Particle physics

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November 6, 2020

SU(3) (contd.)

Three more commutation relations directly form SU(2)

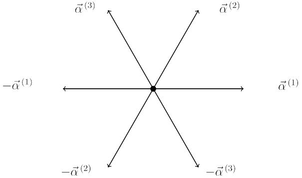
$[I_+, I_-] = 2I_3 \quad [V_+, V_-] = 2V_3 = I_3 + \frac{3}{2}Y \quad [W_+, W_-] = 2W_3 = -I_3 + \frac{3}{2}Y$

Mathematically more convenient to use $\bar{Y} = \frac{\sqrt{3}}{2}Y$ $(= t^8)$

$$\vec{H} = (I_3, \bar{Y}) \qquad E_\pm^{(1)} = I_\pm \qquad E_\pm^{(2)} = V_\pm \qquad E_\pm^{(3)} = W_\pm$$

Compact form of commutation relations:

$$[\vec{H}, E_\pm^{(j)}] = \pm\vec{\alpha}^{(j)} E_\pm^{(j)}$$



$\vec{\alpha}^{(j)}$: *root vectors*, $(\vec{\alpha}^{(j)})^2 = 1$

$\vec{\alpha}^{(1)} = (1, 0) \quad \vec{\alpha}^{(2)} = (\frac{1}{2}, \frac{\sqrt{3}}{2}) \quad \vec{\alpha}^{(3)} = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$

Define linear operator ad_A, $\mathrm{ad}_A X \equiv [A, X]$, acting on the algebra

Simultaneous eigenvectors of $\mathrm{ad}_{H^{1,2}}$:

- $E_\pm^{(j)}$ with eigenvalues $\pm\vec{\alpha}^{(j)}$
- $H^{1,2}$ with both eigenvalues 0

SU(3) (contd.)

Missing: commutators among ladder operators

Can be computed explicitly, but more instructive argument using Jacobi:

$$\begin{aligned}[\vec{H},[E_s^{(i)},E_t^{(j)}]] &= -[E_t^{(j)},[\vec{H},E_s^{(i)}]] - [E_s^{(i)},[E_t^{(j)},\vec{H}]] \\ &= -s\vec{\alpha}^{(i)}[E_t^{(j)},E_s^{(i)}] + t\vec{\alpha}^{(j)}[E_s^{(i)},E_t^{(j)}] \\ &= (s\vec{\alpha}^{(i)} + t\vec{\alpha}^{(j)})[E_s^{(i)},E_t^{(j)}]\end{aligned}$$

$\Rightarrow [E_s^{(i)},E_t^{(j)}] \propto$ simultaneous eigenvector with eigenvalues $s\vec{\alpha}^{(i)} + t\vec{\alpha}^{(j)}$

In general $s\vec{\alpha}^{(i)} + t\vec{\alpha}^{(j)}$ is *not* a vector of eigenvalues

$\Rightarrow [E_s^{(i)},E_t^{(j)}] = 0$ unless $s\vec{\alpha}^{(i)} + t\vec{\alpha}^{(j)} = u\vec{\alpha}^{(k)}$ for some u, k

$$\Longrightarrow \boxed{[E_s^{(i)},E_t^{(j)}] \propto E_u^{(k)}}$$

SU(3) (contd.)

- If SU(3) were exact symmetry, only two generators could be measured simultaneously, but large arbitrariness in choosing them
- Preferred choice exists because SU(3) symmetry is broken: generators used to label the physical states correspond to unbroken

$$\mathrm{SU}(3) \underset{m_s \neq m_u = m_d}{\rightarrow} \mathrm{SU}(2)_I \times \mathrm{U}(1)_Y \underset{\substack{m_u \neq m_d \\ q_u \neq q_d}}{\rightarrow} \mathrm{U}(1)_Q \times \mathrm{U}(1)_Y$$

$\Longrightarrow$ choose I_3, Y, and $\vec{I}^{\,2}$ (*not* an element of the Lie algebra)

Simplest irreducible representations of SU(3)

Trivial representation: $D_T(U) = 1 \; \forall U \in \mathrm{SU}(3)$

- one-dimensional: **1**
- good for any group
- corresponding representation of the algebra: $d(t^a) = 0$

Fundamental (defining) representation $D_F(U) = U$

- three-dimensional: **3**
- good for any matrix Lie groups
- corresponding representation of the algebra: $d_F(t^a) \equiv t_F^a = t^a$

Writing $U = e^{i\alpha\cdot t}$, D_F from d_F

$$\boxed{t^a \Rightarrow D_F(U) = D_F(e^{i\alpha\cdot t}) = U = e^{i\alpha\cdot t}}$$

Complex conjugate representation: $D_C(U) = U^*$

- three-dimensional: $\bar{\mathbf{3}}$
- good for any matrix group, but non necessarily a new rep (e.g., for SU(2) $\mathbf{2} \sim \bar{\mathbf{2}}$)
- for SU(3) $\mathbf{3} \nsim \bar{\mathbf{3}}$
- corresponding algebra rep: $d_C(t^a) \equiv t_C^a = (-t^a)^* = -(t^a)^T$

$$\mathrm{tr}\, t_C^a t_C^b = \mathrm{tr}\,(-t^a)^*(-t^b)^* = \tfrac{1}{2}\delta^{ab}$$

It *is* a representation

- For the group:

$$D_C(U_1U_2) = (U_1U_2)^* = U_1^*U_2^* = D_C(U_1)D_C(U_2) \qquad D_C(\mathbf{1}) = \mathbf{1}$$

- For the algebra:

$$[t^a, t^b]^* = [(t^a)^*, (t^b)^*] = [(-t^a)^*, (-t^b)^*] = -if_{abc}(t^c)^* = if_{abc}(-t^c)^*$$

$$\boxed{t_C^a = -(t^a)^* \Rightarrow D_C(U) = D_C(e^{i\alpha\cdot t}) = U^* = e^{-i\alpha\cdot t^*} = e^{i\alpha\cdot t_C}}$$

Simplest irreducible representations of SU(3) (contd.)

Adjoint representation: $D_A(U) = \mathrm{Ad}_U$

- eight-dimensional: **8**
- exists for any Lie group

Ad_U: linear transformation acting on the algebra: $X = X_a t^a$

$$\mathrm{Ad}_U X \equiv U X U^\dagger \qquad \mathrm{Ad}_U X = X_b \mathrm{Ad}_U t^b = t^a (\mathrm{Ad}_U)_{ab} X_b$$

$(\mathrm{Ad}_U)_{ab}$ are 8×8 matrices, and provide a representation:

$$\mathrm{Ad}_{U_1} \mathrm{Ad}_{U_2} X = U_1 U_2 X U_2^\dagger U_1^\dagger = (U_1 U_2) X (U_1 U_2)^\dagger = \mathrm{Ad}_{U_1 U_2} X$$

What are the generators? For infinitesimal $U \simeq \mathbf{1} + i\alpha \cdot t$

$$\begin{aligned} U X U^\dagger &\simeq (\mathbf{1} + i\alpha \cdot t) X (\mathbf{1} - i\alpha \cdot t) \simeq X + i\alpha \cdot [t, X] \\ &= X + i\alpha_a X_b [t^a, t^b] = (X_b + i\alpha_a X_c i f_{acb}) t^b \\ &= t^b [\delta_{bc} + i\alpha_a (-i f_{abc})] X_c = t^b [\delta_{bc} + i\alpha_a (T^a)_{bc}] X_c \end{aligned}$$

$$(T^a)_{bc} \equiv -i f_{abc} \qquad T^a X = [t^a, X] = \mathrm{ad}_{t^a} X$$

Simplest irreducible representations of SU(3) (contd.)

Abstractly: $\mathrm{ad}_X Y = [X, Y]$ linear transf. of the algebra (as a vector space)

$$
\begin{aligned}
[\mathrm{ad}_X, \mathrm{ad}_Y]Z &= (\mathrm{ad}_X \mathrm{ad}_Y - \mathrm{ad}_Y \mathrm{ad}_X)Z = [X,[Y,Z]] - [Y,[X,Z]] \\
&= [X,[Y,Z]] + [X,[Z,Y]] = [[X,Y],Z] = \mathrm{ad}_{[X,Y]} Z
\end{aligned}
$$

$X \to \mathrm{ad}_X$ is a representation of the algebra

$$\mathrm{ad}_{t^a} X = X_b [t^a, t^b] = t^c(-i f_{acb}) X_b = t^c (T^a)_{cb} X_b$$

More directly from Jacobi identity

$$
\begin{aligned}
(-i) f_{bam} (-i) f_{cmn} - (-i) f_{cam} (-i) f_{bmn} &= i f_{bcm} (-i) f_{man} \\
([T^a, T^b])_{mn} &= i f_{abc} (T^c)_{mn}
\end{aligned}
$$

$$\mathrm{tr}\, T^a T^b = -f_{amn} f_{bnm} = f_{amn} f_{bmn} = 3\delta^{ab}$$

$$\boxed{t_A^a = T^a \Rightarrow D_A(U) = \mathrm{Ad}_{e^{i\alpha\cdot t}} = e^{i \mathrm{ad}_{\alpha \cdot t}}}$$

Weight diagrams

More interested in representation space than in representative matrices: basis corresponds to mass-degenerate particle multiplets

- Convenient basis: eigenvectors of I_3 and Y $\sim$ basis of physical particles with definite Q and S, corresponding to unbroken part of symmetry $U(1)_Y \times U(1)_{I_3} \sim U(1)_Y \times U(1)_Q$

  Ignore weak interactions

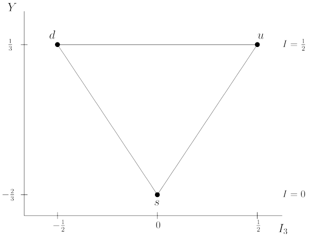
- Find *weights*, pairs of simultaneous eigenvalues (i_3, y) of I_3, Y representatives of in an irrep, plot in $(i_3, \frac{\sqrt{3}}{2}y)$ plane

  $\Rightarrow$ *weight diagram* of the irreducible representation
  - completely and uniquely identifies a representation
  - tells degeneracy of each weight

- Also convenient to organise eigenvectors in isospin multiplets

  $\mathrm{SU}(2)_I$ good approx. symmetry of strong interactions

Fundamental representation

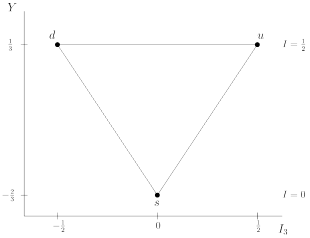


Representation space: $\mathbb{C}^3$
Representatives of generators: $t^a = \frac{\lambda^a}{2}$
$\Rightarrow I_3^F$ and Y^F already diagonal

$$I_3^F = \frac{1}{2}\lambda^3 = \mathrm{diag}\left(\tfrac{1}{2}, -\tfrac{1}{2}, 0\right)$$
$$Y^F = \frac{1}{\sqrt{3}}\lambda^8 = \mathrm{diag}\left(\tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\right)$$

Basis vectors $e_i^{(j=1,2,3)} = \delta_{ij}$ eigenstates of $(I_3^F, Y^F) \sim$ physical states

$$e^{(1)} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad e^{(2)} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad e^{(3)} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$



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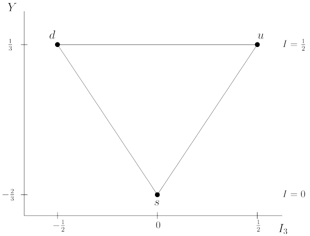
$$(I_3^F, Y^F)e^{(1)} = (\tfrac{1}{2}, \tfrac{1}{3})e^{(1)} \qquad I_+^F e^{(2)} = e^{(1)} \qquad I_-^F e^{(1)} = e^{(2)}$$
$$(I_3^F, Y^F)e^{(2)} = (-\tfrac{1}{2}, \tfrac{1}{3})e^{(2)} \qquad V_+^F e^{(3)} = e^{(1)} \qquad V_-^F e^{(1)} = e^{(3)}$$
$$(I_3^F, Y^F)e^{(3)} = (0, -\tfrac{2}{3})e^{(3)} \qquad W_+^F e^{(3)} = e^{(2)} \qquad W_-^F e^{(2)} = e^{(3)}$$

All other combinations give 0

Fundamental representation



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Using $\vec{I}^2 = I_- I_+ + I_3(I_3+1) = I_+ I_- + I_3(I_3-1)$

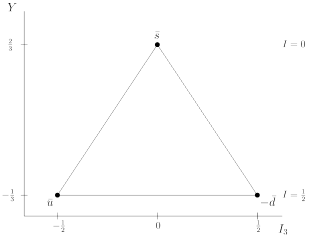
$$\vec{I}_F^2 e^{(1)} = \tfrac{3}{4} e^{(1)} \quad \vec{I}_F^2 e^{(2)} = \tfrac{3}{4} e^{(2)} \quad \vec{I}_F^2 e^{(3)} = 0$$

$\Rightarrow e^{(1)}, e^{(2)}$ isodoublet ($I = \frac{1}{2}$), $e^{(3)}$ isosinglet ($I = 0$)

$$e^{(1)} = |\tfrac{1}{2}\,\tfrac{1}{2}; \tfrac{1}{3}\rangle \quad e^{(2)} = |\tfrac{1}{2}\ -\tfrac{1}{2}; \tfrac{1}{3}\rangle \quad e^{(3)} = |0\,0; -\tfrac{2}{3}\rangle$$

All nonzero matrix elements of $I_\pm$, $V_\pm$, $W_\pm$ are positive

Complex conjugate representation



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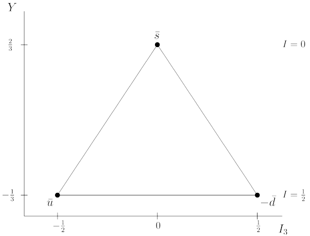

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$$(I_3^C, Y^C)e^{(1)} = (-\tfrac{1}{2}, -\tfrac{1}{3})e^{(1)} \qquad I_+^C e^{(1)} = -e^{(2)} \qquad I_-^C e^{(2)} = -e^{(1)}$$
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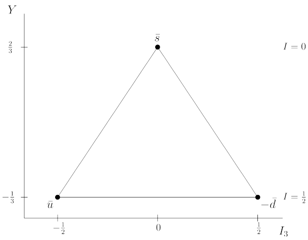
$$(I_3^C, Y^C)e^{(3)} = (0, \tfrac{2}{3})e^{(3)} \qquad W_+^C e^{(2)} = -e^{(3)} \qquad W_-^C e^{(3)} = -e^{(2)}$$

All other combinations give 0

$$I_\pm^C = -(I_\pm^F)^* = -I_\mp^F \qquad V_\pm^C = -(V_\pm^F)^* = -V_\mp^F \qquad W_\pm^C = -(W_\pm^F)^* = -W_\mp^F$$

Complex conjugate representation



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Isomultiplets:

$$\vec{I}_C^{\,2} e^{(1)} = \tfrac{3}{4} e^{(1)} \quad \vec{I}_C^{\,2} e^{(2)} = \tfrac{3}{4} e^{(2)} \quad \vec{I}_C^{\,2} e^{(3)} = 0$$

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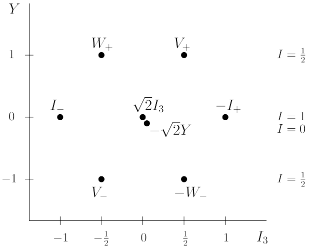
Signs choosen to have $I_\pm$ and $W_\pm$ with positive matrix elements only, cannot have also $V_\pm$

Adjoint representation



Representation space: $\mathbb{C}^8 \sim \mathfrak{su}(3)_{\mathbb{C}}$
Representatives of generators: $(T^a)_{bc} = -if_{abc}$
$\Rightarrow I_3^A$ and Y^A diagonal in the basis

$$\{\vec{H}, E_\pm^{(s)}\} = \{I_3, \bar{Y}, I_\pm, V_\pm, W_\pm\}$$

$$\bar{Y} = t^8$$

$$T^a X \equiv \mathrm{ad}_{t^a} X \equiv [t^a, X]$$

Basis vectors $\{I_3, \bar{Y}, I_\pm, V_\pm, W_\pm\}$ are eigenstates of $(I_3^A, Y^A) \sim$ physical states

$$\vec{H}^A E_\pm^{(s)} = [\vec{H}, E_\pm^{(s)}] = \pm\vec{\alpha}^{(s)} E_\pm^{(s)} \qquad \vec{H}^A H_i = [\vec{H}, H_i] = 0$$

$$T^{3,8} I_3 = 0 \qquad T^{3,8}\bar{Y} = 0 \qquad T^3 I_\pm = \pm I_\pm \qquad T^8 I_\pm = 0$$

$$T^3 V_\pm = \pm\tfrac{1}{2} V_\pm \qquad T^8 V_\pm = \pm\tfrac{\sqrt{3}}{2} V_\pm \qquad T^3 W_\pm = \mp\tfrac{1}{2} W_\pm \qquad T^8 W_\pm = \pm\tfrac{\sqrt{3}}{2} W_\pm$$

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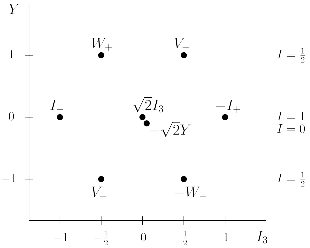

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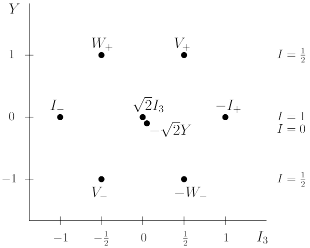
$$\vec{I}_A^2 X = \sum_{j=1}^{3} (T^j)^2 X = \sum_{j=1}^{3} [t^j, [t^j, X]]$$

$\Rightarrow$ (I_-, I_3, I_+) isotriplet $(I = 1)$, $\bar{Y}$ isosinglet $(I = 0)$
(W_+, V_+) and (V_-, W_-) isodoublets $(I = \frac{1}{2})$

Adjoint representation



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Basis vectors $\{I_3, \bar{Y}, I_{\pm}, V_{\pm}, W_{\pm}\}$ are eigenstates of (I_3^A, Y^A) $\sim$ physical states

$$|\tfrac{1}{2}\ -\tfrac{1}{2}; 1\rangle \propto W_+ \qquad |\tfrac{1}{2}\ \tfrac{1}{2}; 1\rangle \propto V_+$$

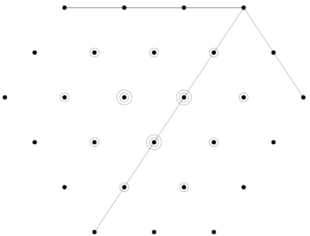
$$|1\ -1; 0\rangle \propto I_- \qquad |1\,0; 0\rangle \propto I_3 \qquad |1\,1; 0\rangle \propto I_+$$

$$|0\,0; 0\rangle \propto Y$$

$$|\tfrac{1}{2}\ -\tfrac{1}{2}; -1\rangle \propto V_- \qquad |\tfrac{1}{2}\ \tfrac{1}{2}; -1\rangle \propto W_-$$

$\Rightarrow$ looks like the baryon octet!

General irreducible finite-dimensional representations



- look for finite-dimensional Hermitian representation of complexified algebra
- diagonalise I_3, Y
- find highest-weight vector $|\psi\rangle$, $I_+|\psi\rangle = V_+|\psi\rangle = W_+|\psi\rangle = 0$ (exists, unique in finite-dim irrep), highest weight $\vec{x}_\psi = (i_0, \frac{\sqrt{3}}{2} y_0)$
- build remaining vectors using I_-, V_-, W_-
- lowering operators are part of $\mathfrak{su}_2$ algebra $\to$ stop after $2i_0$, $2v_0$, $2w_0$ times $\Rightarrow 2i_0, 2w_0$ integers, $\frac{3}{2} y_0 = i_0 + 2w_0$, $v_0 = i_0 + w_0$
- weight diagram reflection-symmetric wrt axis $\perp$ root vectors
- count degeneracies:
  - weights on the boundary are nondegenerate
  - degeneracy increases by one moving from one hexagonal layer to the next
  - degeneracy further increases by one moving to first triangular layer, then constant
- summary: irrep characterised by pair of half-integers (i_0, w_0), dimension $d = (2i_0 + 1)(2w_0 + 1)(i_0 + w_0 + 1)$
- given (i_0, w_0) irrep, (w_0, i_0) is its complex-conjugate ($i_0 = w_0$: real irrep)

The "Eightfold Way"

Baryon octet:

- highest weight = p, $i = \frac{1}{2}, y = 1 \to i_0 = w_0 = \frac{1}{2}$
- $2(i_0 + w_0) + 1 = 3$ lines of constant y with $2, 4, 2$ states
  - isodoublet $i = \frac{1}{2}$ with $y = 1$ $(S = 0) \Rightarrow n, p$
  - isotriplet $i = 1$ and isosinglet $i = 0$ $(S = -1)$ with $y = 0 \Rightarrow \Sigma^{-,0,+}$, Λ
  - isodoublet $i = \frac{1}{2}$ with $y = -1$ $(S = -2) \Rightarrow \Xi^{-,0}$

Meson "septuplet"? Impossible $(d_{\mathrm{irrep}} \neq 7) \Rightarrow$ meson octet

  - isodoublet $i = \frac{1}{2}$ with $y = 1$ $(S = 1) \Rightarrow K^0, K^+$
  - isotriplet $i = 1$ and isosinglet $i = 0$ with $y = 0$ $(S = 0) \Rightarrow \pi^{-,0,+}$, η
  - isodoublet $i = \frac{1}{2}$ with $y = -1$ $(S = -1) \Rightarrow K^-, \bar{K}^0$

Baryon $s = \frac{3}{2}$ resonances "nonuplet"? Impossible $(d_{\mathrm{irrep}} \neq 9) \Rightarrow$ baryon decuplet

- highest weight = Δ^{++}, $i = \frac{3}{2}, y = 1 \to i_0 = \frac{3}{2}, w_0 = 0$
- triangular weight diagram, non-degenerate weights
- $2(i_0 + w_0) + 1 = 4$ lines of constant y with $4, 3, 2, 1$ states
  - isoquartet $i = \frac{3}{2}$ with $y = 1$ $(S = 0) \Rightarrow \Delta^{-,0,+,++}$
  - isotriplet $i = 1$ with $y = 0$ $(S = -1) \Rightarrow \Sigma^{*\,-,0,+}$
  - isodoublet $i = \frac{1}{2}$ with $y = -1$ $(S = -2) \Rightarrow \Xi^{*\,-,0}$
  - isosinglet $i = 0$ with $y = -2$ $(S = -3) \Rightarrow \Omega^-$

The "Eightfold Way"

Baryon octet:

- highest weight $= p$, $i = \frac{1}{2}, y = 1 \to i_0 = w_0 = \frac{1}{2}$
- $2(i_0 + w_0) + 1 = 3$ lines of constant y with $2, 4, 2$ states
  - isodoublet $i = \frac{1}{2}$ with $y = 1$ $(S = 0) \Rightarrow n, p$
  - isotriplet $i = 1$ and isosinglet $i = 0$ $(S = -1)$ with $y = 0 \Rightarrow \Sigma^{-,0,+}$, Λ
  - isodoublet $i = \frac{1}{2}$ with $y = -1$ $(S = -2) \Rightarrow \Xi^{-,0}$

Meson "septuplet"? Impossible $(d_{\rm irrep} \neq 7) \Rightarrow$ meson octet

- isodoublet $i = \frac{1}{2}$ with $y = 1$ $(S = 1) \Rightarrow K^0, K^+$
- isotriplet $i = 1$ and isosinglet $i = 0$ with $y = 0$ $(S = 0) \Rightarrow \pi^{-,0,+}$, η
- isodoublet $i = \frac{1}{2}$ with $y = -1$ $(S = -1) \Rightarrow K^-, \bar{K}^0$

Baryon $s = \frac{3}{2}$ resonances "nonuplet"? Impossible $(d_{\rm irrep} \neq 9) \Rightarrow$ baryon decuplet

- highest weight $= \Delta^{++}$, $i = \frac{3}{2}, y = 1 \to i_0 = \frac{3}{2}, w_0 = 0$
- triangular weight diagram, non-degenerate weights
- $2(i_0 + w_0) + 1 = 4$ lines of constant y with $4, 3, 2, 1$ states
  - isoquartet $i = \frac{3}{2}$ with $y = 1$ $(S = 0) \Rightarrow \Delta^{-,0,+,++}$
  - isotriplet $i = 1$ with $y = 0$ $(S = -1) \Rightarrow \Sigma^{*\,-,0,+}$
  - isodoublet $i = \frac{1}{2}$ with $y = -1$ $(S = -2) \Rightarrow \Xi^{*\,-,0}$
  - isosinglet $i = 0$ with $y = -2$ $(S = -3) \Rightarrow \Omega^-$

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The "Eightfold Way" (contd.)

If SU(3) symmetry is correct, tenth particle had to exist:

$$s_\Omega = \frac{3}{2} \qquad i_\Omega = 0 \qquad Q_\Omega = -1$$

Gell-Mann–Nishijima formula $Q = i_3 + \frac{1}{2}y$

Mass pattern:

$$m_\Delta = 1232 \text{ MeV} \qquad m_{\Sigma^*} = 1384 \text{ MeV} \qquad m_{\Xi^*} = 1533 \text{ MeV}$$

$$\Longrightarrow m(S) \simeq m(\Delta) + 150 \text{ MeV} \cdot |S|$$

$\Rightarrow$ guess $m_\Omega = 1682 \text{ MeV}$
Resonance with $s_\Omega = \frac{3}{2}$, $Q_\Omega = -1$ observed in 1964 with $m_\Omega = 1672 \text{ MeV}$

"*Eightfold way*" (Gell-Mann, 1961; Ne'eman, 1961), i.e., classification of hadron multiplets in terms of irreducible representations of SU(3), works!

From SU(3) invariance to the quark model

Eightfold way works, but why only certain irreps are observed in nature?

- baryons: only octets and decuplets
- mesons: only octets and singlets (e.g.: η')
- no fundamental or complex-conjugate (simplest irreps)

Known result: all irreps of SU(3) are obtained from fundamental (3) and complex conjugate ($\bar{3}$) representations by decomposing tensor products

Fundamental representation actually suffices

$$3 \otimes \bar{3} = 8 \oplus 1 \qquad \Rightarrow \text{meson irreps}$$

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1 \qquad \Rightarrow \text{baryon irreps (except singlet)}$$

What if hadrons are bound states of constituents transforming in fundamental rep 3 (and their antiparticles transforming in $\bar{3}$)?

- supported by Gell-Mann–Nishijima formula $Q = I_3 + \frac{1}{2}Y$
- supported by approximate mass relation
  $m(S) = m(0) + 150\ \mathrm{MeV} \cdot |S|$ for octet and decuplet baryons

Elementary constituents: *quarks* (Gell-Mann) or “aces” (Zweig)

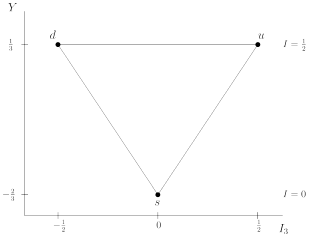
Zweig was the first to believe in the physical existence of quark

From SU(3) invariance to the quark model (contd.)

Can charges be assigned to quarks to reproduce phenomenology?

- I_3, Y come from SU(3) fundamental rep: three states = *flavours*: u, d and s

$$\begin{aligned} u:\ & i_3 = \tfrac{1}{2} && y = \tfrac{1}{3} \\ d:\ & i_3 = -\tfrac{1}{2} && y = \tfrac{1}{3} \\ s:\ & i_3 = 0 && y = -\tfrac{2}{3} \end{aligned}$$



- if baryon=$(q_1 q_2 q_3)$ $\Rightarrow$ $B_u = B_d = B_s = \frac{1}{3}$
- extract $8 \subset 3 \times 3 \times 3$ $\Rightarrow$ content of baryons: $p = (uud)$, $n = (udd)$, $\Lambda = (uds)$
- $Q_{u,d,s}$ and $S_{u,d,s}$ fixed by baryon octet

$$\begin{aligned} p:&\quad 2Q_u + Q_d = 1 &\quad 2S_u + S_d &= 0 \\ n:&\quad Q_u + 2Q_d = 0 &\quad S_u + 2S_d &= 0 \\ \Lambda:&\quad Q_u + Q_d + Q_s = 0 &\quad S_u + S_d + S_s &= -1 \end{aligned}$$

$\Rightarrow$ $Q_u = \frac{2}{3}$, $Q_d = Q_s = -\frac{1}{3}$, and $S_u = S_d = 0$ and $S_s = -1$

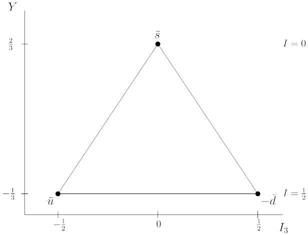
- for each flavour $f = u, d, s$

$$Q_f = I_{3f} + \tfrac{1}{2} Y_f = I_{3f} + \tfrac{1}{2}(B_f + S_f)$$

$\Rightarrow$ GMN and $Y = B + S$ relation automatically satisfied by all baryons

From SU(3) invariance to the quark model (contd.)

Quantum Field Theory requires existence of antiparticles
$\Rightarrow$ *antiquarks* $\bar{u}$, $\bar{d}$, and $\bar{s}$



- charge-conjugation pairs, $Cf = \bar{f}$
- same spin and mass as quarks
- minus all the charges (I_3, Y, Q, B)
- if q transform in rep R of a symmetry group, $\bar{q}$ must transform in $\bar{R}$ (complex-conjugate)
- if meson$=(q_1\bar{q}_2) \Rightarrow B_{\mathrm{meson}} = 0$
- GMN and hypercharge/strangeness relations still satisfied

Light baryons, resonances have $s = \frac{1}{2}$, $s = \frac{3}{2} \Rightarrow$ assign $s_{q,\bar{q}} = \frac{1}{2}$

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$$

- baryons/antibaryons are fermions: spin from three half-integer quark/antiquark spins and two integer relative orbital momenta
- mesons are bosons: spin from two half-integer quark/antiquark spins and one integer relative orbital momentum

	I	I_3	Y	Q	S	B
u	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$
d	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$
s	0	0	$-\frac{2}{3}$	$-\frac{1}{3}$	-1	$\frac{1}{3}$
$\bar{u}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$
$\bar{d}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
$\bar{s}$	0	0	$\frac{2}{3}$	$\frac{1}{3}$	1	$-\frac{1}{3}$