Particle physics

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Noether current: Lorentz transformations

Infinitesimal Lorentz transformation of coordinates

$$x'^{\mu} = x^{\mu} + \tfrac{1}{2}\omega_{\rho\sigma} M^{(\rho\sigma)\,\mu}{}_{\nu} x^{\nu} \qquad M^{(\rho\sigma)\,\mu}{}_{\nu} = \eta^{\rho\mu}\delta^{\sigma}{}_{\nu} - \eta^{\sigma\mu}\delta^{\rho}{}_{\nu}$$

Scalar field $\phi(x)$ left invariant by a Lorentz transformation (by definition)

$$\phi'_a(x') = \phi_a(x)$$

Noether current $J^{(\rho\sigma)\mu}$ (associated to only $\omega_{\rho\sigma} \neq 0$)

$$\begin{aligned} \mathcal{M}^{\mu,\rho\sigma} \equiv J^{(\rho\sigma)\mu} &= M^{(\rho\sigma)\,\mu}{}_{\nu} x^{\nu}\mathcal{L} - M^{(\rho\sigma)\,\nu}{}_{\alpha} x^{\alpha}\partial_{\nu}\phi_a \tfrac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_a)} \\ &= x^{\rho}\left(\partial^{\sigma}\phi_a \tfrac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_a)} - \eta^{\mu\sigma}\mathcal{L}\right) - x^{\sigma}\left(\partial^{\rho}\phi_a \tfrac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_a)} - \eta^{\mu\rho}\mathcal{L}\right) \\ &= x^{\rho}\Theta^{\mu\sigma} - x^{\sigma}\Theta^{\mu\rho} \end{aligned}$$

Conserved charges (generators of Lorentz transformations)

$$\int d^3x\, \mathcal{M}^{0,\rho\sigma} = \int d^3x\, \left[x^{\rho}\Theta^{0\sigma} - x^{\sigma}\Theta^{0\rho}\right] = \mathcal{J}^{(\rho\sigma)}$$

Noether current: internal symmetries

Internal symmetry transformations involve only the fields and not the spacetime coordinates ($M \neq 0$, $\mathcal{A} = 0$)

Simplest case: linear transformations

$$\delta\phi_a(x) = \epsilon \textstyle\sum_b K_{ab}\phi_b(x)$$

Conserved current/charge:

$$J^\mu_{\rm internal} = \textstyle\sum_{ab} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} K_{ab}\phi_b$$

$$Q_{\rm internal} = \int d^3x \textstyle\sum_{ab} \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} K_{ab}\phi_b = \int d^3x \textstyle\sum_{ab} \pi_a K_{ab}\phi_b$$

Noether charges as symmetry generators

Derivation in classical case can be extended to quantum case taking into account operator-ordering problems and short-distance singularities

For free fields normal ordering of operators suffices

Since $Q(t) = Q(0)$ is time-independent

$$[\mathrm{i}\epsilon Q, \phi_i(x)] = \int d^3y \, \Big[\big\{ \textstyle\sum_k [M_k(\phi(y), y) - \mathcal{A}^j(y)\partial_j \phi_k(y)]\pi_k(y)$$

$$- \mathcal{A}^0(y)\mathcal{H}(\phi(y), \pi(y)) - F^0(\phi(y), y) \big\}, \phi_i(x) \Big]_{\mathrm{ET}}$$

$$= i\epsilon \int d^3y \textstyle\sum_k \big\{ M_k(\phi(y), y) - \mathcal{A}^j(y)\partial_j\phi_k(y) - \mathcal{A}^0(y) \frac{\mathcal{H}(\phi(y),\pi(y))}{\partial \pi_k(y)} \big\} [\pi_k(y), \phi_i(x)]_{\mathrm{ET}}$$

$$= \epsilon \int d^3y \textstyle\sum_k \big\{ M_k(\phi(y), y) - \mathcal{A}^j(y)\partial_j\phi_k(y) - \mathcal{A}^0(y)\partial_0\phi_k(y) \big\} \delta_{ik}\delta^{(3)}(\vec{x} - \vec{y}\,)$$

$$= \epsilon[M_i(\phi(x), x) - \mathcal{A}^\nu(x)\partial_\nu\phi_i(x)] = \overline{\delta\phi}_i(x)$$

Conserved-charge operators Q generate the corresponding symmetry transformations on the fields

Noether charges as symmetry generators (contd.)

Q generates symmetry transformation on fields

$$
\begin{aligned}
\text{translations:} \qquad & [P_\mu, \phi(x)] = -\mathrm{i}\partial_\mu \phi(x) \\
\text{Lorentz transformations:} \qquad & [\mathcal{J}^{(\rho\sigma)}, \phi(x)] = \mathrm{i}\,(x^\rho \partial^\sigma - x^\sigma \partial^\rho)\,\phi(x) \\
\text{internal symmetries:} \qquad & [Q_{\rm internal}, \phi_a(x)] = -\mathrm{i}K_{ab}\phi_b(x)
\end{aligned}
$$

Q are Hermitean (after dealing with ordering problems) as long as Lagrangian is Hermitean (starting from real Lagrangian at classic level)

Noether's theorem in quantum case entails existence of Hermitean generators of the continuous symmetries of the Lagrangian

$\Rightarrow$ unitary representation of Poincaré and other symmetry groups, transforming quantum field operators like their classical counterpart

$\Rightarrow$ quantum system exhibits the desired symmetry

Interacting fields

Free fields are of limited utility to describe the real world, need a theory

- describing interacting particles
- complying with Poincaré invariance and locality

Use fields (local, simple transf. properties) and canonical quantisation:

- Lagrangian exhibits symmetries of the theory manifestly
- imposing canonical commutation (CCR) gives microcausality/locality

  or anticommutation relations (or CAR)
- Noether's theorem gives symmetry generators

Unfortunately almost never possible to complete the program: nonlinear EOM, hard to solve $\Rightarrow$ need approximation technique

In many cases of practical interest, Hamiltonian $H = H_0 + V$, free (solvable) H_0 + interaction part V

Treat V as a perturbation to the free Hamiltonian H_0, evaluate effects in successive steps $\Rightarrow$ *perturbative quantisation*

Interaction picture

Let $\phi(x)$, $\pi(x)$ be field and conjugate momentum realising canonical quantisation program (solve EOM + obey CCR)

$$\phi(x) = \phi(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt} \qquad \dot{\phi}(t,\vec{x}) = i[H,\phi(t,\vec{x})]$$
$$\pi(x) = \pi(t,\vec{x}) = e^{iHt}\pi(0,\vec{x})e^{-iHt} \qquad \dot{\pi}(t,\vec{x}) = i[H,\pi(t,\vec{x})]$$

Full Hamiltonian

$$H = \int d^3x \, [\pi(t,\vec{x})\dot{\phi}(t,\vec{x}) - \mathcal{L}(\phi(t,\vec{x}), \partial\phi(t,\vec{x}))]$$

$\partial_0\phi$ expressed as a function of ϕ and π

Assume $\mathcal{L}(\phi,\partial\phi) = \mathcal{L}_0(\phi,\partial\phi) + \mathcal{L}_I(\phi)$ (no derivative interaction)

$$\pi(\phi,\partial\phi) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \frac{\partial\mathcal{L}_0}{\partial(\partial_0\phi)} = \pi_0(\phi,\partial\phi)$$

Same functional form as conjugate momentum π_0 of free theory
$\Leftrightarrow$ $\partial_0\phi$ in full theory is the same function of ϕ and π as in the free theory

Interaction picture (contd.)

In the cases of interest H is time-independent

$$H = H[\phi,\pi] = \int d^3x\,[\pi(t,\vec{x})\dot{\phi}(t,\vec{x}) - \mathcal{L}_0(\phi(t,\vec{x}),\partial\phi(t,\vec{x})) - \mathcal{L}_I(\phi(t,\vec{x}))]$$
$$= H_0[\phi,\pi] - \int d^3x\,\mathcal{L}_I(\phi(t,\vec{x})) = H_0[\phi,\pi] + V[\phi]$$

H_0 and V are separately time dependent; do splitting at $t=0$

$$H[\phi(t,\vec{x}),\pi(t,\vec{x})] = H[\phi(0,\vec{x}),\pi(0,\vec{x})] = H_0[\phi(0,\vec{x}),\pi(0,\vec{x})] + V[\phi(0,\vec{x})]$$

Fields in the interaction picture:

- evolve in time with the free Hamiltonian
- coincide with fully interacting fields (in Heisenberg picture) at $t=0$

$$\phi_{\rm int}(t,\vec{x}) \equiv e^{iH_0t}\phi_{\rm int}(0,\vec{x})e^{-iH_0t} \qquad \phi_{\rm int}(0,\vec{x}) = \phi(0,\vec{x})$$
$$\pi_{\rm int}(t,\vec{x}) \equiv e^{iH_0t}\pi_{\rm int}(0,\vec{x})e^{-iH_0t} \qquad \pi_{\rm int}(0,\vec{x}) = \pi(0,\vec{x})$$

Interaction picture (contd.)

$\phi_{\rm int}$, $\pi_{\rm int}$ CCR at $t = 0$, free unitary evolution $\Rightarrow$

- obey CCR at all times
- obey Hamilton equations of motion of free theory

$$\dot{\phi}_{\rm int} = i[H_0, \phi_{\rm int}] = \frac{\delta H_0}{\delta \pi_{\rm int}}, \qquad \dot{\pi}_{\rm int} = i[H_0, \pi_{\rm int}] = -\frac{\delta H_0}{\delta \phi_{\rm int}}$$

$\phi_{\rm int}$, $\pi_{\rm int}$ are free fields $\Rightarrow$ we know exactly what they are

Example: H_0 free Hamiltonian for charged (non-Hermitian) scalar field

$$\phi_{\rm int}(t,\vec{x}\,) = \int d\Omega_p \left\{ a(\vec{p}\,)e^{-ip\cdot x} + b(\vec{p}\,)^\dagger e^{ip\cdot x} \right\} \qquad \pi_{\rm int}(t,\vec{x}\,) = \dot{\phi}_{\rm int}(t,\vec{x}\,)$$

$a(\vec{p}\,)$, $a(\vec{p}\,)^\dagger$, $b(\vec{p}\,)$, $b(\vec{p}\,)^\dagger$ usual annihilation and creation operators

Interaction Hamiltonian in the interaction picture $V_I(t)$

$$V_I(t) \equiv e^{iH_0 t} V[\phi_{\rm int}(0,\vec{x}\,)] e^{-iH_0 t} = V[\phi_{\rm int}(t,\vec{x}\,)] = -\int d^3x\, \mathcal{L}_I(\phi_{\rm int}(t,\vec{x}\,))$$

We have not solved anything yet, but we have set up the stage for solving the theory iteratively (spectrum, S-matrix)

Perturbation theory

If V (and so V_I) is a small perturbation $\Rightarrow$ power-expand S and compute S-matrix elements order by order in the perturbation

$$
\begin{aligned}
S &= \mathrm{Texp}\left\{-i\int_{-\infty}^{+\infty} dt\, V_I(t)\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty} d\tau_1 \ldots \int_{-\infty}^{+\infty} d\tau_n T\left\{V_I(\tau_1)\ldots V_I(\tau_n)\right\}
\end{aligned}
$$

Products of fields at same spacetime point lead to problems (infinities) $\Rightarrow$ get rid of (part of) them taking V_I to be normal-ordered

Does not change the symmetries of S

$$
\begin{aligned}
S &= \mathrm{Texp}\left\{i\int d^4x\ :\mathcal{L}_I(\phi_{\rm int}(x)):\right\} \\
&= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1 \ldots \int d^4x_n\, T\left\{:\mathcal{L}_I(\phi_{\rm int}(x_1)):\ldots:\mathcal{L}_I(\phi_{\rm int}(x_n)):\right\}
\end{aligned}
$$

Can now compute scattering cross sections in perturbation theory (PT)

Perturbation theory (contd.)

Final state coinciding initial state practically impossible to observe in experiments (would interfere with experimental setup) $\Rightarrow i \neq f$

Basic assumption of PT: exact S-matrix elements $\langle \varphi_f | S | \varphi_i \rangle$ well approximated with lowest-order terms of the expansion of S

$$\langle \varphi_f | S - \mathbf{1} | \varphi_i \rangle = \Big\langle \varphi_f \Big| i \int d^4x : \mathcal{L}_I(x) : + \frac{i^2}{2} \int d^4x_1 \int d^4x_2 \, T \{ : \mathcal{L}_I(x_1) :: \mathcal{L}_I(x_2) : \} + \ldots \Big| \varphi_i \Big\rangle$$

Object of interest: matrix element

$$\Big\langle \varphi_f \Big| \int d^4x_1 \int d^4x_2 \ldots \int d^4x_n \, T \{ : \mathcal{L}_I(x_1) :: \mathcal{L}_I(x_2) : \ldots : \mathcal{L}_I(x_n) : \} \Big| \varphi_i \Big\rangle$$

$|\varphi_{i,f}\rangle$: energy/momentum eigenstates – eigenstates of free Hamiltonian H_0 and free spatial momentum operators built out of free fields $\phi_{\rm int}$, created out of vacuum $|0\rangle$ by corresponding creation operators

Perturbation theory (contd.)

Energy-momentum conservation from translation invariance: for total initial/final momentum $p_{i,f}$

$$
\begin{aligned}
&\langle\varphi_f| \int d^4x_1 \ldots \int d^4x_n T\{:\mathcal{L}_I(x_1):\ldots:\mathcal{L}_I(x_n):\}|\varphi_i\rangle \\
&= \int d^4x_1 \ldots \int d^4x_n \langle\varphi_f| \underbrace{e^{-ix_n\cdot P}}_{\text{operator}} T\{:\mathcal{L}_I(x_1-x_n):\ldots:\mathcal{L}_I(0):\} \underbrace{e^{ix_n\cdot P}}_{\text{operator}} |\varphi_i\rangle \\
&= \int d^4x_n \underbrace{e^{-ix_n\cdot(p_f-p_i)}}_{\text{phase factor}} \\
&\quad \times \langle\varphi_f| \int d^4y_1 \ldots \int d^4y_{n-1} T\{:\mathcal{L}_I(y_1):\ldots:\mathcal{L}_I(y_{n-1})::\mathcal{L}_I(0):\}|\varphi_i\rangle \\
&= (2\pi)^4\delta^{(4)}(p_f-p_i) \\
&\quad \times \langle\varphi_f| \int d^4y_1 \ldots \int d^4y_{n-1} T\{:\mathcal{L}_I(y_1):\ldots:\mathcal{L}_I(y_{n-1})::\mathcal{L}_I(0):\}|\varphi_i\rangle
\end{aligned}
$$

$$
\Rightarrow \boxed{S_{fi} = \mathbf{1}_{fi} + (2\pi)^4\delta^{(4)}(p_f-p_i)\,\mathrm{i}\,\mathcal{M}_{fi}}
$$

Factor i conventional

Wick's theorem

Matrix elements simplified using *Wick's theorem*
For a single Hermitean scalar field:

$$T\left(\varphi(x_1)\dots\varphi(x_n)\right) = \sum_{m=0}^{\left[\frac{n}{2}\right]}\Big\{ :\varphi(x_1)\dots\varphi(x_{n-2m}): D(x_{n-2m+1},x_{n-2m+2})\dots D(x_{n-1},x_n) + \text{other pairings}\Big\}$$

Sum over number m of pairings of coordinates $\{x_1,\dots,x_n\}$ (from 0 to maximal possible value $\left[\frac{n}{2}\right]$) and over all possible such pairings

$D(x,y)$: *contraction* of two fields, or *propagator*

$$D(x,y) = \langle 0|\,T\left(\varphi(x)\varphi(y)\right)|0\rangle = \langle 0|e^{-iy\cdot P}\,T\left(\varphi(x-y)\varphi(0)\right)e^{iy\cdot P}|0\rangle = \langle 0|\,T\left(\varphi(x-y)\varphi(0)\right)|0\rangle = D(x-y)$$

For Hermitean scalar field also

$$D(x-y) = \langle 0|\,T\left(\varphi(x)\varphi(y)\right)|0\rangle = \langle 0|\,T\left(\varphi(y)\varphi(x)\right)|0\rangle = D(y-x)$$

Wick's theorem (contd.)

Sketch of proof: for $n = 2$ we know already

$$\varphi(x_1)\varphi(x_2) =: \varphi(x_1)\varphi(x_2) : + \langle 0|\varphi(x_1)\varphi(x_2)|0\rangle$$

Imposing time ordering

$$T\left(\varphi(x_1)\varphi(x_2)\right) =: \varphi(x_1)\varphi(x_2) : + \langle 0| T\left(\varphi(x_1)\varphi(x_2)\right)|0\rangle =: \varphi(x_1)\varphi(x_2) : + D(x_1, x_2)$$

$: \varphi(x_1)\varphi(x_2) :=: \varphi(x_2)\varphi(x_1) :$

For general n: use induction (prove true for n fields $\Rightarrow$ true for $n+1$)

- write field product in terms of positive/negative-frequency parts and bring them to normal order
- for any ϕ_- passing from right to left of a ϕ_+, pay by adding term with pair of fields replaced by their contraction
- doing combinatorics right, the result follows

Formula valid for any set of ($\mathbb{R}$ or $\mathbb{C}$) scalar fields φ_a: nonzero contractions only for φ_a and $\varphi_a^\dagger$, $a = b$ ($[\varphi_a, \varphi_b] = [\varphi_a, \varphi_b^\dagger] = 0$ for $a \neq b$)
When fields in T product are already partially NO, contractions have to be considered only among fields belonging to different NO blocks

No ϕ_--ϕ_+ exchange needed within NO block

Feynman diagrams

Perturbative calculation of S-matrix elements:

1. expand S in powers of V_I
2. use Wick's theorem to write T-product as sums of NO products
3. normal-ordered products matrix elements = coefficients of string of annihilation/creation operators destroying particles in initial/final state summed over pairings with initial/final particles

Feynman diagrams: graphic device to accomplish the task

Example: ϕ^4 theory

$$\mathcal{L}_I(\phi) = \frac{\lambda}{4!}\phi^4 \Rightarrow S = \mathrm{Texp}\left\{\frac{i\lambda}{4!}\int d^4x \; :\phi_{\rm int}(x)^4:\right\}$$

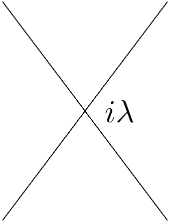
Free field $\phi_{\rm int}(x)$ describes free neutral scalar particles
Consider elastic scattering of two particles: assume λ small, to lowest order

$$\langle \vec{p}_1{}'\vec{p}_2{}'|S - \mathbf{1}|\vec{p}_1\vec{p}_2\rangle = \frac{i\lambda}{4!}\int d^4x \, \langle \vec{p}_1{}'\vec{p}_2{}'| \, :\phi_{\rm int}(x)^4: \, |\vec{p}_1\vec{p}_2\rangle + \mathcal{O}(\lambda^2)$$

Feynman diagrams (contd.)

Expanding $:\phi_{\rm int}(x)^4:$ in creation/annihilation operators

- nonzero contribution only when matching initial/final particles (2 creation, 2 annihilation)
- can pick any $a/a^\dagger$ from any of the fields $\Rightarrow$ 4! ways
- draw 4-pronged *vertex* for each $\mathcal{L}_I$, contract lines with external particles or among themselves



$$\langle \vec{p}_1{}'\vec{p}_2{}'|S-\mathbf{1}|\vec{p}_1\vec{p}_2\rangle \underset{\mathcal{O}(\lambda)}{=} i\lambda\int d^4x \langle \vec{p}_1{}'|\phi_{\rm int}(x)|0\rangle\langle \vec{p}_2{}'|\phi_{\rm int}(x)|0\rangle\langle 0|\phi_{\rm int}(x)|\vec{p}_1\rangle\langle 0|\phi_{\rm int}(x)|\vec{p}_2\rangle$$

Remaining matrix elements correspond to value of external lines

$$\langle 0|\phi_{\rm int}(x)|\vec{p}\,\rangle = \int d\Omega_q\, e^{-iq\cdot x}\langle 0|a(\vec{q}\,)|\vec{p}\,\rangle = \int d\Omega_q\, e^{-iq\cdot x}\langle 0|a(\vec{q}\,)a(\vec{p}\,)^\dagger|0\rangle$$

$$\boxed{\text{initial particle}} = \int d\Omega_q\, e^{-iq\cdot x}\langle 0|[a(\vec{q}\,),a(\vec{p}\,)^\dagger]|0\rangle = e^{-ip\cdot x}$$

$$\langle \vec{p}\,|\phi_{\rm int}(x)|0\rangle = \int d\Omega_q\, e^{iq\cdot x}\langle \vec{p}\,|a(\vec{q}\,)^\dagger|0\rangle = \int d\Omega_q\, e^{iq\cdot x}\langle 0|a(\vec{p}\,)a(\vec{q}\,)^\dagger|0\rangle$$

$$\boxed{\text{final particle}} = \int d\Omega_q\, e^{iq\cdot x}\langle 0|[a(\vec{p}\,),a(\vec{q}\,)^\dagger]|0\rangle = e^{ip\cdot x}$$

Feynman diagrams (contd.)

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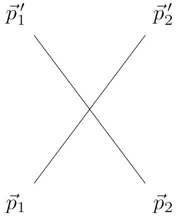

$$\langle \vec{p}_1^{\,\prime}\vec{p}_2^{\,\prime}|S-\mathbf{1}|\vec{p}_1\vec{p}_2\rangle \underset{\mathcal{O}(\lambda)}{=} i\lambda\int d^4x \langle \vec{p}_1^{\,\prime}|\phi_{\rm int}(x)|0\rangle\langle \vec{p}_2^{\,\prime}|\phi_{\rm int}(x)|0\rangle\langle 0|\phi_{\rm int}(x)|\vec{p}_1\rangle\langle 0|\phi_{\rm int}(x)|\vec{p}_2\rangle$$

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$$\langle \vec{p}\,|\phi_{\rm int}(x)|0\rangle = \int d\Omega_q\, e^{iq\cdot x}\langle \vec{p}\,|a(\vec{q}\,)^\dagger|0\rangle = \int d\Omega_q\, e^{iq\cdot x}\langle 0|a(\vec{p}\,)a(\vec{q}\,)^\dagger|0\rangle$$

$$\boxed{\text{final particle}} = \int d\Omega_q\, e^{iq\cdot x}\langle 0|[a(\vec{p}\,),a(\vec{q}\,)^\dagger]|0\rangle = e^{ip\cdot x}$$

Feynman diagrams (contd.)

S-matrix element to lowest order:

$$\begin{aligned}\langle \vec{p}_1'\vec{p}_2'|S-\mathbf{1}|\vec{p}_1\vec{p}_2\rangle &= i(2\pi)^4\delta^{(4)}(p_1'+p_2'-p_1-p_2)\mathcal{M}(\vec{p}_1,\vec{p}_2;\vec{p}_1',\vec{p}_2')\\ &= i\lambda\int d^4x\, e^{i(p_1'+p_2'-p_1-p_2)\cdot x}+\mathcal{O}(\lambda^2)\\ &= i\lambda(2\pi)^4\delta^{(4)}(p_1'+p_2'-p_1-p_2)+\mathcal{O}(\lambda^2)\Rightarrow \mathcal{M}(\vec{p}_1,\vec{p}_2;\vec{p}_1',\vec{p}_2')=\lambda+\mathcal{O}(\lambda^2)\end{aligned}$$

Example: ϕ^3 theory

$$\mathcal{L}_I(\phi)=\frac{\lambda}{3!}\phi^3$$

Neutral scalar particles, same $2\to 2$ elastic scattering process as above

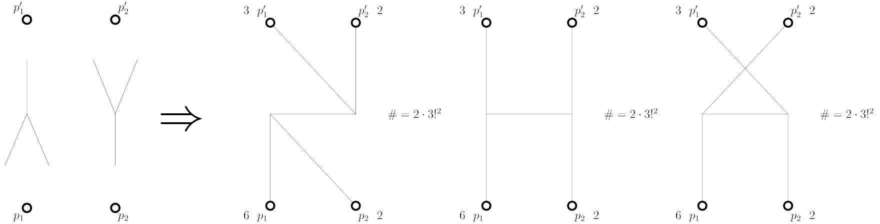
$\mathcal{O}(\lambda)$: term does not have enough fields, vanishing contribution

$\mathcal{O}(\lambda^2)$: $\langle \vec{p}_1'\vec{p}_2'|S-\mathbf{1}|\vec{p}_1\vec{p}_2\rangle$

$$=\frac{1}{2}\left(\frac{i\lambda}{3!}\right)^2\int d^4x\int d^4y\,\langle \vec{p}_1'\vec{p}_2'|\,T\left\{:\phi_{\rm int}(x)^3:\,:\phi_{\rm int}(y)^3:\right\}|\vec{p}_1\vec{p}_2\rangle+\mathcal{O}(\lambda^3)$$

Two vertices at points x and y

Feynman diagrams (contd.)



From Wick's theorem, only term with NO of 4 fields for i/f particles
$\Rightarrow$ one contraction (between fields from different vertices $\Rightarrow$ 3^2 ways)

$$\langle \vec{p}_1'\vec{p}_2'|S-\mathbf{1}|\vec{p}_1\vec{p}_2\rangle \underset{\mathcal{O}(\lambda^3)}{=} \frac{1}{2}\left(\frac{i\lambda}{3!}\right)^2 3^2 \int d^4x \int d^4y\, D(x-y)\langle \vec{p}_1'\vec{p}_2'|:\phi_{\rm int}(x)^2\phi_{\rm int}(y)^2:|\vec{p}_1\vec{p}_2\rangle$$

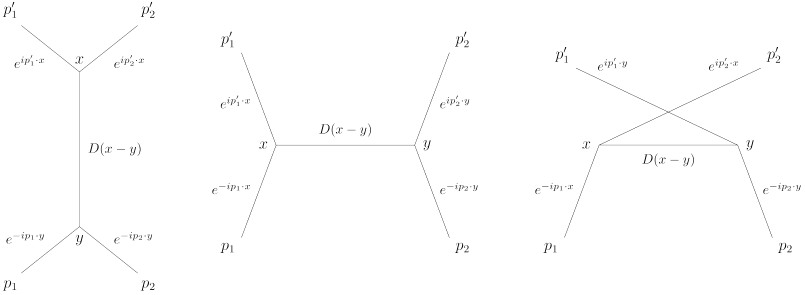
Expand normal-ordered product term, need two a and two $a^\dagger$

- both a from one vertex, both $a^\dagger$ from other vertex (2 ways)
- or one a and one $a^\dagger$ from each vertex (2^2 ways)

Vertices are equivalent since integrated over

- associate $a/a^\dagger$ to particles in all possible ways (2^2 ways or 2+2 ways)
  $\Rightarrow$ factors $e^{-ip_{1|2}\cdot x|y}$ (initial) and $e^{ip'_{1|2}\cdot x|y}$ (final)

Feynman diagrams (contd.)



$$\langle \vec{p}_1{}'\vec{p}_2{}'|S-\mathbf{1}|\vec{p}_1\vec{p}_2\rangle = \frac{1}{2}\left(\frac{i\lambda}{3!}\right)^2 (3!)^2 2\int d^4x\int d^4y\, D(x-y)$$
$$\times\left\{e^{i(p_1'+p_2')\cdot x}e^{-i(p_1+p_2)\cdot y} + e^{i(p_1'-p_1)\cdot x}e^{i(p_2'-p_2)\cdot y} + e^{i(p_2'-p_1)\cdot x}e^{i(p_1'-p_2)\cdot y}\right\} + \mathcal{O}(\lambda^3)$$
$$= (i\lambda)^2\int d^4x\int d^4y\, D(x-y)$$
$$\times\left\{e^{i(p_1'+p_2')\cdot x}e^{-i(p_1+p_2)\cdot y} + e^{i(p_1'-p_1)\cdot x}e^{i(p_2'-p_2)\cdot y} + e^{i(p_2'-p_1)\cdot x}e^{i(p_1'-p_2)\cdot y}\right\} + \mathcal{O}(\lambda^3)$$

Feynman diagrams (contd.)

Go over to momentum space: $D(x) = \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot x}\tilde{D}(q)$

$$\langle \vec{p}_1{}'\vec{p}_2{}'|S-\mathbf{1}|\vec{p}_1\vec{p}_2\rangle$$

$$\underset{\mathcal{O}(\lambda^2)}{=} (i\lambda)^2 \int d^4x \int d^4y \int \frac{d^4q}{(2\pi)^4}\, \tilde{D}(q)$$

$$\times \left\{ e^{i(p_1'+p_2'-q)\cdot x} e^{-i(p_1+p_2-q)\cdot y} + e^{i(p_1'-p_1-q)\cdot x} e^{i(p_2'-p_2+q)\cdot y} + e^{i(p_2'-p_1-q)\cdot x} e^{i(p_1'-p_2+q)\cdot y} \right\}$$

Integrate over x and y $\Rightarrow$ momentum conservation at each vertex

$$\langle \vec{p}_1{}'\vec{p}_2{}'|S-\mathbf{1}|\vec{p}_1\vec{p}_2\rangle$$

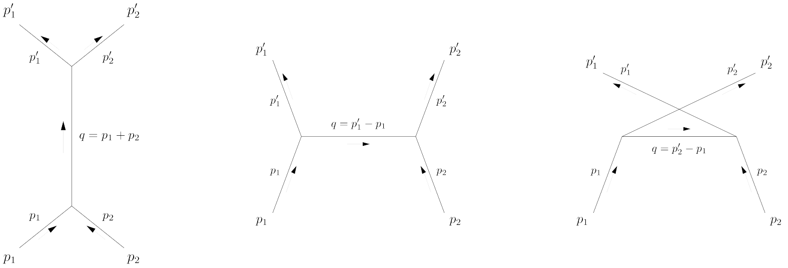
$$= (i\lambda)^2 \int \frac{d^4q}{(2\pi)^4}\, \tilde{D}(q) \big\{ (2\pi)^4\delta^{(4)}(p_1'+p_2'-q)(2\pi)^4\delta^{(4)}(p_1+p_2-q)$$

$$+ (2\pi)^4\delta^{(4)}(p_1'-p_1-q)(2\pi)^4\delta^{(4)}(p_2'-p_2+q)$$

$$+ (2\pi)^4\delta^{(4)}(p_2'-p_1-q)(2\pi)^4\delta^{(4)}(p_1'-p_2+q)\big\} + \mathcal{O}(\lambda^3)$$

$$= i(2\pi)^4\delta^{(4)}(p_1'+p_2'-p_1-p_2)\left\{ i\lambda^2 \left[\tilde{D}(p_1+p_2) + \tilde{D}(p_1'-p_1) + \tilde{D}(p_2'-p_1) \right] \right\} + \mathcal{O}(\lambda^3)$$

Feynman diagrams (contd.)



Feynman rules:

- one point for each vertex, as many lines out of it as fields in $\mathcal{L}_I$
- pair particles with lines (*external*), remaining lines with each other (*internal*)
- draw all the possible *topologically inequivalent* diagrams with prescribed external particles and number of vertices; count multiplicity of diagram
- associate a momentum to each line of the graph

  Flowing in/out for i/f particles/external lines; for internal lines arbitrary
- for each external line factor 1; for each internal line factor $\tilde{D}(q)$
- conserve momentum at each vertex $(2\pi)^4\delta^{(4)}(\sum_j p_j)$

  Here p_j flowing *into* the vertex
- integrate over internal momenta