Particle physics

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Strong interactions, colour conservation, and confinement

Colour conserved at vertices, what are the consequences?

Essentially none: quarks and antiquarks are confined in hadrons *with net colour zero* $\Rightarrow$ conservation law is zero colour in, zero colour out

Confinement not (yet) proved in QCD, detailed mechanism not fully understood, but basic idea simple: static quark-antiquark potential approx.

$$V_s = -\frac{4}{3}\frac{\alpha_s}{r} + \sigma r$$

static = limit of infinite quark masses

$\alpha_s = \frac{g_s^2}{4\pi}$, g_s the strong coupling constant; σ: *string tension*

Pulling q and $\bar{q}$ apart, energy stored in the system increases linearly $\Rightarrow$ infinite energy needed to free them (at $r = \infty$) $\Rightarrow$ confinement

For finite quark masses, when $\sigma r > 2m_q$ energy stored in the system sufficient to create $q\bar{q}$ pair out of the vacuum $\Rightarrow$ *string breaking*, new q and $\bar{q}$ bind to the old ones and get confined again

Symmetries

Symmetry: change in the experimenter's point of view that does not change the results of possible experiments (according to [5])

Observers $\mathcal{O}$ and $\mathcal{O}'$ making measurements on the same physical system

- use *same* operative rules concerning measurements
- use in general *different* reference frames
- find different values for physical quantities, providing different descriptions of the same system

In QM: $\mathcal{O}$ and $\mathcal{O}'$ assign different state vectors to the same system (*different* expectation values of the *same* operators), physics encoded in Hilbert space *rays* $\mathcal{R}$ and $\mathcal{R}'$ associated to the state of the system

$\mathcal{R} = [\psi]$ equivalence class of ψ under relation $\psi \sim \psi'$ if $\psi' = e^{i\phi}\psi$

For certain pairs of observers descriptions are different but *equivalent*: same physical laws implied by measurements, impossible for an observer to determine her/his reference frame using only her/his measurements

Symmetries (contd.)

If $\mathcal{O}$ and $\mathcal{O}'$ give equivalent descriptions, then

- same set of possible physical states that they can observe
- if $\mathcal{O}$ sees two states of the system as different, so must do $\mathcal{O}'$

If not, then one could distinguish the two observers $\rightarrow$ not equivalent

Establishing a relation between $\mathcal{O}$ and $\mathcal{O}'$ = defining a mapping $\mathcal{M}$ from the space of rays $\underline{\mathcal{H}} = \{\mathcal{R}\}$ to $\{\mathcal{R}'\} = \underline{\mathcal{H}}$, i.e., itself

- to each $\mathcal{R}$ observed by $\mathcal{O}$ there corresponds one and only one $\mathcal{R}'$ observed by $\mathcal{O}'$ (domain of definition = $\underline{\mathcal{H}}$)
- every $\mathcal{R}'$ possible observation of $\mathcal{O}'$ $\Rightarrow$ $\mathcal{M}$ must be surjective (onto)
- $\mathcal{R}_1 \neq \mathcal{R}_2$ must be mapped into $\mathcal{R}'_1 \neq \mathcal{R}'_2 \Rightarrow \mathcal{M}$ must be injective (one-to-one) $\Longrightarrow$ bijective, invertible mapping

More directly: if $\mathcal{O}$, $\mathcal{O}'$ equivalent, and mapping exists from $\mathcal{O}$ to $\mathcal{O}'$, then inverse mapping from $\mathcal{O}'$ to $\mathcal{O}$ must exist too (otherwise not equivalent)

$$\begin{aligned} \mathcal{M} : \underline{\mathcal{H}} &\rightarrow \underline{\mathcal{H}} \\ \mathcal{R} &\mapsto \mathcal{R}' = \mathcal{M}\mathcal{R} \end{aligned}$$

Wigner's theorem

Suppose we perform an experiment on the system, and $\mathcal{O}$ and $\mathcal{O}'$ see the system transition between two states

$$\mathcal{O} : \mathcal{R}_i \longrightarrow \mathcal{R}_f \qquad \mathcal{O}' : \mathcal{R}'_i \longrightarrow \mathcal{R}'_f$$

occurring with probabilities P and P',

$$P = (\mathcal{R}_i \cdot \mathcal{R}_f)^2 \qquad P' = (\mathcal{R}'_i \cdot \mathcal{R}'_f)^2$$

$\mathcal{R}_1 \cdot \mathcal{R}_2 \equiv |(\psi_1, \psi_2)|$ with $\psi_{1,2}$ normalised vectors belonging to $\mathcal{R}_{1,2}$
$\mathcal{O}$ and $\mathcal{O}'$ looking at same process $\Rightarrow P = P'$ for any pair of states

$$\mathcal{R}_i \cdot \mathcal{R}_f = \mathcal{R}'_i \cdot \mathcal{R}'_f = (\mathcal{M}\mathcal{R}_i) \cdot (\mathcal{M}\mathcal{R}_f)$$

Wigner's theorem: invertible transformation $\mathcal{M} : \underline{\mathcal{H}} \to \underline{\mathcal{H}}$ that conserves probabilities can be implemented as a transformation on the space of vectors $\mathcal{H}$ that is either linear and unitary or antilinear and antiunitary:

$$U(\alpha\psi + \beta\phi) = \alpha U\psi + \beta U\phi\,, \qquad (U\psi, U\phi) = (\psi, \phi)\,,$$
$$T(\alpha\psi + \beta\phi) = \alpha^* T\psi + \beta^* T\phi\,, \qquad (T\psi, T\phi) = (\psi, \phi)^*$$

See [5] for a proof

Wigner's theorem

Wigner's theorem implies that we can search for symmetry transformations looking only at unitary and antiunitary transformations in $\mathcal{H}$

(Anti)unitary transformations: norm-preserving, onto (anti)linear transformations

- linear/antilinear:

$$U(\alpha\psi + \beta\phi) = \alpha U\psi + \beta U\phi \qquad T(\alpha\psi + \beta\phi) = \alpha^* T\psi + \beta^* T\phi$$

- preserve the norm $\|\psi\|$ of vectors; using polarisation identity

$$4(\psi,\phi) = \|\psi+\phi\| + \|\psi-\phi\| + i\|\psi - i\phi\| - i\|\psi + i\phi\|$$

$$\Longrightarrow (U\psi, U\phi) = (\psi,\phi) \qquad (T\psi, T\phi) = (\psi,\phi)^*$$

adjoint operator: $(\psi, U\phi) = (U^\dagger\psi, \phi) \qquad (\psi, T\phi) = (T^\dagger\psi,\phi)^*$

$$\Longrightarrow U^\dagger U = T^\dagger T = \mathbf{1}$$

- whole Hilbert space as their image: $\forall\psi\ \exists\phi_{U,T}$ s.t. $\psi = U\phi_U = T\psi_T$

$$\psi = U(U^\dagger U)\phi = (UU^\dagger)U\phi = (UU^\dagger)\psi \Longrightarrow UU^\dagger = TT^\dagger = \mathbf{1}$$

Conversely: linear U with $U^\dagger U = UU^\dagger = \mathbf{1}$ implies unitary, antilinear T with $T^\dagger T = TT^\dagger = \mathbf{1}$ implies antiunitary

Symmetry: dynamical requirements

Equivalence of observers = same space of states and same laws of physics

Same laws of physics means equations of motion have the same form in both reference frames (i.e., for both observers)

- $\Rightarrow$ Same Hamiltonian (= same dynamical evolution) for both observers $\Rightarrow$ transformed of the evolved = evolved of the transformed
- $\Leftarrow$ Transformed of the evolved = evolved of the transformed $\Rightarrow$ same Hamiltonian (= same dynamical evolution) for both observers

For time-independent transformations M, temporal evolution $U(t) = e^{-iHt}$ with time-independent H:

$$U(t)M\psi(0) = MU(t)\psi(0) \Rightarrow [U(t), M] = 0 \Rightarrow [H, M] = 0$$

Symmetry group

Given symmetry transformations M_i on $\mathcal{H}$

- $M_2 M_1$ still a symmetry
- $\mathbf{1}$ (identity) is a symmetry
- M_i^{-1} (inverse) exists and is a symmetry
- $M_3(M_2 M_1) = M_3 M_2 M_1 = (M_3 M_2) M_1$ (associativity)

$\Rightarrow$ symmetry transformations of a physical system form a group

Two types of symmetry transformations:

- *continuous*: transformations $M = M(\alpha)$ are part of a continuous family dependent on a set of real parameters α
  (e.g.: translations, rotations, Lorentz boosts, isospin)
- *discrete*: no such family
  (e.g.: parity, charge conjugation, time reversal)

If $M = M(\alpha)$ is an element of a continuous family of symmetry transformations connected to the identity $\mathbf{1}$ then it must be unitary

Continuous symmetries

Consider one-parameter group of (unitary) symmetries $M(\alpha)$

Unitary: $M(\alpha)^\dagger M(\alpha) = M(\alpha)M(\alpha)^\dagger = \mathbf{1}$
Symmetry: $[M(\alpha), H] = 0$

$$M(\alpha_1)M(\alpha_2) = M(\alpha_1 + \alpha_2) \qquad M(0) = \mathbf{1}$$

Such a parameterisation can always be found under general, reasonable conditions

$$\begin{aligned} M(d\alpha)M(\alpha) &= M(\alpha + d\alpha) \\ \left(\mathbf{1} + d\alpha \tfrac{dM}{d\alpha}(0)\right) M(\alpha) &= M(\alpha) + d\alpha \tfrac{dM}{d\alpha}(\alpha) \\ \frac{dM}{d\alpha}(0) M(\alpha) &= \frac{dM}{d\alpha}(\alpha) \end{aligned}$$

$$M(\alpha) = \exp\left\{\alpha \tfrac{dM}{d\alpha}(0)\right\} = \exp\left\{i\alpha(-i)\tfrac{dM}{d\alpha}(0)\right\} = \exp\{i\alpha Q\}$$

$Q = Q^\dagger$ Hermitean and $[H, Q] = 0 \Rightarrow$ conserved physical quantity

E.g.: translations $\Rightarrow$ four-momentum, rotations $\Rightarrow$ angular momentum

Conserved quantities and Schrödinger/Heisenberg pictures

- *Schrödinger picture*:
  - time-independent observables $Q = Q_S$
  - time-dependent states $|\psi_S(t)\rangle = U(t)|\psi_S(0)\rangle = U(t)|\psi\rangle$
- *Heisenberg picture*:
  - time-dependent observables $Q_H(t)$
  - time-independent states $|\psi_H\rangle = |\psi_S(0)\rangle = |\psi\rangle$

Relation between pictures: expectation values should be the same

$$\begin{aligned}\langle Q\rangle_\psi(t) &\equiv \langle\psi_S(t)|Q_S|\psi_S(t)\rangle = \langle\psi_S(0)|U(t)^\dagger Q_S U(t)|\psi_S(0)\rangle \\ &= \langle\psi_S(0)|U(t)^\dagger Q_S U(t)|\psi_S(0)\rangle = \langle\psi_H|Q_H(t)|\psi_H\rangle \\ Q_H(t) &= U(t)^\dagger Q U(t)\end{aligned}$$

Equation of motion $\dot{Q}_H(t) = \frac{dQ_H(t)}{dt} = i[H, Q_H(t)]$
For a conserved quantity $\dot{Q}_H(t) = 0 \Leftrightarrow Q_H(t) = Q \Leftrightarrow [H, Q] = 0$

Independently of the picture: $\langle Q\rangle_\psi(t) = \langle Q\rangle_\psi(0)$

Free particles

Free particles (empirical definition):

localised objects travelling on straight lines at constant speed

- simple system
- physically relevant: initial and final states in scattering experiments
  - initial state: particles are far away from each other, not interacting yet
  - final state: measurements on final products far away from each other, not interacting any more

Free particle state characterised by

- type = mass m, spin s, electric charge q + other (compatible) charges
- energy E, momenta $\vec{p}$ and spin component in some direction (conventionally: s_z)

$\Rightarrow$ complete set of observables (overcomplete since $E^2 = \vec{p}^{\,2} + m^2$)

One-particle states: type $\alpha = (m, s, \ldots)$, momentum $\vec{p}$, spin z-comp. s_z

$$|\vec{p}, s_z; \alpha\rangle$$

Parity

Parity (P): change sign to all spatial coordinates

Non-relativistic case: use quantum mechanics, wave function changes as

$$P\psi_{s_z}(\vec{x}) = \psi_{s_z}(-\vec{x})$$

$\vec{x} \to -\vec{x}$, $t \to t$, $s_z \to s_z$ $\Rightarrow$ momenta change sign, angular momenta do not

Relativistic case: cannot use wave functions in coordinate space to describe a system, characterise particles by momenta and spin

$$P|\vec{p}, s_z; \alpha\rangle = \eta_\alpha | -\vec{p}, s_z; \alpha\rangle$$

P: (anti)unitary operator on the Hilbert space of the system
η_α: phase factor (*intrinsic parity*) included for generality (does not change the physical meaning of the transformation); if P is a symmetry a consistent assignment of phases can be made

Heisenberg picture: quantum states given once and for all, effect of symmetry transformations shifted on *operators*

$$P^\dagger \vec{p} P = -\vec{p} \qquad P^\dagger \vec{s} P = \vec{s}$$

Parity (contd.)

Is parity a unitary or antiunitary symmetry?
Invariance (dynamical part) $\Rightarrow PU(t) = U(t)P \Rightarrow [P, iH] = 0$

1. linear unitary $\Rightarrow PH - HP = [P, H] = 0$
2. antilinear antiunitary $\Rightarrow PH + HP = \{P, H\} = 0$, so given ψ_E, $H\psi_E = E\psi_E$, then $H(P\psi_E) = -PH\psi_E = (-E)(P\psi_E)$

Negative energy particle states not found in nature $\Rightarrow$ empirically forced to choose option 1

P diagonalisable with H, but assignment of intrinsic parities not unique

- Consider only strong and EM interactions $\Rightarrow$ particle-type numbers conserved for each type, associated to U(1) symmetries
- Restrict to u, d, e^- + antiparticles $\rightarrow Q, \mathcal{B}, L$ conserved

If $P^{(0)}$ is a parity operator ($=$ flips $\vec{p}$ not s_z) with intrinsic parities $\eta_\alpha^{(0)}$ then $P = P^{(0)} e^{i(aB+bL+cQ)}$ is still a parity operator with different η_α

Parity (contd.)

Choose a, b, c such that

$$p\colon \eta_p = \eta_p^{(0)} e^{i(a+c)} = 1 \qquad n\colon \eta_n = \eta_n^{(0)} e^{ia} = 1 \qquad e^-\colon \eta_e = \eta_e^{(0)} e^{i(b-c)} = 1$$

- all other intrinsic parities fixed by consistency
- can choose arbitrarily one intrinsic parity for each conserved quantity
- for truly neutral particles (e.g. γ, π^0) intrinsic parity cannot be redefined through phase transformation, has genuine intrinsic meaning

QFT imposes relations between intrinsic P of particle α and antiparticle $\bar{\alpha}$

$$\boxed{\text{bosons: } \eta_\alpha \eta_{\bar{\alpha}} = 1} \qquad \boxed{\text{fermions: } \eta_\alpha \eta_{\bar{\alpha}} = -1}$$

$P^2 =$ phase transformation $(\vec{p} \to \vec{p}, s_z \to s_z)$, and one can set $P^2 = 1$

- elementary $\alpha \neq \bar{\alpha} \Rightarrow$ particle-type charge, use the above to fix $\eta_\alpha^2 = \eta_{\bar{\alpha}}^2 = 1$
- for self-conjugate bosons $P^2 = 1$
- for self-conjugate (Majorana) fermions $P^2 = -1$, but unobserved in nature

Parity (contd.)

Empirical assignement of intrinsic parities using parity conservation

Ignoring weak interactions where $\not{P}$

Transition probability for scattering/decay processes

$$a\ b \to c\ d \qquad \text{transition probability} \propto |\langle c\ d|H_I|a\ b\rangle|^2$$
$$a \to b\ c \qquad \text{transition probability} \propto |\langle b\ c|H_I|a\rangle|^2$$

$H_I = H - H_0$: interaction Hamiltonian

- $[P, H] = 0$, diagonalise parity and energy together
- If $[P, H] = 0$ since $[P, H_0] = 0$ then $[P, H_I] = 0 \Rightarrow$ selection rule, nonzero matrix element only if same initial and final parity
- Since process takes place $\Rightarrow$ nonzero matrix elements $\Rightarrow$ same parity

But processes could take place to higher orders in perturbation theory!
Accurate statement is in terms of scattering, decay operators S, Γ instead of H_I

References

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