Particle physics

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Isospin and hadron multiplets

Isospin invariance $[\vec{I}, H_s] = 0$ implies degenerate energy eigenspaces are representation spaces for irreducible representations of SU(2)

Up to accidental degeneracies

Eigenspace $\{|E;n\rangle,\ n = 1,\dots,N\}$ of energy E, $H_s|E;n\rangle = E|E;n\rangle$, left invariant by $U(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{\mathbf{I}}}$

$U(\vec{\alpha})$: family of unitary transformations on $\mathcal{H}$, exists thanks to Wigner's theorem
$\vec{\mathbf{I}}$: operators, representatives of isospin generators in the full (infinite-dimensional) representation on $\mathcal{H}$

$$H_s\left(U|E;n\rangle\right) = UH_s|E;n\rangle = E\left(U|E;n\rangle\right) \Rightarrow U|E;n\rangle = \mathcal{U}_{n'n}|E;n'\rangle$$

Since $\mathcal{U}$ provide an irreducible representation $\Rightarrow N = 2i+1$, $2i \in \mathbb{N}_0$
Experimental observation of hadron multiplets: almost degenerate in mass, same baryon number/strangeness but different electric charge

- assign i from dimension $2i+1$ of multiplet
- assign increasing i_3 to increasing electric charge (no same-charge states in multiplets observed)

Isospin and hadron multiplets (contd.)

Empirical formula for el. charge Q (*Gell-Mann–Nishijima formula*)

$$Q = I_3 + \tfrac{1}{2}(B+S) = I_3 + \tfrac{1}{2}Y$$

B: baryon number
S: strangeness
$Y = B + S$: *hypercharge*

Examples:

- p, n: nucleon doublet
- $\pi^{-,0,+}$: pion triplet
- K^+, K^0 and $\bar{K}^0, K^-$: kaon doublets
- $\Sigma^{-,0,+}$: Sigma triplet
- $\Delta^{-,0,+,++}$: Delta quartet

particle	i	i_3	Q	B	S	Y
p	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	1
n	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	0	1
π^+	1	1	1	0	0	0
π^0	1	0	0	0	0	0
π^-	1	-1	-1	0	0	0
K^+	$\frac{1}{2}$	$\frac{1}{2}$	1	0	1	1
K^0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1	1
$\bar{K}^0$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	-1	-1
K^-	$\frac{1}{2}$	$-\frac{1}{2}$	-1	0	1	-1
Σ^+	1	1	1	1	-1	0
Σ^0	1	0	0	1	-1	0
Σ^-	1	-1	-1	1	-1	0
Δ^{++}	$\frac{3}{2}$	$\frac{3}{2}$	2	1	0	1
Δ^+	$\frac{3}{2}$	$\frac{1}{2}$	1	1	0	1
Δ^0	$\frac{3}{2}$	$-\frac{1}{2}$	0	1	0	1
Δ^-	$\frac{3}{2}$	$-\frac{3}{2}$	-1	1	0	1

Isospin and hadron multiplets (contd.)

Modern perspective: isomultiplets understood in terms of u, d quarks
Light quark/antiquark isodoublet, flavour wave function

$$q = \psi_u u + \psi_d d \qquad \bar{q} = \psi_{\bar{u}} \bar{u} + \psi_{\bar{d}} \bar{d}$$

$$u = \bar{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad d = \bar{d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Under SU(2) transformation $q \to Uq \qquad \bar{q} \to U^* \bar{q}$

Required by QFT

U: **fundamental** (defining), U^*: **complex-conjugate** representations

Equivalent for SU(2)

Strange quark: isosinglet ($I = 0$), under SU(2) $s, \bar{s} \to s, \bar{s}$, trivial rep.
Composing the representations of q, s and $\bar{q}, \bar{s} \to$ meson multiplets

$$\underset{q}{\tfrac{1}{2}} \otimes \underset{\bar{q}}{\tfrac{1}{2}} = 0 \oplus 1 \qquad \underset{q}{\tfrac{1}{2}} \otimes \underset{\bar{s}}{0} = \tfrac{1}{2} \qquad \underset{s}{0} \otimes \underset{\bar{q}}{\tfrac{1}{2}} = \tfrac{1}{2}$$

$$\text{dimension:} \quad 2 \times 2 = 1 + 3 \qquad 2 \times 1 = 2 \qquad 2 \times 1 = 2$$

Composing three representations of $q, s \to$ baryon multiplets

Isospin and hadron multiplets (contd.)

Fundamental $U = e^{i\vec{\alpha}\cdot\vec{I}}$ and complex-conjugate $U^* = e^{-i\vec{\alpha}\cdot\vec{I}^*} = e^{i\vec{\alpha}\cdot\vec{I}_C}$ representations $\Rightarrow$ fundamental and complex-conjugate rep. of $\mathfrak{su}_2$

$$\vec{I}_C = -\vec{I}^* \implies I_{C3} = -I_3 \qquad I_{C+} = -I_- \qquad I_{C-} = -I_+$$

$I^*_{1,3} = I_{1,3}, I^*_2 = -I_2$

$$\begin{array}{cccc} I_3 u = \frac{1}{2} u & I_3 d = -\frac{1}{2} d & I_- u = d & I_+ d = u \\ I_{C3}\bar{u} = -I_3\bar{u} & I_{C3}\bar{d} = -I_3\bar{d} & I_{C-}\bar{d} = -I_+\bar{d} & I_{C+}\bar{u} = -I_-\bar{u} \\ = -\frac{1}{2}\bar{u} & = \frac{1}{2}\bar{d} & = -\bar{u} & = -\bar{d} \end{array}$$

Relation between isospin eigenstates and flavour eigenstates

$$\boxed{u = |q; \tfrac{1}{2}\ +\tfrac{1}{2}\rangle \quad d = |q; \tfrac{1}{2}\ -\tfrac{1}{2}\rangle} \qquad \boxed{\bar{u} = |\bar{q}; \tfrac{1}{2}\ -\tfrac{1}{2}\rangle \quad \bar{d} = -|\bar{q}; \tfrac{1}{2}\ +\tfrac{1}{2}\rangle}$$

Representation composition $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$, triplet $I = 1$

$$\boxed{\begin{array}{rcll} -u\bar{d} = & |q; \tfrac{1}{2}\ \tfrac{1}{2}\rangle \otimes |\bar{q}; \tfrac{1}{2}\ \tfrac{1}{2}\rangle & = |1\,1\rangle & = \pi^+ \\ \frac{u\bar{u}-d\bar{d}}{\sqrt{2}} = & \frac{1}{\sqrt{2}}\left[|q; \tfrac{1}{2}\ \tfrac{1}{2}\rangle \otimes |\bar{q}; \tfrac{1}{2}\ -\tfrac{1}{2}\rangle + |q; \tfrac{1}{2}\ -\tfrac{1}{2}\rangle \otimes |\bar{q}; \tfrac{1}{2}\ \tfrac{1}{2}\rangle\right] & = |1\,0\rangle & = \pi^0 \\ d\bar{u} = & |q; \tfrac{1}{2}\ -\tfrac{1}{2}\rangle \otimes |\bar{q}; \tfrac{1}{2}\ -\tfrac{1}{2}\rangle & = |1\ -1\rangle & = \pi^- \end{array}}$$

Isospin and hadron multiplets (contd.)

- Three quarks for a baryon,
  $n_u + n_d + n_s = 3$
  $\Rightarrow \sum_f \left(b_f - \frac{1}{3}\right) n_f = 0 \Rightarrow b_f = \frac{1}{3}$
- Isospin assignment
  $\Rightarrow I_3 = \frac{1}{2}\left(n_u - n_d\right)$
- Strangeness associated to presence of strange quarks
  $\Rightarrow S = -n_s$

Quantum numbers

	I	I_3	Q	B	S	Y
u	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$
d	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$
$\bar{u}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$
$\bar{d}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$
s	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	-1	$-\frac{2}{3}$
$\bar{s}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{2}{3}$

- From p, n isospin and strangeness $\Rightarrow n_u^{(p)} = n_d^{(n)} = 2,\ n_u^{(n)} = n_d^{(p)} = 1$, from charge $\Rightarrow q_u = \frac{2}{3},\ q_d = -\frac{1}{3}$
- From Λ $(Q = 0, S = -1, I = 0) \Rightarrow q_s = -\frac{1}{3}$

Gell-Mann–Nishijima satisfied at quark level since

$$I_3 = \frac{1}{2}(n_u - n_d) \quad Q = \frac{2}{3}n_u - \frac{1}{3}(n_d + n_s) \quad B = \frac{1}{3}(n_u + n_d + n_s) \quad S = -n_s$$

A bit *ad hoc* construction, will be fully justified in the quark model

Isospin conservation and scattering processes

$\pi N \to \pi' N'$ scattering, $\mathcal{M}_{\pi N \to \pi' N'} = \langle \pi' \, N' | H_I | \pi N \rangle$

$$p\,\pi^+ \to p\,\pi^+ \qquad p\,\pi^- \to p\,\pi^- \qquad p\,\pi^- \to n\,\pi^0$$

Pion-nucleon state decomposed in eigenstates of total isospin $\vec{I} = \vec{I}_\pi + \vec{I}_N$

$$|\pi^-\rangle = |1\ -1\rangle \qquad |\pi^0\rangle = |1\,0\rangle \qquad |\pi^+\rangle = |1\ +1\rangle$$
$$|n\rangle = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle \qquad |p\rangle = |\tfrac{1}{2}\ +\tfrac{1}{2}\rangle$$

Decompose $1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$ using Clebsch-Gordan coefficients

$$|p\,\pi^+\rangle = |\tfrac{3}{2}\ \tfrac{3}{2}\rangle$$
$$|p\,\pi^-\rangle = \sqrt{\tfrac{1}{3}}|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$
$$|n\,\pi^0\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

Isospin conservation and scattering processes (contd.)

Isospin conservation $[H_I, \vec{I}] = 0 \Rightarrow \langle i'\, i_3' | H_I | i\, i_3 \rangle = \delta_{i'i} \delta_{i_3' i_3} \mathcal{M}_i$

Special case of Wigner-Eckart theorem, can be proved using only $[I_\pm, H_I] = 0$

$$\begin{aligned}
\mathcal{M}_{p\pi^+ \to p\pi^+} &= \langle p\pi^+ | H_I | p\pi^+ \rangle = \langle \tfrac{3}{2}\, \tfrac{3}{2} | H_I | \tfrac{3}{2}\, \tfrac{3}{2} \rangle = \mathcal{M}_{\frac{3}{2}} \\
\mathcal{M}_{p\pi^- \to p\pi^-} &= \langle p\pi^- | H_I | p\pi^- \rangle = \tfrac{1}{3} \left(\langle \tfrac{3}{2}\ {-\tfrac{1}{2}} | H_I | \tfrac{3}{2}\ {-\tfrac{1}{2}} \rangle + 2 \langle \tfrac{1}{2}\ {-\tfrac{1}{2}} | H_I | \tfrac{1}{2}\ {-\tfrac{1}{2}} \rangle \right) \\
&= \tfrac{1}{3} \left(\mathcal{M}_{\frac{3}{2}} + 2\mathcal{M}_{\frac{1}{2}} \right) \\
\mathcal{M}_{p\pi^- \to n\pi^0} &= \langle n\pi^0 | H_I | p\pi^- \rangle = \tfrac{\sqrt{2}}{3} \left(\langle \tfrac{3}{2}\ {-\tfrac{1}{2}} | H_I | \tfrac{3}{2}\ {-\tfrac{1}{2}} \rangle - \langle \tfrac{1}{2}\ {-\tfrac{1}{2}} | H_I | \tfrac{1}{2}\ {-\tfrac{1}{2}} \rangle \right) \\
&= \tfrac{\sqrt{2}}{3} \left(\mathcal{M}_{\frac{3}{2}} - \mathcal{M}_{\frac{1}{2}} \right) = \mathcal{M}_{n\pi^0 \to p\pi^+}
\end{aligned}$$

πN amplitudes all depend on only two independent amplitudes $\mathcal{M}_{\frac{1}{2},\frac{3}{2}} \Rightarrow$ relations among physical amplitudes, cross sections

Isospin conservation and scattering processes (contd.)

Experimental fact: $\sigma_{p\pi^+\,\mathrm{tot}} \propto \sum_f |\mathcal{M}_{p\pi^+\to\mathrm{anything}}|^2$ has a peak at $\sqrt{s} = 1.232\ \mathrm{GeV} = m_\Delta \sim \Delta^{++}$ resonance

At $\sqrt{s} \simeq m_\Delta$

- $p\pi^+$ scattering proceeds through the formation of unstable particle Δ^{++} and its subsequent decay, $I_\Delta = \frac{3}{2} \Rightarrow$ expect $|\mathcal{M}_{\frac{3}{2}}| \gg |\mathcal{M}_{\frac{1}{2}}|$
- Ratios of cross sections (same proportionality factors between amplitude square and cross section)

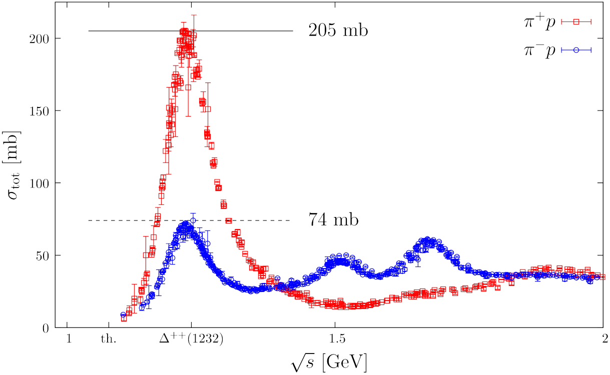
$$\frac{\sigma_{p\pi^+\to p\pi^+}}{\sigma_{p\pi^-\to p\pi^-}} \simeq 9 \qquad \frac{\sigma_{p\pi^+\to p\pi^+}}{\sigma_{p\pi^-\to n\pi^0}} \simeq \frac{9}{2}$$

- For $p\pi^+$ scattering, elastic channel essentially only available channel

  Very little phase space for an extra π^0
- For $p\pi^-$ scattering, elastic channel and inelastic channel $p\pi^- \to n\pi^0$

  Again very little phase space for an extra π^0

Isospin conservation and scattering processes (contd.)



$$\frac{\sigma_{p\pi^+ \mathrm{tot}}}{\sigma_{p\pi^- \mathrm{tot}}} = \frac{\sigma_{p\pi^+ \to p\pi^+}}{\sigma_{p\pi^- \to p\pi^-} + \sigma_{p\pi^- \to n\pi^0}} \underset{\sqrt{s}=m_\Delta}{\simeq} \frac{1}{\frac{1}{9} + \frac{2}{9}} = 3$$

Quark model

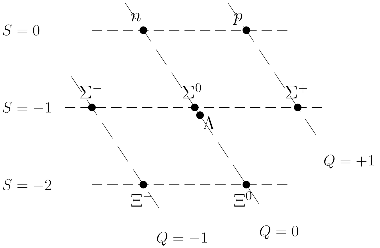
End of '50s: hadron "zoo" lacking organisation, besides baryons/mesons and isospin multiplets + strangeness

Patterns were present: plot in (I_3, S) plane:

Use only hadrons known in late '50s

eight lightest $s = \frac{1}{2}$ baryons

- fit nicely in a hexagonal array
- fit into isospin multiplets with small (permille) mass splittings
- larger but regular splitting between baryons with different S: $\Delta S = -1 \leftrightarrow \Delta M = +150\mathrm{MeV}$



Quark model

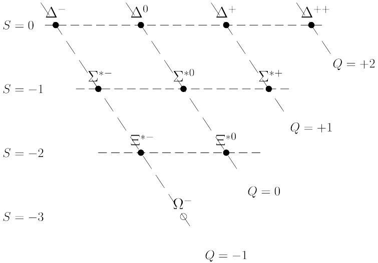
End of '50s: hadron "zoo" lacking organisation, besides baryons/mesons and isospin multiplets + strangeness

Patterns were present: plot in (I_3, S) plane:

Use only hadrons known in late '50s

nine $s = \frac{3}{2}$ baryon resonances

- fit in almost-triangular array
- single isospin multiplet for each S
- same regularity in splittings between states with different S



Quark model

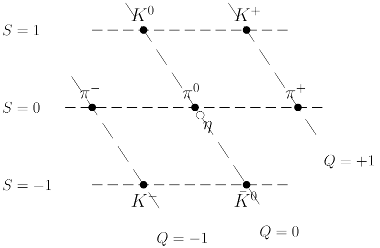
End of '50s: hadron "zoo" lacking organisation, besides baryons/mesons and isospin multiplets + strangeness

Patterns were present: plot in (I_3, S) plane:

Use only hadrons known in late '50s

seven lightest pseudoscalar mesons

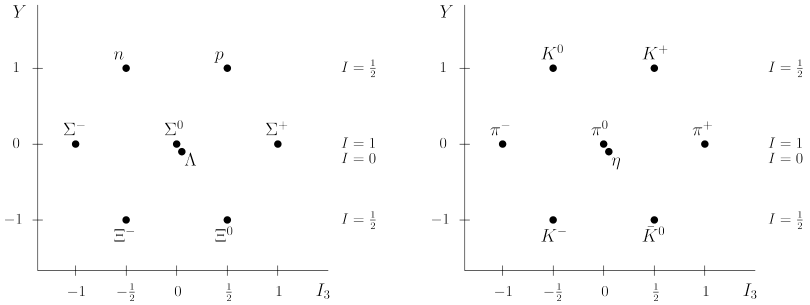
- hexagonal array similar to light baryons with a state missing in the centre
- shifted by one unit in S
- patterns overlap if one uses hypercharge $Y = B + S$



Plot in (I_3, Y) plane, isomultiplets identified

Including all hadrons known now

baryon and meson octets

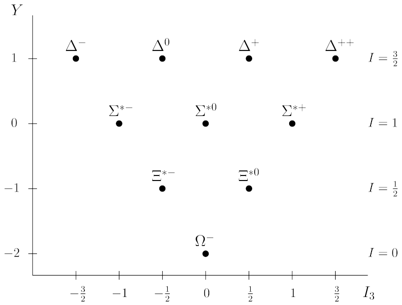


Quark model (contd.)

Plot in (I_3, Y) plane, isomultiplets identified

Including all hadrons known now

baryon decuplet



Quark model (contd.)

Possible explanation:

- approximate internal symmetry exists extending $SU(2)_I \times U(1)_Y$ (isospin and hypercharge $\vec{I}$-Y) $\Rightarrow$ explain patterns
- symmetry breaking $\Rightarrow$ explain mass differences

Assume strong Hamiltonian $H_s = H_0 + H_I$:

- H_0 symmetric, yields degenerate multiplets $\sim$ unitary irreps of internal (continuous) symmetry group $G \Rightarrow$ should match observed multiplets
- H_I symmetry-breaking term $\Rightarrow$ should reproduce the splittings

Reasonable assumptions:

- internal symmetry group, compact Lie group
- looking for extension of $\vec{I}$-Y symmetry $\Rightarrow G \supset \mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$

  Full symmetry group contains $\mathrm{U}(1)_B \sim$ baryon number conservation, expected to commute with G since no baryon/meson degeneracy
- I, I_3, Y fully classify light hadrons (besides $\vec{p}$, s_z, $\mathcal{B}$)

Simplest group satisfying requirements, with 8-dim irrep having desired decomposition into I-Y multiplets: $\boxed{\text{SU(3)}}$

SU(3)

SU(3): group of 3×3 unitary unimodular matrices, $U^\dagger U = \mathbf{1}$, $\det U = 1$

General form $U = e^{iA}$ with $A^\dagger = A$, $\mathrm{tr}\, A = 0$

Hermitean traceless matrices = 8-dim linear space, choose basis $t^a = \frac{1}{2}\lambda^a$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

λ^a: *Gell-Mann matrices*

Scalar product $(t^a, t^b) \equiv 2\,\mathrm{tr}\, t^a t^b = \delta^{ab}$, extended by (anti)linearity

Commutator of tr.less Hermitean matrices = tr.less anti-Hermitean matrix

$$[t^a, t^b] = i f_{abc} t^c$$

Real totally antisymmetric/cyclic structure constants $f_{abc} = -2i\,\mathrm{tr}\,[t^a, t^b] t^c$

$U = e^{i\alpha_a t^a} \Rightarrow t^a$ generators of $\mathfrak{su}(3)$ algebra

SU(3) (contd.)

Jacobi identity: for any matrices A, B, C

$$[[A, B], C] + [[C, A], B] + [[B, C], A] = 0$$

Applied to generators

$$[[t^a, t^b], t^c] + [[t^c, t^a], t^b] + [[t^b, t^c], t^a] = 0$$
$$\Longrightarrow f_{bcm} f_{amn} + f_{abm} f_{cmn} + f_{cam} f_{bmn} = 0$$

Structure constant satisfy

$$f_{abc} f_{abd} = 3\delta_{cd}$$

• Changing basis of the algebra $\Rightarrow$ equally good set of generators, but different structure constants and normalisation

• Possibility to choose t^a so that f_{abc} totally antisymmetric and $f_{abc} f_{abd} \propto \delta_{cd}$ are consequences of semisimplicity and compactness of SU(3):

- semisimple G = direct product of simple groups
- simple G = non-Abelian group w/out normal subgroups $N \subset G$, $gng^{-1} \subseteq N \forall n \in N, g \in G$
- compact G = compact as a manifold

SU(3) (contd.)

$t^{1,2,3} \sim$ generators $\vec{I}$ of $\mathrm{SU}(2)_I$ isospin subgroup
$t^8 \sim$ generator Y of $\mathrm{U}(1)_Y$ subgroup (up to a numerical factor)

SU(3) commutation relations

$$[t^i, t^j] = i\epsilon_{ijk} t^k \quad i,j,k = 1,2,3 \qquad\qquad [t^8, t^i] = 0 \quad i = 1,2,3$$

reproduce commutation relations

$$[I_i, I_j] = i\epsilon_{ijk} I_k \quad i,j,k = 1,2,3 \qquad\qquad [Y, I_i] = 0 \quad i = 1,2,3$$

Identify

$$\boxed{t^i = I_i \quad i = 1,2,3 \qquad t^8 = \tfrac{\sqrt{3}}{2} Y}$$

See below for normalisation of Y

Also $t^{4,5}$ and $t^{6,7}$ correspond to (different) SU(2) subgroups

$$t^4 = V_1 \qquad t^5 = V_2 \qquad \tfrac{1}{2} t^3 + \tfrac{\sqrt{3}}{2} t^8 = V_3 = \tfrac{1}{2} I_3 + \tfrac{3}{4} Y \qquad Y_V = I_3 - \tfrac{1}{2} Y$$

$$t^6 = W_1 \qquad t^7 = W_2 \qquad -\tfrac{1}{2} t^3 + \tfrac{\sqrt{3}}{2} t^8 = W_3 = -\tfrac{1}{2} I_3 + \tfrac{3}{4} Y \qquad Y_W = -I_3 - \tfrac{1}{2} Y$$

$$[V_i, V_j] = i\epsilon_{ijk} V_k \qquad [W_i, W_j] = i\epsilon_{ijk} W_k \qquad [Y, I_i] = [Y_V, V_i] = [Y_W, W_i] = 0$$

$I_3, V_3, W_3, Y, Y_v, Y_W$ not independent!

SU(3) (contd.)

Same commutation relations of ladder operators

$$[I_3, I_\pm] = \pm I_\pm \qquad [V_3, V_\pm] = \pm V_\pm \qquad [W_3, W_\pm] = \pm W_\pm$$

Ladder operators

$$I_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad V_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad W_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad V_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad W_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Together with $Y_{V,W}$ commutation relation $\Longrightarrow$

$$[I_3, I_\pm] = \pm I_\pm \qquad [I_3, V_\pm] = \pm\tfrac{1}{2} V_\pm \qquad [I_3, W_\pm] = \mp\tfrac{1}{2} W_\pm$$
$$[Y, I_\pm] = 0 \qquad [Y, V_\pm] = \pm V_\pm \qquad [Y, W_\pm] = \pm W_\pm$$

- SU(3) is a *rank-2* group: at most two generators can be diagonalised simultaneously – we can and will take I_3, Y
- $I_\pm$, $V_\pm$, $W_\pm$ on simultaneous eigenvector of I_3, Y produces new eigenvector
- Y changes by $\pm 1 \Rightarrow$ corresponds to observed integer differences