Particle physics

Matteo Giordano

Eötvös Loránd University (ELTE)
Budapest

ELTE
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Dirac equation

How to describe fermions with fields?

- set up relativistic EOM
- solve and quantise normal modes/canonical anticommutation relations
- Klein-Gordon equation transposes $E^2 = \vec{p}^{\,2} + m^2$ in QM language and has to be obeyed, but does not involve spin

Problems of KG equation (within relativistic quantum mechanics):

- negative energy states
- no covariant probability current giving positive-definite probability density

Dirac approach: find "square root" of KG

- first order in time, expected to avoid negative energy states
- automatically implies KG

Dirac equation (contd.)

First-order equation

$$b^\mu \partial_\mu \psi = a\psi \Longrightarrow [-(b^\mu \partial_\mu)^2 + a^2]\psi = 0$$

Impose

$$\Box = -b^\mu b^\nu \partial_\mu \partial_\nu = -\tfrac{1}{2}\{b^\mu, b^\nu\}\partial_\mu \partial_\nu \qquad m^2 = a^2$$

Satisfied if

$$\{b^\mu, b^\nu\} = -2\eta^{\mu\nu} \qquad a^2 = m^2$$

Cannot be solved by complex numbers: $(b^\mu)^2 = -1 \ \forall \mu \Rightarrow b^\mu b^\nu \neq 0 \ \forall \mu, \nu$
Simplest solution requires 4×4 matrices, $b^\mu = i\gamma^\mu$, $a = m 1_4$

$$\gamma^0 = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix} \qquad \gamma^j = \begin{pmatrix} 0_2 & \sigma_j \\ -\sigma_j & 0_2 \end{pmatrix} \qquad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Dirac equation: $$\boxed{(i\partial\!\!\!/ - m)\psi = 0}$$

$\partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$

Dirac equation (contd.)

Does it accomplish its tasks?

1. Probability current/density — success

Since $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$

$$\bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0 \qquad \bar{\psi} \equiv \psi^\dagger\gamma^0$$

Probability current $J^\mu = \bar{\psi}\gamma^\mu\psi$: covariant and conserved

$$\partial_\mu\bar{\psi}\gamma^\mu\psi = \bar{\psi}\partial\!\!\!/\psi + \bar{\psi}\overleftarrow{\partial\!\!\!/}\psi = \bar{\psi}(\partial\!\!\!/ - im)\psi + \bar{\psi}(\overleftarrow{\partial\!\!\!/} + im)\psi = 0$$

Probability density $\rho = J^0 = \psi^\dagger\psi \geq 0$

2. Negative energies — failure

Dirac Hamiltonian: recast

$$i\partial_0\psi = (m\gamma^0 - i\vec{\nabla}\cdot\gamma^0\vec{\gamma})\psi \equiv H_{\mathrm{Dirac}}\psi$$

Energy eigenfunctions: plane waves $\psi = \psi_0 e^{-ip\cdot x}$

$$\gamma^0(m + \vec{p}\cdot\vec{\gamma})\psi_0 = p^0\psi_0 \longrightarrow (p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} - m)\psi_0 = (\not{p} - m)\psi_0 = 0$$

Four solutions: two with positive energy $E = \sqrt{\vec{p}^{\,2} + m^2}$, two with negative energy $E = -\sqrt{\vec{p}^{\,2} + m^2}$

Positive-energy solutions: $E = p^0 = \sqrt{\vec{p}^{\,2} + m^2}$, momentum $\vec{p}$

$$\psi_+ = u(\vec{p}\,)e^{-ip\cdot x} \qquad (\not{p} - m)u(\vec{p}\,) = 0 \qquad u = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

Negative-energy solutions: $E = -p^0$, momentum $-\vec{p}$

$$\psi_- = v(\vec{p}\,)e^{ip\cdot x} \qquad (\not{p} + m)v(\vec{p}\,) = 0 \qquad v = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

u, v: four-component *bispinors*
$\xi_{1,2}$, $\eta_{1,2}$: two-component *spinors*

$$0 = (\not{p} - m)u = \begin{pmatrix} (p^0 - m)\xi_1 - \vec{p}\cdot\vec{\sigma}\xi_2 \\ -(p^0 + m)\xi_2 + \vec{p}\cdot\vec{\sigma}\xi_1 \end{pmatrix} \Longrightarrow \xi_2 = \frac{\vec{p}\cdot\vec{\sigma}}{p^0 + m}\xi_1$$

$$0 = (\not{p} + m)v = \begin{pmatrix} (p^0 + m)\eta_1 - \vec{p}\cdot\vec{\sigma}\eta_2 \\ -(p^0 - m)\eta_2 + \vec{p}\cdot\vec{\sigma}\eta_1 \end{pmatrix} \Longrightarrow \eta_1 = \frac{\vec{p}\cdot\vec{\sigma}}{p^0 + m}\eta_2$$

$$u_s(\vec{p}\,) = \sqrt{p^0 + m}\begin{pmatrix} \chi_s \\ \frac{\vec{p}\cdot\vec{\sigma}}{p^0+m}\chi_s \end{pmatrix} \qquad v_s(\vec{p}\,) = \sqrt{p^0 + m}\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{p^0+m}\tilde{\chi}_s \\ \tilde{\chi}_s \end{pmatrix}$$

$\chi_{s'}^\dagger\chi_s = \tilde{\chi}_{s'}^\dagger\tilde{\chi}_s = \delta_{s's}$, $s = 1, 2$

Dirac equation (contd.)

Normalisation:

$$\bar{u}_{s'}(\vec{p}\,)u_s(\vec{p}\,) = 2m\delta_{s's} \qquad \bar{v}_{s'}(\vec{p}\,)v_s(\vec{p}\,) = -2m\delta_{s's}$$

$$\bar{u}_{s'}(\vec{p}\,)v_s(\vec{p}\,) = 0 \qquad \bar{v}_{s'}(\vec{p}\,)u_s(\vec{p}\,) = 0$$

$\bar{u}_s(\vec{p}\,) \equiv u_s(\vec{p}\,)^\dagger\gamma^0 \qquad \bar{v}_s(\vec{p}\,) \equiv v_s(\vec{p}\,)^\dagger\gamma^0$

Important relations:

$$\bar{u}_s(\vec{p}\,)\gamma^\mu u_t(\vec{p}\,) = \bar{v}_s(\vec{p}\,)\gamma^\mu v_t(\vec{p}\,) = 2p^\mu\delta_{st} \qquad u_s(\vec{p}\,)^\dagger v_t(-\vec{p}\,) = v_s(\vec{p}\,)^\dagger u_t(-\vec{p}\,) = 0$$

Completeness of the solutions entails the relations

$$\sum_s u_s(\vec{p}\,)\bar{u}_s(\vec{p}\,) = \not{p} + m \qquad \sum_s v_s(\vec{p}\,)\bar{v}_s(\vec{p}\,) = \not{p} - m$$

Double degeneracy of energy levels explains two spin states of an electron: low-energy limit $\vec{p} \to 0$ of positive-energy solution

$$u_s(\vec{p}\,) \to \sqrt{2m}\begin{pmatrix}\chi_s \\ 0\end{pmatrix}$$

Two surviving components $\approx$ two components of electron wave function

Usual choice: $\chi_1 = \begin{pmatrix}1 \\ 0\end{pmatrix} \qquad \chi_2 = \begin{pmatrix}0 \\ 1\end{pmatrix}$

$s = 1 \leftrightarrow s_z = \frac{1}{2}$, $s = 2 \leftrightarrow s_z = -\frac{1}{2}$

Dirac equation (contd.)

How to interpret the negative-energy solutions? Antiparticle states

Negative-energy solution of momentum $-\vec{p}$ for the electron reintepreted as positive-energy solution of momentum $\vec{p}$ for positron

Low-energy limit:

$$v_s(\vec{p}\,) \to \sqrt{2m}\begin{pmatrix} 0 \\ \tilde{\chi}_s \end{pmatrix}$$

Change of sign of E from $-$ to $+$ can be obtained by changing the direction of time ("negative-energy state travelling backward in time = positive-energy state travelling forward in time"), consistency requires to change both $\vec{p}$ and spin

$$\tilde{\chi}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \tilde{\chi}_2 = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$s = 1 \leftrightarrow s_z = \frac{1}{2}$, $s = 2 \leftrightarrow s_z = -\frac{1}{2}$

Relation between χ_s and $\tilde{\chi}_s$ consequence of particles and antiparticles transforming in complex-conjugate irreducible representations of $\mathrm{SU}(2)$ (rotations)

Quantisation of the Dirac field

General solution of Dirac equation:

$$\psi(x) = \int d\Omega_p \sum_{s=1,2} \left\{ u_s(\vec{p}\,)b_s(\vec{p}\,)e^{-ip\cdot x} + v_s(\vec{p}\,)d_s(\vec{p}\,)^\dagger e^{ip\cdot x} \right\}$$

Heuristic approach As for scalar field promote $b_s(\vec{p}\,)$, $d_s(\vec{p}\,)^\dagger$ to fermionic creation/annihilation operators imposing *anticommutation* relations

$$\{b_s(\vec{p}\,), b_t(\vec{q}\,)\} = \{b_s(\vec{p}\,)^\dagger, b_t(\vec{q}\,)^\dagger\} = \{d_s(\vec{p}\,), d_t(\vec{q}\,)\} = \{d_s(\vec{p}\,)^\dagger, d_t(\vec{q}\,)^\dagger\} = 0$$

$$\{b_s(\vec{p}\,), b_t(\vec{q}\,)^\dagger\} = \{d_s(\vec{p}\,), d_t(\vec{q}\,)^\dagger\} = (2\pi)^3 2p^0 \delta^{(3)}(\vec{p} - \vec{q}\,)\delta_{st}$$

Imply equal-time anticommutation relations for fields

$$\{\psi_\alpha(x), \psi_\beta(y)\}_{\mathrm{ET}} = \{\psi_\alpha(x)^\dagger, \psi_\beta^\dagger(y)\}_{\mathrm{ET}} = 0$$

$$\begin{aligned}
\{\psi_\alpha(x), \psi_\beta^\dagger(y)\}_{\mathrm{ET}} &= \int d\Omega_p \int d\Omega_q \sum_{s,t} \Big[e^{-i(p\cdot x - q\cdot y)} u_{s\alpha}(\vec{p}\,) u_{t\beta}(\vec{q}\,)^\dagger \{b_s(\vec{p}\,), b_t(\vec{q}\,)^\dagger\} \\
&\qquad + e^{i(p\cdot x - q\cdot y)} v_{s\alpha}(\vec{p}\,) v_{t\beta}(\vec{q}\,)^\dagger \{d_s(\vec{p}\,)^\dagger, d_t(\vec{q}\,)\} \Big] \\
&= \int d\Omega_p \sum_s \left[e^{-ip\cdot(x-y)} u_{s\alpha}(\vec{p}\,) u_{s\beta}(\vec{p}\,)^\dagger + e^{ip\cdot(x-y)} v_{s\alpha}(\vec{p}\,) v_{s\beta}(\vec{p}\,)^\dagger \right] \\
&= \int d\Omega_p \Big[e^{i\vec{p}\cdot(\vec{x}-\vec{y})}(\not{p} + m) + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}(\not{p} - m) \Big] \gamma^0 \\
&= \int d\Omega_p \, p^0 \Big[e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \Big] = \delta^{(3)}(\vec{x} - \vec{y})
\end{aligned}$$

Quantisation of the Dirac field (contd.)

Canonical quantisation

Dirac equation obtained via variational principle from Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \qquad \bar{\psi} = \psi^\dagger\gamma^0$$

Performing variation wrt $\bar{\psi}$

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\partial\!\!\!/ - m)\psi = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0$$

Canonical momentum conjugate to ψ

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\psi^\dagger$$

Imposing canonical *anti*commutation relations

$$\{\psi_\alpha(x), \psi_\beta(y)\}_{\rm ET} = 0 \qquad \{\pi_\alpha(x), \pi_\beta(y)\}_{\rm ET} = -\{\psi_\alpha(x)^\dagger, \psi_\beta(y)^\dagger\}_{\rm ET} = 0$$

$$\{\psi_\alpha(x), \pi_\beta(y)\}_{\rm ET} = \{\psi_\alpha(x), i\psi_\beta^\dagger(y)\}_{\rm ET} = i\delta^{(3)}(\vec{x} - \vec{y}\,)$$

$\Rightarrow$ anticommutation relations for creation/annihilation operators

Quantisation of the Dirac field (contd.)

Normal-ordered product for fermion fields:

expand in creation/annihilation operators, reorder strings moving creation operators to the left, multiply by $(-1)^{\sigma_{\mathrm{P}}}$

$$: b_s(\vec{p}) b_t(\vec{q})^\dagger : = - b_t(\vec{q})^\dagger b_s(\vec{p}) = - : b_t(\vec{q})^\dagger b_s(\vec{p}) :$$
$$: b_s(\vec{p}) b_t(\vec{q}) : = \quad b_s(\vec{p}) b_t(\vec{q}) \quad = - \, b_t(\vec{q}) b_s(\vec{p}) \qquad = - : b_t(\vec{q}) b_s(\vec{p}) :$$

Order of operators matters for overall sign of the normal-ordered product!

Positive- and negative-frequency parts of ψ and $\bar{\psi}$

$$\psi_+(x) = \int d\Omega_p \sum_s u_s(\vec{p}) b_s(\vec{p}) e^{-ip\cdot x} \qquad \psi_-(x) = \int d\Omega_p \sum_s v_s(\vec{p}) d_s(\vec{p})^\dagger e^{ip\cdot x}$$
$$\bar{\psi}_+(x) = \int d\Omega_p \sum_s \bar{v}_s(\vec{p}) d_s(\vec{p}) e^{-ip\cdot x} \qquad \bar{\psi}_-(x) = \int d\Omega_p \sum_s \bar{u}_s(\vec{p}) b_s(\vec{p})^\dagger e^{ip\cdot x}$$

For two fields $\psi_{1,2} = \psi, \bar{\psi}$

$$\begin{aligned} :\psi_1\psi_2: &= \psi_{1+}\psi_{2+} - \psi_{2-}\psi_{1+} + \psi_{1-}\psi_{2+} + \psi_{1-}\psi_{2-} \\ &= -\psi_{2+}\psi_{1+} - \psi_{2-}\psi_{1+} + \psi_{1-}\psi_{2+} - \psi_{2-}\psi_{1-} = - :\psi_2\psi_1: \end{aligned}$$

Hamiltonian

$$H = \int d^3x \left[\pi(x)\partial_0\psi(x) - \bar{\psi}(x)\left(i\partial\!\!\!/ - m\right)\psi(x)\right]$$

$$= \int d^3x \left[i\bar{\psi}(x)\gamma^0\partial_0\psi(x) - \bar{\psi}(x)\left(i\partial\!\!\!/ - m\right)\psi(x)\right] = \int d^3x\, \bar{\psi}(x)\left(-i\vec{\nabla}\cdot\vec{\gamma} + m\right)\psi(x)$$

Imposing normal-ordering

$$H = \int d^3x \; :\bar{\psi}(x)\left(-i\vec{\nabla}\cdot\vec{\gamma} + m\right)\psi(x):$$

$$= \int d^3x \int d\Omega_p \int d\Omega_q \sum_{s,t} :\left\{\left[b_s(\vec{p}\,)^\dagger e^{ip\cdot x}\bar{u}_s(\vec{p}\,) + d_s(\vec{p}\,)e^{-ip\cdot x}\bar{v}_s(\vec{p}\,)\right]\right.$$
$$\left.\times \left[(m + \vec{q}\cdot\vec{\gamma})u_t(\vec{q}\,)e^{-iq\cdot x}b_t(\vec{q}\,) + (m - \vec{q}\cdot\vec{\gamma})v_t(\vec{q}\,)e^{iq\cdot x}d_s(\vec{q}\,)^\dagger\right]\right\}:$$

(Use $(p\!\!\!/ - m)u_s(\vec{p}) = 0$, $(p\!\!\!/ + m)v_s(\vec{p}) = 0$)

$$= \int d^3x \int d\Omega_p \int d\Omega_q \sum_{s,t} :\left\{\left[b_s(\vec{p}\,)^\dagger e^{ip\cdot x}\bar{u}_s(\vec{p}\,) + d_s(\vec{p}\,)e^{-ip\cdot x}\bar{v}_s(\vec{p}\,)\right]\right.$$
$$\left.\times \left[q^0\gamma^0 u_t(\vec{q}\,)e^{-iq\cdot x}b_t(\vec{q}\,) - q^0\gamma^0 v_t(\vec{q}\,)e^{iq\cdot x}d_s(\vec{q}\,)^\dagger\right]\right\}:$$

(Use $u_s(\vec{p})^\dagger v_s(-\vec{p}) = v_s(\vec{p})^\dagger u_s(-\vec{p}) = 0$)

$$= \int d\Omega_p\, p^0 \sum_s :\left\{b_s(\vec{p}\,)^\dagger b_s(\vec{p}\,) - d_s(\vec{p}\,)d_s(\vec{p}\,)^\dagger\right\}:$$

$$= \int d\Omega_p\, p^0 \sum_s \left\{b_s(\vec{p}\,)^\dagger b_s(\vec{p}\,) + d_s(\vec{p}\,)^\dagger d_s(\vec{p}\,)\right\}$$

Quantisation of the Dirac field (contd.)

$$H = \int d^3x \; : \bar{\psi}(x)\big(-i\vec{\nabla}\cdot\vec{\gamma} + m\big)\psi(x) :$$
$$= \int d\Omega_p \, p^0 \sum_s \big\{ b_s(\vec{p}\,)^\dagger b_s(\vec{p}\,) + d_s(\vec{p}\,)^\dagger d_s(\vec{p}\,) \big\}$$

- Without normal ordering, Hamiltonian would be divergent since $-d_s(\vec{p}\,)d_s(\vec{p}\,)^\dagger = d_s(\vec{p}\,)^\dagger d_s(\vec{p}\,) - \{d_s(\vec{p}\,), d_s(\vec{p}\,)^\dagger\}$
- Using commutators instead of anticommutators, the normal-ordered Hamiltonian would not be positive-definite

Requirement of positive-definite Hamiltonian $\Rightarrow$

- anticommutation relations for half-integer-spin (fermionic) creation and annihilation operators $\Rightarrow$ Fermi-Dirac statistics
- commutation relations for integer-spin (bosonic) creation and annihilation operators $\Rightarrow$ Bose-Einstein statistics

$\Rightarrow$ *spin-statistics theorem*

Wick's theorem for fermions

Product of two fermion fields

$$\psi_1(x)\psi_2(y) = :\psi_1(x)\psi_2(y): + \{\psi_{1+}(x), \psi_{2-}(y)\}$$

Anticommutator is c-number equal to its vacuum expectation value, $\neq 0$ only for $(\psi_1, \psi_2) = (\psi, \bar{\psi}) \Rightarrow$ ordinary = NO product in other cases

$$\begin{aligned}\psi(x)\bar{\psi}(y) &= :\psi(x)\bar{\psi}(y): + \{\psi_+(x), \bar{\psi}_-(y)\} = :\psi(x)\bar{\psi}(y): + \langle 0|\{\psi_+(x), \bar{\psi}_-(y)\}|0\rangle \\ &= :\psi(x)\bar{\psi}(y): + \langle 0|\psi_+(x)\bar{\psi}_-(y)|0\rangle = :\psi(x)\bar{\psi}(y): + \langle 0|\psi(x)\bar{\psi}(y)|0\rangle\end{aligned}$$

Time-ordered product for fermion fields

$$T\left(\psi_1(x)\psi_2(y)\right) = \theta(x^0 - y^0)\psi_1(x)\psi_2(y) - \theta(y^0 - x^0)\psi_2(y)\psi_1(x) = -T\left(\psi_2(y)\psi_1(x)\right)$$

Wick's theorem ($n = 2$):

$$T\left(\psi(x)\bar{\psi}(y)\right) = :\psi(x)\bar{\psi}(y): + \langle 0|T\left(\psi(x)\bar{\psi}(y)\right)|0\rangle \equiv :\psi(x)\bar{\psi}(y): + S(x-y)$$

Field contraction/propagator $S(x-y)$: 4×4 matrix, $S = S_{\alpha\beta}$, depends only on $x - y$ due to translation invariance

Wick's theorem for fermions (contd.)

Generalisation to n fields by induction, final result:
same formula as for scalar fields, but each term multiplied by the parity $(-1)^{\sigma_{\rm P}}$ of the permutation required to bring the fields from the original order to the order in which they appear (including in the contractions)

$(-1)^{\sigma_{\rm P}} = (-1)^{\text{n. of transpositions}}$

Example ($n = 4$):

$$
\begin{aligned}
&T\left(\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\gamma(w)\bar{\psi}_\delta(z)\right) \\
&= :\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\gamma(w)\bar{\psi}_\delta(z): \\
&\quad + S_{\alpha\beta}(x-y) :\psi_\gamma(w)\bar{\psi}_\delta(z): - S_{\alpha\delta}(x-z) :\psi_\gamma(w)\bar{\psi}_\beta(y): \\
&\quad + :\psi_\alpha(x)\bar{\psi}_\beta(y): S_{\gamma\delta}(w-z) \\
&\quad + S_{\alpha\beta}(x-y)S_{\gamma\delta}(w-z) - S_{\alpha\delta}(x-z)S_{\gamma\beta}(w-y)
\end{aligned}
$$

Interacting theories with fermions

Reminder: canonical programme to quantise interacting theories

- add $\mathcal{L}_I$ to free Lagrangian
- solve Euler-Lagrange equations of motion
- impose canonical (anti)commutation relations

Fermions do not make it any easier, have to find approximations

Perturbative approach:

- go over to interaction picture
- expand S-matrix in powers of $\mathcal{L}_I$
- use Wick's theorem/Feynman rules

Example: *Yukawa interaction*

$$\mathcal{L}_I = g :\phi\bar{\psi}\psi:$$

ψ: Dirac field, ϕ: Hermitean scalar field

Fermion-fermion elastic scattering to lowest order

$$\langle f|S-1|i\rangle = \frac{(ig)^2}{2}\int d^4x \int d^4y \langle f|T\left(:\phi(x)\bar{\psi}(x)\psi(x)::\phi(y)\bar{\psi}(y)\psi(y):\right)|i\rangle$$

$$|i\rangle = |\vec{p}_1\, s_{1\,z}; \vec{p}_2\, s_{2\,z}\rangle \qquad |f\rangle = |\vec{p}_1'\, s_{1\,z}'; \vec{p}_2'\, s_{2\,z}'\rangle$$

Interacting theories with fermions (contd.)

Using Wick's theorem for bosonic fields

$$\langle f|S-1|i\rangle = \frac{(ig)^2}{2}\int d^4x \int d^4y\, D(x-y)\langle f|\, T\left(:\bar{\psi}(x)\psi(x)::\bar{\psi}(y)\psi(y):\right)|i\rangle$$

As many particles as fermionic fields $\Rightarrow$ only no-contraction term in the fermionic Wick expansion contributes

$$\langle f|S-1|i\rangle = \frac{(ig)^2}{2}\int d^4x \int d^4y\, D(x-y)\langle f|\,:\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y):\,|i\rangle$$

Annihilation operator (for particles in initial state) in ψ, creation operator (for particles in final state) in $\bar{\psi}$

$$\begin{aligned}
&\langle f|\,:\bar{\psi}_\alpha(x)\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\beta(y):\,|i\rangle = \langle f|\,:\bar{\psi}_\alpha(x)\bar{\psi}_\beta(y)\psi_\beta(y)\psi_\alpha(x):\,|i\rangle \\
&= \langle f|\,:\bar{\psi}_\alpha(x)\bar{\psi}_\beta(y):\,|0\rangle\langle 0|\,:\psi_\beta(y)\psi_\alpha(x):\,|i\rangle \\
&= \Big(\langle \vec{p}_1{}' \, s_{1\,z}'|\bar{\psi}_\alpha(x)|0\rangle\langle \vec{p}_2{}' \, s_{2\,z}'|\bar{\psi}_\beta(y)|0\rangle - \langle \vec{p}_2{}' \, s_{2\,z}'|\bar{\psi}_\alpha(x)|0\rangle\langle \vec{p}_1{}' \, s_{1\,z}'|\bar{\psi}_\beta(y)|0\rangle\Big) \\
&\quad \times \Big(\langle 0|\psi_\alpha(x)|\vec{p}_1\, s_{1\,z}\rangle\langle 0|\psi_\beta(y)|\vec{p}_2\, s_{2\,z}\rangle - \langle 0|\psi_\alpha(x)|\vec{p}_2\, s_{2\,z}\rangle\langle 0|\psi_\beta(y)|\vec{p}_1\, s_{1\,z}\rangle\Big)
\end{aligned}$$

Minus signs due to anticommuting nature of Dirac field

Interacting theories with fermions (contd.)

Source of minus sign:

$$b_t(\vec{q}\,)b_s(\vec{p}\,)|\vec{p}_1\, s_{1\,z}; \vec{p}_2\, s_{2\,z}\rangle = b_t(\vec{q}\,)b_s(\vec{p}\,)b_{s_{1\,z}}(\vec{p}_1)^\dagger|\vec{p}_2\, s_{2\,z}\rangle$$

$$= \delta_{ss_{1\,z}}(2\pi)^3 2p_1^0\delta^{(3)}(\vec{p}-\vec{p}_1)\delta_{ts_{2\,z}}(2\pi)^3 2p_2^0\delta^{(3)}(\vec{p}-\vec{p}_2)|0\rangle$$

$$- b_t(\vec{q}\,)b_{s_{1\,z}}(\vec{p}_1)^\dagger b_s(\vec{p}\,)|\vec{p}_2\, s_{2\,z}\rangle$$

$$= \delta_{ss_{1\,z}}(2\pi)^3 2p_1^0\delta^{(3)}(\vec{p}-\vec{p}_1)\delta_{ts_{2\,z}}(2\pi)^3 2p_2^0\delta^{(3)}(\vec{p}-\vec{p}_2)|0\rangle$$

$$- b_t(\vec{q}\,)b_{s_{1\,z}}(\vec{p}_1)^\dagger \delta_{ss_{2\,z}}(2\pi)^3 2p_2^0\delta^{(3)}(\vec{p}-\vec{p}_2)|0\rangle$$

$$= (2\pi)^6 4p_1^0p_2^0[\delta_{ss_{1\,z}}\delta^{(3)}(\vec{p}-\vec{p}_1)\delta_{ts_{2\,z}}\delta^{(3)}(\vec{p}-\vec{p}_2) - \delta_{ts_{2\,z}}\delta^{(3)}(\vec{p}-\vec{p}_1)\delta_{ss_{2\,z}}\delta^{(3)}(\vec{p}-\vec{p}_2)]|0\rangle$$

One-particle matrix elements

$$\langle 0|\psi(x)|\vec{p}\, s_z\rangle = \int d\Omega_q \sum_s e^{-iq\cdot x}u_s(\vec{q}\,)\langle 0|b_s(\vec{q}\,)b_{s_z}(\vec{p}\,)^\dagger|0\rangle$$

$$= \int d\Omega_q \sum_s e^{-iq\cdot x}u_s(\vec{q}\,)\langle 0|\{b_s(\vec{q}\,), b_{s_z}(\vec{p}\,)^\dagger\}|0\rangle$$

$$= \int d\Omega_q \sum_s e^{-iq\cdot x}u_s(\vec{q}\,)\delta_{ss_z}(2\pi)^3 2q^0\delta^{(3)}(\vec{p}-\vec{q}\,) = e^{-ip\cdot x}u_{s_z}(\vec{p}\,)$$

$$\langle \vec{p}\, s_z|\bar{\psi}(x)|0\rangle = \int d\Omega_q \sum_s e^{iq\cdot x}\bar{u}_s(\vec{q}\,)\langle 0|b_{s_z}(\vec{p}\,)b_s(\vec{q}\,)^\dagger|0\rangle$$

$$= \int d\Omega_q \sum_s e^{iq\cdot x}\bar{u}_s(\vec{q}\,)\langle 0|\{b_{s_z}(\vec{p}\,), b_s(\vec{q}\,)^\dagger\}|0\rangle$$

$$= \int d\Omega_q \sum_s e^{iq\cdot x}\bar{u}_s(\vec{q}\,)\delta_{ss_z}(2\pi)^3 2q^0\delta^{(3)}(\vec{p}-\vec{q}\,) = e^{ip\cdot x}\bar{u}_{s_z}(\vec{p}\,)$$

Interacting theories with fermions (contd.)

x, y are integrated over $\Rightarrow$ equivalent

$$\begin{aligned}\langle f|S-1|i\rangle &= \frac{(ig)^2}{2}\int d^4x \int d^4y\, D(x-y) \\ &\quad\times 2\big(\langle \vec{p}_1{}'\, s'_{1z}|\bar{\psi}_\alpha(x)|0\rangle\langle \vec{p}_2{}'\, s'_{2z}|\bar{\psi}_\beta(y)|0\rangle\langle 0|\psi_\alpha(x)|\vec{p}_1\, s_{1z}\rangle\langle 0|\psi_\beta(y)|\vec{p}_2\, s_{2z}\rangle \\ &\qquad - \langle \vec{p}_2{}'\, s'_{2z}|\bar{\psi}_\alpha(x)|0\rangle\langle \vec{p}_1{}'\, s'_{1z}|\bar{\psi}_\beta(y)|0\rangle\langle 0|\psi_\alpha(x)|\vec{p}_1\, s_{1z}\rangle\langle 0|\psi_\beta(y)|\vec{p}_2\, s_{2z}\rangle\big) \\ &= (ig)^2\int d^4x \int d^4y\, D(x-y) \\ &\quad\times \big(e^{i(p'_1-p_1)\cdot x}e^{i(p'_2-p_2)\cdot y}\bar{u}_{s'_{1z}}(\vec{p}_1{}')u_{s_{1z}}(\vec{p}_1)\bar{u}_{s'_{2z}}(\vec{p}_2{}')u_{s_{2z}}(\vec{p}_2) \\ &\qquad - e^{i(p'_2-p_1)\cdot x}e^{i(p'_1-p_2)\cdot y}\bar{u}_{s'_{2z}}(\vec{p}_2{}')u_{s_{1z}}(\vec{p}_1)\bar{u}_{s'_{1z}}(\vec{p}_1{}')u_{s_{2z}}(\vec{p}_2)\big)\end{aligned}$$

Integrating over x, y

$$\begin{aligned}\langle f|S-1|i\rangle &= i(2\pi)^4\delta^{(4)}(p'_1+p'_2-p_1-p_2) \\ &\quad\times ig^2\big(\tilde{D}(p'_1-p_1)\bar{u}_{s'_{1z}}(\vec{p}_1{}')u_{s_{1z}}(\vec{p}_1)\bar{u}_{s'_{2z}}(\vec{p}_2{}')u_{s_{2z}}(\vec{p}_2) \\ &\qquad - \tilde{D}(p'_2-p_1)\bar{u}_{s'_{2z}}(\vec{p}_2{}')u_{s_{1z}}(\vec{p}_1)\bar{u}_{s'_{1z}}(\vec{p}_1{}')u_{s_{2z}}(\vec{p}_2)\big)\end{aligned}$$

$$\mathcal{M}_{fi} = -g^2\Big[\frac{1}{t-m^2}\bar{u}_{s'_{1z}}(\vec{p}_1{}')u_{s_{1z}}(\vec{p}_1)\bar{u}_{s'_{2z}}(\vec{p}_2{}')u_{s_{2z}}(\vec{p}_2) - \frac{1}{u-m^2}\bar{u}_{s'_{2z}}(\vec{p}_2{}')u_{s_{1z}}(\vec{p}_1)\bar{u}_{s'_{1z}}(\vec{p}_1{}')u_{s_{2z}}(\vec{p}_2)\Big]$$

Relative minus sign reflects fermionic nature of colliding particles

Feynman rules

S-matrix faster with Feynman diagrams, extra rules for fermions

- lines corresponding to Dirac fields (or charged fields in general) are *oriented*: $\psi/\bar{\psi}$ enters/exits the vertex
- once fields are paired with incoming/outgoing particles add
  - line entering diagram for particle in initial state = factor $u_s(\vec{p})e^{-ip\cdot x}$

    p, s: quantum numbers of the particle, x: vertex to which the line is attached
  - line exiting diagram for particle in final state = factor $\bar{u}_s(\vec{p})e^{ip\cdot x}$
  - line exiting diagram for antiparticle in initial state = factor $\bar{v}_s(\vec{p})e^{-ip\cdot x}$
  - line entering diagram for antiparticle the final state = $v_s(\vec{p})e^{ip\cdot x}$
- for each field contraction add an internal line, oriented from vertex of $\bar{\psi}(y)$ to vertex of $\psi(x)$ = propagator $S(x-y)$
- put relative minus signs between diagrams corresponding to same process differing by permutations of fermionic lines (reflect signs in Wick's theorem)

  a minus sign for an antiparticle line fully crossing a diagram;
  a minus sign for a fermion loop
- integrate over position of vertices to get momentum-conserving delta function at each vertex

Feynman rules (contd.)

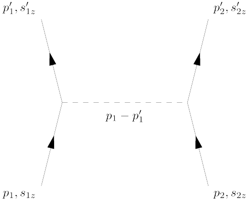
Assigning momenta flowing in/out of the diagram for incoming/outgoing particles (irrespectively of line orientation) and to internal lines (in the same direction of the arrow, conventionally), rules in momentum space:

- factors from $\mathcal{L}_I$ for each of n vertices $\times \frac{i^n}{n!}$ from Dyson's formula
- $u_s(\vec{p}\,)$ for a particle in the initial state
- $\bar{u}_s(\vec{p}\,)$ for a particle in the final state
- $\bar{v}_s(\vec{p}\,)$ for an antiparticle in the initial state
- $v_s(\vec{p}\,)$ for an antiparticle in the final state
- $\tilde{S}(q) = \int d^4x\, e^{iq\cdot x} S(x)$ for each internal line
- $(2\pi)^4\delta(\sum p)$ for each vertex (p inflowing)
- contract Dirac indices along uninterrupted fermionic lines (starting from end)

$$(\bar{u}_{\rm out}|\bar{v}_{\rm in})_{s'\alpha_1}(\vec{p}\,'\,)\tilde{S}_{\alpha_1\alpha_2}(q_1)\tilde{S}_{\alpha_2\alpha_3}(q_2)\ldots\tilde{S}_{\alpha_{n-1}\alpha_n}(q_{n-1})(u_{\rm in}|v_{\rm out})_{s\alpha_n}(\vec{p}\,)$$

- put minus factor when exchanging fermionic lines/making a fermion loop
- count realisations for each topologically distinct diagram
- integrate over internal momenta

Feynman rules (contd.)



Fermion-fermion scattering in Yukawa theory:

- two topologically distinct diagrams
- diagrams related by exchanging two fermion lines $\Rightarrow$ relative $-$
- counting: each diagram realised in 2 ways since vertices equivalent

Detailed counting: ψ for first e^- chosen in two ways, other pairings unique once diagram topology is fixed

$$i(2\pi)^4\delta^{(4)}(p_f - p_i)\mathcal{M}_{fi} = \frac{1}{2!}(ig)^2 2\,[\underbrace{\bar{u}_{s_{1z}'}(\vec{p}_1{}')u_{s_{1z}}(\vec{p}_1)}_{\text{diag. 1, 1 -- 1' line}}\tilde{D}(p_1' - p_1)\underbrace{\bar{u}_{s_{2z}'}(\vec{p}_2{}')u_{s_{2z}}(\vec{p}_2)}_{\text{diag. 1, 2 -- 2' line}}$$
$$- \underbrace{\bar{u}_{s_{2z}'}(\vec{p}_2{}')u_{s_{1z}}(\vec{p}_1)}_{\text{diag. 2, 1 -- 2' line}}\tilde{D}(p_2' - p_1)\underbrace{\bar{u}_{s_{1z}'}(\vec{p}_1{}')u_{s_{2z}}(\vec{p}_2)}_{\text{diag. 2, 2 -- 1' line}}]$$

Fermion propagator

$$
\begin{aligned}
&S_{\alpha\beta}(x-y) = \langle 0| T\left(\psi_\alpha(x)\bar{\psi}_\beta(y)\right)|0\rangle \\
&= \int d\Omega_p \int d\Omega_q \sum_{s,t} \theta(x^0-y^0) e^{-i(p\cdot x - q\cdot y)} u_{s\,\alpha}(\vec{p}\,)\bar{u}_{t\,\beta}(\vec{q}\,)\langle 0|b_s(\vec{p}\,)b_t(\vec{q}\,)^\dagger|0\rangle \\
&\qquad - \theta(y^0-x^0) e^{-i(p\cdot y - q\cdot x)} v_{t\,\alpha}(\vec{q}\,)\bar{v}_{s\,\beta}(\vec{p}\,)\langle 0|d_s(\vec{p}\,)d_t(\vec{q}\,)^\dagger|0\rangle \\
&= \int d\Omega_p \sum_s \theta(x^0-y^0)e^{-ip\cdot(x-y)}u_{s\,\alpha}(\vec{p}\,)\bar{u}_{s\,\beta}(\vec{p}\,) - \theta(y^0-x^0)e^{-ip\cdot(y-x)}v_{s\,\alpha}(\vec{p}\,)\bar{v}_{s\,\beta}(\vec{p}\,) \\
&= \int d\Omega_p \left[\theta(x^0-y^0)e^{-ip\cdot(x-y)}(\not{p}+m)_{\alpha\beta} - \theta(y^0-x^0)e^{-ip\cdot(y-x)}(\not{p}-m)_{\alpha\beta}\right]
\end{aligned}
$$

Fermion propagator obeys inhomogeneous Dirac equation

$$
\begin{aligned}
&(i\not{\partial}^x - m)S(x-y) = (i\not{\partial}^x - m)\langle 0|T\left(\psi(x)\bar{\psi}(y)\right)|0\rangle \\
&= \langle 0|T\left((i\vec{\gamma}\cdot\vec{\nabla}_x - m)\psi(x)\bar{\psi}(y)\right)|0\rangle + \partial_0^x\langle 0|T\left(i\gamma^0\psi(x)\bar{\psi}(y)\right)|0\rangle \\
&= \langle 0|T\left((i\not{\partial}^x - m)\psi(x)\bar{\psi}(y)\right)|0\rangle + i\delta(x^0-y^0)\langle 0|\{\gamma^0\psi(x),\bar{\psi}(y)\}_{\rm ET}|0\rangle
\end{aligned}
$$

ψ obeys Dirac equation + canonical anticommutation relations

$$
\begin{aligned}
&\gamma^0_{\alpha\beta}\{\psi_\beta(x),\bar{\psi}_\delta(y)\}_{\rm ET} = \gamma^0_{\alpha\beta}\{\psi_\beta(x),\psi^\dagger_\gamma(y)\}_{\rm ET}\gamma^0_{\gamma\delta} = \delta_{\alpha\delta}\delta^{(3)}(\vec{x}-\vec{y}\,) \\
&\Rightarrow (i\not{\partial}^x - m)S(x-y) = i\delta^{(4)}(x-y)
\end{aligned}
$$

Fermion propagator

Setting $S(x) = \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot x}\tilde{S}(q)$

$$(\not{q} - m)\tilde{S}(q) = i$$

Since $(\not{q} + m)(\not{q} - m) = q^2 - m^2$
$\Rightarrow (\not{q} - m)^{-1} = (\not{q} + m)/(q^2 - m^2)$ except for $q^2 = m^2$

Correct prescription for pole yielding $S(x)$:

$$\tilde{S}(q) = i\frac{\not{q} + m}{q^2 - m^2 + i\epsilon}$$

Using the residue theorem

Beware: $q^0 \neq \sqrt{\vec{q}^{\,2} + m^2}$,
is a free integration variable

$$\begin{aligned} S(x) &= i\int \tfrac{d^4q}{(2\pi)^4}\, e^{-iq\cdot x}\tfrac{\not{q}+m}{q^2-m^2+i\epsilon} \\ &= i\int \frac{d^4q}{(2\pi)^4}\left\{\theta(x^0)\frac{(-2\pi i)}{2q^0}e^{-i(q^0x^0+\vec{q}\cdot\vec{x})}(q^0\gamma^0 - \vec{q}\cdot\vec{\gamma} + m)\right. \\ &\qquad \left. + \theta(-x^0)\tfrac{2\pi i}{(-2q^0)}e^{-i(-q^0x^0+\vec{q}\cdot\vec{x})}(-q^0\gamma^0 - \vec{q}\cdot\vec{\gamma} + m)\right\}_{q^0=\sqrt{\vec{q}^{\,2}+m^2}} \\ &= \int d\Omega_q \left\{\theta(x^0)e^{-iq\cdot x}(\not{q} + m) - \theta(-x^0)e^{iq\cdot x}(\not{q} - m)\right\} \end{aligned}$$

THE END.

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