Particle physics

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Parity (contd.)

Instead of momentum eigenstates $|\vec{p}\rangle$
$\to$ use energy, orbital angular momentum eigenstates $|E;\ell\ell_z\rangle$

$|\ell\ell_z\rangle$ are also P eigenstates: $P|\ell\ell_z\rangle = (-1)^\ell|\ell\ell_z\rangle$ as in quantum mechanics

$$\begin{aligned} 0 &= \langle \ell', \ell'_z; c\ d|[P, H_I]|\ell, \ell_z; a\ b\rangle \\ &= [(-1)^{\ell'}\eta_c\eta_d - (-1)^\ell\eta_a\eta_b]\langle \ell', \ell'_z; c\ d|H_I|\ell, \ell_z; a\ b\rangle \\ 0 &= \langle \ell, \ell_z; b\ c|[P, H_I]|a\rangle = [(-1)^\ell\eta_b\eta_c - \eta_a]\langle \ell, \ell_z; b\ c|H_I|a\rangle \end{aligned}$$

Look at decay process in the rest frame of the decaying particle, no orbital angular momentum

Non-vanishing matrix elements $\Rightarrow$

$$(-1)^{\ell'}\eta_c\eta_d = (-1)^\ell\eta_a\eta_b \qquad (-1)^\ell\eta_b\eta_c = \eta_a$$

$\Rightarrow$ assign intrinsic parity to one of the particles involved in the process using conventional intrinsic parities and those already determined

Parity (contd.)

Parity of the charged pion: study pion capture by deuteron (d)

$$\pi^- \; d \to (\pi^- d) \to n \; n$$

$d = (pn)$ bound state with orbital angular momentum $\ell_d = 0$, intrinsic parity $\eta_d = \eta_p \eta_n (-1)^0 = 1$, spin $s_d = 1$
π^- captured, $(\pi^- d)$ atom formed, decays from ground state into neutrons

- NR final state, wave f. $R_\ell(r) Y_\ell^m(\theta, \varphi) |S, S_z\rangle$
- ground state: $\ell_G = 0$ (+ a little $\ell_G = 2$)

$$\eta_G = \eta_\pi (-1)^{\ell_G} = \eta_\pi = \eta_n^2 (-1)^\ell = (-1)^\ell$$

- $\ell = 0, s_\pi = 0, s_d = 1 \Rightarrow \boxed{J = 1}$
- antisymmetric wave f. $\boxed{(-1)^{S+\ell+1} = -1}$
  - spin wf: $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \Rightarrow$ sign $(-1)^{S+1}$
  - space wf: $\vec{x} \to -\vec{x} \Rightarrow$ sign $(-1)^\ell$

$$\boxed{\eta_\pi = -1}$$

$$|\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\rangle = |11\rangle$$
$$\tfrac{1}{\sqrt{2}}(|\text{-}\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\rangle + |\tfrac{1}{2}\rangle \otimes |\text{-}\tfrac{1}{2}\rangle) = |10\rangle$$
$$|\text{-}\tfrac{1}{2}\rangle \otimes |\text{-}\tfrac{1}{2}\rangle = |1\text{-}1\rangle$$
$$\tfrac{1}{\sqrt{2}}(|\text{-}\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\rangle - |\tfrac{1}{2}\rangle \otimes |\text{-}\tfrac{1}{2}\rangle) = |00\rangle$$

ℓ	S	J	$(-1)^{S+\ell+1}$
0	1	1	1
1	0	1	1
1	1	$0 \oplus 1 \oplus 2$	-1
2	1	$1 \oplus 2 \oplus 3$	1

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Parity (contd.)

Intrinsic parity of Δ^{++} from $\Delta^{++} \to p\,\pi^+$

- $s_{\Delta^{++}} = \frac{3}{2}$, $s_p = \frac{1}{2}$, $s_{\pi^+} = 0 \Rightarrow$ final state must have $\ell = 1, 2$
- $\eta_{\Delta^{++}} = \eta_p \eta_{\pi^+} (-1)^\ell = (-1)^{\ell+1}$
- angular distribution of decay products implies $\ell = 1 \Rightarrow \eta_{\Delta^{++}} = 1$

For γ intrinsic parity assigned on the basis of theoretical considerations

- classically $\vec{E}$ (true) vector, $\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A} \Rightarrow$ vector potential $\vec{A}$ (true) vector
- photon states enconded in $\vec{A}$ after quantisation $\Rightarrow \eta_\gamma = -1$
- alternatively: in QED γ-e coupling encoded in $A_\mu J^\mu$, electric current J^μ Lorentz vector $\Rightarrow A_\mu$ Lorentz vector, $\eta_\gamma = -1$

Charge conjugation

Charge conjugation (C): exchange all particles with corresponding antiparticles keeping momenta and spins unchanged

$$C|\vec{p}, s_z; \alpha\rangle = \xi_\alpha |\vec{p}, s_z; \bar{\alpha}\rangle$$

ξ_α: intrinsic charge conjugation phase, meaningful only for truly neutral particles (e.g., γ, π^0, but not n)

C unitary and $[C, H] = 0$ (same argument used for P)

- changes the sign of all internal quantum numbers $\mathcal{O}_{\rm int}$, $\{C, \mathcal{O}_{\rm int}\} = 0$ (el. charge, baryon/lepton/lepton family/quark flavour/lepton flavour)
- changes sign to magnetic moment $\vec{\mu} \propto q\vec{s}$

QFT imposes relations between intrinsic C of particle α and antiparticle $\bar{\alpha}$

$$\boxed{\text{bosons and fermions: } \xi_\alpha \xi_{\bar{\alpha}} = 1}$$

$C^2 =$ phase transformation, one can set $C^2 = 1 \Rightarrow \xi_\alpha = \pm 1$

$$C^2|\vec{p}, s_z; \alpha\rangle = \xi_\alpha \xi_{\bar{\alpha}} |\vec{p}, s_z; \alpha\rangle$$

Charge conjugation (contd.)

How to assign ξ_α to a self-conjugate particle?

- theoretical arguments: e.g., γ from Maxwell eqs.

$$\vec{\nabla}\cdot\vec{E}\propto\rho\,,\qquad \vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}$$
$$\vec{\nabla}\wedge\vec{B}\propto\vec{J}\,,\qquad \vec{B}=\vec{\nabla}\wedge\vec{A}$$

  - exchanging $\pm$ charges $\Rightarrow \rho\to-\rho$, $\vec{J}\to-\vec{J}$
    $\Rightarrow$ change sign of $\vec{E},\vec{B}$ $\Rightarrow A_\mu=(\phi,\vec{A})\to -A_\mu$
  - for the photon quantum field $C^\dagger A_\mu C=-A_\mu \Rightarrow \xi_\gamma=-1$
- selection rule implied by C invariance: e.g., π^0
  - $\pi^0\to\gamma\,\gamma \Rightarrow \xi_{\pi^0}=\xi_\gamma^2=1$
  - can assign $\xi_{\pi^\pm}=1$ as well but just a matter of convention: no selection rule for them, cannot fix $\xi_{\pi^\pm}$
  - if C exact $\Rightarrow \pi^0\to\gamma\gamma\gamma$ strictly forbidden;
    C violations from WI, quite unrelated to this process
    $\Rightarrow$ expect strong suppression (expt.: $\Gamma_{\pi^0\to 3\gamma}/\Gamma_{\pi^0\to 2\gamma}<3.1\cdot 10^{-8}$)

Time reversal

Time reversal (T): inversion of the arrow of time $t \to t' = -t$

Under T both momentum and spin components change sign

$$T|\vec{p}, s_z; \alpha\rangle = \zeta_{\alpha,s_z}| - \vec{p}, -s_z; \alpha\rangle$$

Intrinsic phase $\zeta_{\alpha,s_z} = (-1)^{s-s_z}\zeta_\alpha$

T is an antiunitary symmetry: from invariance

$$TU(t)\psi(0) = U(t')T\psi(0) = U(-t)T\psi(0) \Rightarrow TiH = -iHT \Rightarrow \{T, iH\} = 0$$

1. linear unitary $\Rightarrow \{T, H\} = 0$
2. antilinear antiunitary $\Rightarrow Ti = -iT \Rightarrow [T, H] = 0$

Option 1 excluded by absence of negative-energy states

Antiunitarity implies residual phase ζ_α has no physical meaning (can be reabsorbed in a redefinition of the states)

CPT theorem

General theorem of quantum field theory: the antiunitary transformation $\Theta = CPT$ is a symmetry for any translation and Lorentz-invariant theory of local quantum fields

$$\Theta|\vec{p}, s_z; \alpha\rangle = CPT|\vec{p}, s_z; \alpha\rangle = CP\zeta_{\alpha,s_z}|-\vec{p}, -s_z; \alpha\rangle = C\eta_\alpha\zeta_{\alpha,s_z}|\vec{p}, -s_z; \alpha\rangle$$
$$= \xi_\alpha\eta_\alpha\zeta_{\alpha,s_z}|\vec{p}, -s_z; \bar{\alpha}\rangle = \theta_{\alpha,s_z}|\vec{p}, -s_z; \bar{\alpha}\rangle$$

- CPT good also for weak interactions where P, C, CP not conserved
- if violations of CPT observed, QFT inadequate to explain them

CPT theorem $\Rightarrow (m,s)_\alpha = (m,s)_{\bar{\alpha}}$ since $[\Theta, p^2] = [\Theta, \vec{J}^2] = 0$

$$\Theta p^2|\vec{p}, s_z; \alpha\rangle = m_\alpha^2\theta_{\alpha,s_z}|\vec{p}, -s_z; \bar{\alpha}\rangle = p^2\Theta|\vec{p}, s_z; \alpha\rangle = m_{\bar{\alpha}}^2\theta_{\alpha,s_z}|\vec{p}, -s_z; \bar{\alpha}\rangle$$

Also $\tau_\alpha = \tau_{\bar{\alpha}}$ for unstable particles: in Born approximation

$$\tau_\alpha^{-1} = \Gamma_\alpha = \sum_f c_f|\langle f|H_I|\alpha\rangle|^2 = \sum_f c_f|\langle f|\Theta^\dagger H_I\Theta|\alpha\rangle|^2$$
$$= \sum_f c_f|\langle\bar{f}|H_I|\bar{\alpha}\rangle|^2 = \sum_f c_{\bar{f}}|\langle\bar{f}|H_I|\bar{\alpha}\rangle|^2 = \Gamma_{\bar{\alpha}} = \tau_{\bar{\alpha}}^{-1}$$

$c_f = c_{\bar{f}}$: kinematical factors dependent on masses and spins of final state

Isospin

1932: Chadwick discovers the neutron, solves the puzzle of mass/charge mismatch in nuclei

- nuclear electric charge $= e \times$ number of protons in the nucleus
- nuclear mass is very accurately $m_p \times$ number of protons and neutrons

$$m_n = 939.57 \text{ MeV} \quad m_p = 938.28 \text{ MeV} \qquad (m_n - m_p)/m_p \simeq 0.0014$$

1932: Heisenberg proposes

- p, n are two different states of the same particle, the *nucleon*
- affected in the same way by the strong interactions

Heisenberg's view:
$\Rightarrow$ strong interactions exactly invariant under $p \leftrightarrow n$
$\Rightarrow$ small mass difference attributed to EM effects
What we know now:
$\Rightarrow$ symmetry approximate even if EM interactions switched off
$\Rightarrow$ important role played by $m_d - m_u$ (EM alone would lead to $m_p > m_n$)

Isospin (contd.)

- Nucleon N has internal degree of freedom, two states corresponding to p, n

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Superposition principle: $N(\alpha, \beta) = \alpha p + \beta n$ also a possible state
- Assumption: all states $N(\alpha, \beta)$ look the same to strong interactions
- Mathematically speaking: strong interactions invariant under SU(2) rotations of the nucleon state – *isospin* symmetry
- Generalised to SU(2) symmetry of strong interaction Hamiltonian (nucleons, pions, kaons. . .)
- Isospin symmetry not exact but very good approximate symmetry of strong interactions (we will soon see why)

Isospin (contd.)

Invariance of strong interactions under internal $SU(2)_I$ symmetry group

SU(2) group:

- Lie group (group which is also a manifold)
- three generators $\vec{I} = (I_1, I_2, I_3)$
- same Lie algebra as SO(3)

$$[I_a, I_b] = i\varepsilon_{abc} I_c$$

Invariance of strong Hamiltonian $H_s \Rightarrow [\vec{I}, H_s] = 0$

In Heisenberg picture: $\frac{d}{dt}\vec{I}(t) = i[H_s, \vec{I}(t)] = 0$

Important consequences:

- spectrum of the theory organised in degenerate isospin multiplets (corresponding to irreducible representations of SU(2))
- conservation of isospin in dynamical hadronic processes (decay, scattering)

Isospin (contd.)

Isospin symmetry traces back to symmetry under SU(2) rotations in the space of up and down quarks, broken only by small mass difference

Broken also by EM effects

Analogy: u and d two states of the "light quark" q

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Strong interaction Hamiltonian

$$H_s = H_0 + H'$$

- H_0: invariant under SU(2) rotations in (u, d) space
- H': symmetry-breaking term $\propto m_u - m_d$
- $m_u - m_d \ll 0.1 \div 1\,\mathrm{GeV}$ (typical strong scale)
  $\Rightarrow$ isospin good approximate symmetry

SU(2) and Lie groups

Unitary unimodular 2×2 complex matrices

$$U^\dagger U = UU^\dagger = \mathbf{1} \qquad \det U = 1$$

Group: $(U_2 U_1)^\dagger (U_2 U_1) = U_1^\dagger U_2^\dagger U_2 U_1 = \mathbf{1}$, $\det(U_2 U_1) = \det U_2 \det U_1 = 1$
U. matrix $U = e^{iH}$, $H = H^\dagger$ Hermitean, $1 = \det U = e^{i \operatorname{tr} H} \Rightarrow \operatorname{tr} H = 0$

$$H = \frac{1}{2} \begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -\alpha_3 \end{pmatrix} = \vec{\alpha} \cdot \frac{\vec{\sigma}}{2}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c \qquad \{\sigma_a, \sigma_b\} = 2\delta_{ab}$$

Continuous group

$$U(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} = \cos\frac{|\vec{\alpha}|}{2}\mathbf{1} + i\sin\frac{|\vec{\alpha}|}{2}\hat{\alpha}\cdot\vec{\sigma} = u_0\mathbf{1} + i\vec{u}\cdot\vec{\sigma} \qquad u_0^2 + \vec{u}^2 = 1$$

Also a manifold: point in SU(2) $\leftrightarrow$ point on the 4d-sphere S^3

$\Rightarrow$ **Lie group**

SU(2) and Lie groups (contd.)

- Group reconstructed from **generators** $\vec{I} = \frac{\vec{\sigma}}{2}$ by exponentiation

$$U(\vec{\alpha}) = e^{i\vec{\alpha}\vec{I}}$$

- For small $|\vec{\alpha}| \ll 1$, $U(\vec{\alpha}) \simeq \mathbf{1} + i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2} = \mathbf{1} + i\vec{\alpha}\cdot\vec{I}$
- Generators: $I_a = -i\frac{\partial}{\partial\alpha_a}U(\vec{\alpha})|_{\vec{\alpha}=\vec{0}}$
- **Lie algebra** $\mathfrak{su}(2)$:
  - real vector space spanned by $\{I_a\}$, $X \in \mathfrak{su}(2): \quad X = X^a I_a$
  - commutator $[X, Y] = XY - YX \in \mathfrak{su}(2)$ (antisymm. bilinear form)
- Basic commutation relations: $[X, Y] = X^a Y^b [I_a, I_b]$,

$$[I_a, I_b] = i\epsilon_{abc} I_c$$

- In general: for any Lie group
  - elements $U = U(\alpha_1, \ldots, \alpha_n) = U(\alpha)$
  - generators $t_a = -i\frac{\partial}{\partial\alpha_a}U(\alpha)|_{\alpha_a=0}$ satisfy $[t_a, t_b] = iC_{ab}{}^c t_c$ for some real **structure constants** $C_{ab}{}^c$, yield a Lie algebra
  - for compact groups ($\sim$ compact manifolds), structure constants are totally antisymmetric and one writes $C_{ab}{}^c = f_{abc}$

SU(2) representations

Group representation: mapping $G \to {}_n\mathbb{C}_n$, $g \mapsto D(g)$ respecting group composition law, $\mathbb{C}^n$: representation space

$$\boxed{D(g_2)D(g_1) = D(g_2 g_1)}$$

- $D(g)D(e) = D(g) \Rightarrow D(e) = \mathbf{1}$
- $D(g^{-1})D(g) = D(e) \Rightarrow D(g^{-1}) = D(g)^{-1}$

Unitary representation: $D(g)$ unitary, $D(g)^{-1} = D(g)^\dagger$

Reducible representation: proper subspace $\exists S \subset \mathbb{C}^n$ left invariant by the representation, $D(g)S = S\ \forall g \in G$

For unitary representation, if S invariant then $S^\perp$ invariant as well

$$0 = (s_\perp, D(g^{-1})s) = (D(g)s_\perp, D(g)D(g^{-1})s) = (D(g)s_\perp, s),\quad \forall g \in G \Rightarrow D(g)S^\perp = S^\perp$$

Repeat until no invariant subspace is left $\Rightarrow$ **completely reducible**, decomposes in **irreducible** representations: $\nexists S \subset \mathbb{C}^n$ left invariant

SU(2) representations (contd.)

Lie algebra representation: linear mapping $\mathfrak{g} \to {}_n\mathbb{C}_n$, $X \mapsto d(X)$ respecting commutators

$$d(\alpha X + \beta Y) = \alpha d(X) + \beta d(Y) \ \alpha, \beta \in R \qquad d([X, Y]) = [d(X), d(Y)]$$

Unitary representation of SU(2) $\leftrightarrow$ Hermitean representation of $\mathfrak{su}_{\mathbb{C}}(2)$

Representation of the *complexified* algebra $\mathfrak{su}_{\mathbb{C}}(2)$

- if rep. of $\mathfrak{su}(2)$ exists $\Rightarrow$ extend by linearity to complex coefficients $\Rightarrow$ get rep. of $\mathfrak{su}_{\mathbb{C}}(2)$
- if rep. of $\mathfrak{su}_{\mathbb{C}}(2)$ exists $\Rightarrow$ restrict to real coefficients $\Rightarrow$ get rep. of $\mathfrak{su}(2)$

Irreducible representation of Lie algebra: does not leave any subspace invariant, S s.t. $d(X)S \subset S \ \forall X \in \mathfrak{g}$

Irreducible representation of SU(2) $\leftrightarrow$ irreducible representation of $\mathfrak{su}_{\mathbb{C}}(2)$

Theorem: for compact Lie groups all finite-dim. representations equivalent to unitary representations

After change of basis all $M^{-1}D(g)M$ become unitary

Task: classify unitary irreps of SU(2) $\approx$ classify Hermitean irreps of $\mathfrak{su}(2)$

SU(2) representations (contd.)

Finite-dimensional Hermitean representations of $\mathfrak{su}(2)$

Hermitean rep. of generators $d(I_a) = d(I_a)^\dagger$, $[d(I_a), d(I_b)] = i\epsilon_{abc} d(I_c)$
$\Rightarrow$ algebra representation by linearity
$\Rightarrow$ group representation by exponentiation

Denote representatives with I_a, can diagonalise (only) one of them, I_3
$\Rightarrow$ representation space spanned by eigenvectors $I_3|i_3\rangle = i_3|i_3\rangle$

$[I_3, X^a I_a] = \mathrm{i}\epsilon_{3ab} X^a I_b = 0 \Leftrightarrow \mathrm{i}\epsilon_{3ab} X^a = 0\ \forall b \Leftrightarrow X^a = 0\ \forall a$

Raising/lowering operators $I_\pm = I_1 \pm iI_2$, $I_+^\dagger = I_-$, obey comm. relations

$$[I_3, I_\pm] = \pm I_\pm \qquad [I_+, I_-] = 2I_3$$

Here we use the complexified algebra

$$I_3 I_\pm |i_3\rangle = (I_\pm I_3 + [I_3, I_\pm])|i_3\rangle = (i_3 I_\pm \pm I_\pm)|i_3\rangle = (i_3 \pm 1) I_\pm |i_3\rangle$$

$\Rightarrow$ If $|i_3\rangle$ eigenvector then $I_\pm|i_3\rangle$ are eigenvectors too

SU(2) representations (contd.)

- There must be a unique eigenvector $|i\rangle$ such that $I_+|i\rangle = 0$ ($\langle i|i\rangle = 1$)
  - existence: finite-dimensional representation requires the chain $I_+|i_3\rangle$, $I_+^2|i_3\rangle$, $I_+^3|i_3\rangle$, ..., to stop
  - uniqueness: if more than one existed rep. would not be irreducible

    If $I_+|i'\rangle = 0$, make $\langle i'|i\rangle = 0$, then use commutation relations
    $\langle i'|I_{a_1}\dots I_{a_n}|i\rangle = \sum_{k_1+k_2\leq n} c_{k_1k_2}\langle i'|I_-^{k_1}\dots I_+^{k_2}|i\rangle = 0$
    $\Rightarrow \langle i'|e^{i\vec{\alpha}\cdot\vec{I}}|i\rangle = 0$, $|i\rangle, |i'\rangle$ belong to different invariant subspaces
  - also eigenvector of $\vec{I}^2 = \sum_a I_a^2 = I_-I_+ + I_3 + I_3^2 = I_+I_- - I_3 + I_3^2$, eigenvalue $\vec{I}^2|i\rangle = i(i+1)|i\rangle$
- Construct vectors $|i_3\rangle$ from $|i\rangle$ via I_-, $I_-^{i-i_3}|i\rangle = C|i_3\rangle$ with $\langle i_3|i_3\rangle = 1$
  - Condon-Shortley convention: choose C real positive
  - automatically eigenvectors of $\vec{I}^2$ since $[I_\pm, \vec{I}^2] = [I_3, \vec{I}^2] = 0$
  - chain must stop, $I_-|i_*\rangle = 0$,
    $0 = \langle i_*|I_+I_-|i_*\rangle = \langle i_*|\vec{I}^2 + I_3 - I_3^2|i_*\rangle = i(i+1) + i_*(i_* - 1) \Rightarrow i_* = -i$

    Other solution $i_* = i + 1 > i$ unacceptable

Irreps: $(2i+1)$-d, rep. space spanned by $\{|i\rangle, |i-1\rangle, \dots, |-i+1\rangle, |-i\rangle\}$ eigenvectors of I_3, with constant $\vec{I}^2 = i(i+1)$, $2i \in \mathbb{N}_0$ (spin-i reps.)

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