\bar{q}$ pair $\Rightarrow$ mesons
- $\epsilon_{i_1 \ldots i_{N_c}}$: antisymmetric, singlet in $\underbrace{N_c \otimes \ldots \otimes N_c}_{N_c \text{ times}} \Rightarrow$ baryons if $N_c = 3$

Solves two problems with baryons at once...

- explain why it takes three quarks to make a baryon
- solves representation puzzle:
  - combine with symmetric flavour/spin wf to get antisymmetric total wf both for octet and decuplet
  - since $\nexists$ totally antisymmetric spin wf out of $qqq \Rightarrow$ cannot use flavour singlet wf for baryons

Colour (contd.)

…and if colour is made a dynamical degree of freedom
$\Rightarrow$ QCD (Gell-Mann, Leutwyler, Fritsch, 1972), fundamental dynamical theory of strong interactions

Is really $N_c = 3$? Experimental confirmation from:

- Drell-Yan process $\pi N \to \mu^+\mu^- X$ (X= anything)
  - q_i from nucleon and $\bar{q}_i$ from pion with the same colour undergo

    $$q_i\,\bar{q}_i \to \gamma \to \mu^+\mu^-$$

  - same annihilation probability for any colour $\to$ cross section $\propto N_c$
- neutral pion decay $\pi^0 \to \gamma\gamma$
  - $q_i\,\bar{q}_i \to \gamma\gamma$, scattering amplitude colour-independent
  - wave function $\propto \frac{\delta_{ij}}{\sqrt{N_c}} \to \Gamma \propto (N_c/\sqrt{N_c})^2 = N_c$

Hadron masses: the Gell-Mann–Okubo formula

How to break SU(3) symmetry to reproduce experimental results?

- Pre-QCD: breaking has to preserve isospin and strangeness, smallest representation with $I = Y = 0$ state is the adjoint **8**
- QCD: $m_s \gg m_u \simeq m_d \Rightarrow$ strong Hamiltonian in quark rest frame

$$\begin{aligned} \langle q_i | H | q_j \rangle &= m_i \delta_{ij} = \mathrm{diag}(m_{ud}, m_{ud}, m_s) \\ &= \tfrac{2m_{ud}+m_s}{3}\mathbf{1} + \tfrac{m_{ud}-m_s}{3}\mathrm{diag}(1,1,-2) \\ &= \tfrac{2m_{ud}+m_s}{3}\mathbf{1} + \tfrac{m_{ud}-m_s}{\sqrt{3}}\lambda_8 \end{aligned}$$

Pre-QCD suggests, and QCD predicts:

$$H = H_0 + H_8$$

H_0: SU(3) singlet, symmetric
H_8: transforms as the eighth component in the adjoint representation

Hadron masses: the Gell-Mann–Okubo formula (contd.)

For quantitative estimate: assume H_8 small perturbation, use 1st-order PT

- 0th-order: degenerate multiplets of baryons

$$H_0|B^{(0)};a\rangle = m_a^{(0)}|B^{(0)};a\rangle$$

- 1st order: diagonalise

$$\langle B^{(0)\prime};b|H_8|B^{(0)};a\rangle$$

  - ground-state baryons (octet/decuplet) not mixed by perturbation
  - ignore contributions from higher states
  - $\Rightarrow$ diagonalise $\langle B^{(0)\prime};a|H_8|B^{(0)};a\rangle$ within each multiplet

- perturbation diagonal in isospin-hypercharge basis, baryon masses

$$m_a(B) = m_a^{(0)} + \Delta m_a(B) \qquad \Delta m_a(B) = \langle B^{(0)};a|H_8|B^{(0)};a\rangle$$

- representation theory determines $\Delta m_a(B)$ in a multiplet up to two unknown, a-dependent coefficients (depend on details of interaction)

Hadron masses: the Gell-Mann–Okubo formula (contd.)

$\langle B'(R)|H_a|B(R)\rangle$: matrix elements transforming in

$$
\begin{aligned}
R\otimes\bar{R}: \qquad &= \langle\tilde{B}'(R)|H_a|\tilde{B}(R)\rangle\mathcal{U}^{R}_{\tilde{B}B}\mathcal{U}^{R*}_{\tilde{B}'B'} \\
8: \qquad &= \langle B'(R)|U^\dagger H_a U|B(R)\rangle = \mathcal{U}^{8}_{ba}\langle B'(R)|H_b|B(R)\rangle
\end{aligned}
$$

Decompose $R\otimes\bar{R} = \bigoplus_{\tilde{R}}\tilde{R}$, find how many $\tilde{R}=8$

From representation theory: for SU(3), 8 found at most twice
$\Rightarrow$ only two possible independent tensorial structures

Most general form:

$$\langle B'(R)|H_a|B(R)\rangle = \delta m_1(R)(\mathcal{T}_a^{(8,1)})_{B'B} + \delta m_2(R)(\mathcal{T}_a^{(8,2)})_{B'B}$$

Coefficients $\delta m_j(R)$ depend on multiplet

Hadron masses: the Gell-Mann–Okubo formula (contd.)

First structure: $\mathcal{T}_a^{(8,1)} = T_a^R$

$$\mathcal{U}^{R\dagger} T_a^R \mathcal{U}^R = (\mathcal{U}^8)_{ab} T_b^R$$

Second structure: $\mathcal{T}_a^{(8,2)} = D_a^R \equiv d_{abc} T_b^R T_c^R$

- $\lambda^a\lambda^b = k_0\mathbf{1} + k_c\lambda^c$ since it is a 3×3 complex matrix

$$\lambda^a\lambda^b = \tfrac{1}{2}\{\lambda^a,\lambda^b\} + \tfrac{1}{2}[\lambda^a,\lambda^b] = \tfrac{2}{3}\delta_{ab} + (if_{abc} + d_{abc})\lambda^c$$

$$d_{abc} = \tfrac{1}{4}\mathrm{tr}\,\{\lambda^a,\lambda^b\}\lambda^c = 2\mathrm{tr}\,\{t^a,t^b\}t^c$$

- d_{abc} totally symmetric, invariant under adjoint transformation

$$(\mathcal{U}^8)_{a'a}(\mathcal{U}^8)_{b'b}(\mathcal{U}^8)_{c'c}d_{a'b'c'} = d_{abc}$$

$\Rightarrow D_a^R$ transforms in the adjoint: since $(\mathcal{U}^8)^T\mathcal{U}^8 = 1$

$$\begin{aligned}\mathcal{U}^{R\dagger} D_a^R \mathcal{U}^R &= \mathcal{U}^R d_{abc} T_b^R T_c^R \mathcal{U}^R = d_{abc}(\mathcal{U}^8)_{bb'}(\mathcal{U}^8)_{cc'} T_{b'}^R T_{c'}^R \\ &= (\mathcal{U}^8)_{aa'} d_{a''bc}(\mathcal{U}^8)_{a''a'}(\mathcal{U}^8)_{bb'}(\mathcal{U}^8)_{cc'} T_{b'}^R T_{c'}^R \\ &= (U^{(8)})_{aa'} d_{a'b'c'} T_{b'}^R T_{c'}^R = (U^{(8)})_{aa'} D_{a'}^R\end{aligned}$$

Hadron masses: the Gell-Mann–Okubo formula (contd.)

Perturbation:

$$\langle B'(R)|H_8|B(R)\rangle = \delta m_1(R)(T_8^R)_{B'B} + \delta m_2(R)(D_8^R)_{B'B}$$

$$\begin{aligned} D_8^R &= d_{8bc} T_b^R T_c^R \\ &= -\tfrac{1}{2\sqrt{3}} \textstyle\sum_a (T_a^R)^2 + \tfrac{\sqrt{3}}{2}[(T_1^R)^2 + (T_2^R)^2 + (T_3^R)^2] - \tfrac{1}{2\sqrt{3}}(T_8^R)^2 \\ &= -\tfrac{1}{2\sqrt{3}} C^R + \tfrac{\sqrt{3}}{2}\left(\vec{I}^2 - \tfrac{1}{4}Y^2\right) \end{aligned}$$

$C^R = \sum_a (T_a^R)^2$: quadratic *Casimir operator*, commutes with all T_a^R
$\Rightarrow$ must be $\propto 1$ within multiplet (Schur's lemma)

Perturbation diagonal within a multiplet, diagonal terms read

$$\Delta m(B) = \langle B^{(0)}|H_8|B^{(0)}\rangle = \tfrac{\sqrt{3}}{2}\left[\delta m_1 Y + \delta m_2\left(-\tfrac{C_R}{3} + I(I+1) - \tfrac{1}{4}Y^2\right)\right]$$

Redefining unknown constants $\Rightarrow$ *Gell-Mann–Okubo mass formula*:

$$m(B) = m^{(0)} + \langle B^{(0)}|H_8|B^{(0)}\rangle = \tilde{m}^{(0)} + \delta\tilde{m}_1 Y + \delta\tilde{m}_2\left[I(I+1) - \tfrac{1}{4}Y^2\right]$$

$m^{(0)}, \tilde{m}^{(0)}, \delta\tilde{m}_{1,2}$ depend on irreducible multiplet

Hadron masses: the Gell-Mann–Okubo formula (contd.)

Notation: $X_{I,Y}$ is particle X with isospin I and hypercharge Y

Baryon octet:

$$\begin{aligned}
\Lambda_{0,0} : &\qquad m_\Lambda = \tilde{m}^{(0)} \\
N_{\frac{1}{2},1} : &\qquad m_N = \tilde{m}^{(0)} + \delta\tilde{m}_1 + \tfrac{1}{2}\delta\tilde{m}_2 \\
\Sigma_{1,0} : &\qquad m_\Sigma = \tilde{m}^{(0)} + 2\delta\tilde{m}_2 \\
\Xi_{\frac{1}{2},-1} : &\qquad m_\Xi = \tilde{m}^{(0)} - \delta\tilde{m}_1 + \tfrac{1}{2}\delta\tilde{m}_2
\end{aligned}$$

Four equations with three unknowns $\Rightarrow$ one relation among masses, e.g.

$$m_N + m_\Xi = \tfrac{3}{2}m_\Lambda + \tfrac{1}{2}m_\Sigma$$

Exp.: LHS$=2257\ \mathrm{MeV}$ vs. RHS$=2270.5\ \mathrm{MeV}$ (accurate to percent level)

Hadron masses: the Gell-Mann–Okubo formula (contd.)

Baryon decuplet:

I and Y linearly related: $2I - Y = 2$

$$I(I+1) - \tfrac{1}{4}Y^2 = 2 + \tfrac{3}{2}Y \quad \Longrightarrow \quad m_B = \tilde{m}^{(0)} + \delta m Y$$

$$\begin{aligned}
\Delta_{\frac{3}{2},1} :& \qquad m_\Delta = \tilde{m}^{(0)} + \delta m \\
\Sigma^*_{1,0} :& \qquad m_{\Sigma^*} = \tilde{m}^{(0)} \\
\Xi^*_{\frac{1}{2},-1} :& \qquad m_{\Xi^*} = \tilde{m}^{(0)} - \delta m \\
\Omega_{0,-2} :& \qquad m_\Omega = \tilde{m}^{(0)} - 2\delta m
\end{aligned}$$

Four equations with two unknowns $\Rightarrow$ two mass relations, e.g.

$$m_\Delta + m_{\Xi^*} = 2m_{\Sigma^*} \qquad 2m_\Delta + m_\Omega = 3m_{\Sigma^*}$$

LHS=2765 MeV vs. RHS=2768 MeV,
LHS=4136 MeV vs. RHS=4152 MeV (accuracy of permille)

Mass of Ω predicted by means of this type of formula

Hadron masses: the Gell-Mann–Okubo formula (contd.)

Meson octet

Gell-Mann–Okubo formula fails disastrously for pseudoscalar meson octet

We did not take into account mixing of singlet and octet, but this is small for pseudoscalars

Using instead the *square* of the masses one gets

$$4m_K^2 = 3m_\eta^2 + m_\pi^2$$

LHS$=$0.98 GeV^2 vs. RHS$=$0.92 GeV^2 (percent accuracy)

- Why does original formula fail? Perturbation not small here, same order of unperturbed masses $\Rightarrow$ PT does not work
- Why does modified formula work? Spontaneous breaking of chiral symmetry in QCD (beyond the scope of this course)

Weak interactions

- Strong interactions:
  most symmetric (P, C, I, Q, B, flavour symmetries)
- EM interactions:
  almost as symmetric (break I)
- Weak interactions:
  least symmetric (only Q, B, L, lepton family – for massless ν)

  New effects:

  $\Rightarrow$ many new decay channels available:
  - $\pi^- \to \ell^- \bar{\nu}_\ell$, $\pi^+ \to \ell^+ \nu_\ell$ (violates flavour symmetries)
  - $K \to 2\pi, 3\pi$ (violate S, I – also P)

  $\Rightarrow$ P,C-violation effects

  $\Rightarrow$ CP-violation effects

  $\Rightarrow$ K^0-$\bar{K}^0$ oscillations

Parity violation

P, C violations in weak interactions:

- 1956: Yang and Lee suggest P violations to solve θ-τ puzzle
- 1956: Wu's experiment demonstrates it

β-decay of polarised nuclei of cobalt 60, ${}^{60}_{27}\mathrm{Co}$ ($J = 5$)
$\rightarrow$ excited nichel 60, ${}^{60}_{28}\mathrm{Ni}^*$ ($J = 4$)
$\rightarrow$ electromagnetic decay to ground state emitting two photons

$$
\begin{aligned}
{}^{60}_{27}\mathrm{Co} \quad \longrightarrow \quad & {}^{60}_{28}\mathrm{Ni}^* + e^- + \bar{\nu}_e \\
& \;\hookrightarrow {}^{60}_{28}\mathrm{Ni} + \gamma + \gamma
\end{aligned}
$$

Fundamental process: β-decay $n \rightarrow p\ e^-\ \bar{\nu}_e$

- cool down Co, put in uniform magnetic to polarise their spin in (say) up direction
- e, $\bar{\nu}_e$ are spin-$\frac{1}{2}$, angular momentum conservation requires Ni, e,$\bar{\nu}_e$ all polarised in the up direction
- Ni essentially at rest, e and $\bar{\nu}_e$ emitted back-to-back to conserve momentum

Parity violation (contd.)

If P were a symmetry: $\vec{s} \underset{P}{\rightarrow} \vec{s}$, equally probable to find electrons emitted in the direction of nuclear spin and in the direction opposite to it

Experiment shows preferential emission opposite to nuclear spin $\Rightarrow$ P violation

Why?

- $\bar{\nu}_e$ only exist with positive helicity $h = \frac{\vec{p} \cdot \vec{s}}{|\vec{p}\,|}$
- if e emitted opposite to nuclear spin $\Rightarrow$ $h_e = -1$, $h_{\bar{\nu}} = +1$ all right
- if e emitted along nuclear spin $\Rightarrow$ $h_e = +1$, $h_{\bar{\nu}} = -1$ impossible

${}^{60}_{27}\overline{\mathrm{Co}} \rightarrow {}^{60}_{28}\overline{\mathrm{Ni}}^* + e^+ + \nu_e$: e^+ emitted preferentially along nuclear spin since $h_\nu = -1 \Rightarrow C$ violated, but CP apparently not

Is CP a symmetry of weak interactions? Optimal place to look for CP violations is the K^0–$\bar{K}^0$ system