# Particle physics

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# Quantisation of the scalar field (contd.)

In momentum space

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\tilde{\phi}(\vec{p}) \qquad \pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\tilde{\pi}(\vec{p})$$

$$\phi(\vec{x})^* = \phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\tilde{\phi}(-\vec{p})^* \qquad \pi(\vec{x})^* = \pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\tilde{\pi}(-\vec{p})^*$$

Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}\int d^3x \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3} e^{i(\vec{p}+\vec{q})\cdot\vec{x}}[\tilde{\pi}(\vec{p})\tilde{\pi}(\vec{q}) + (-\vec{p}\cdot\vec{q} + m^2)\tilde{\phi}(\vec{p})\tilde{\phi}(\vec{q})] \\ &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}[\tilde{\pi}(\vec{p})\tilde{\pi}(-\vec{p}) + (\vec{p}^{\,2} + m^2)\tilde{\phi}(\vec{p})\tilde{\phi}(-\vec{p})] \\ &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}[|\tilde{\pi}(\vec{p})|^2 + (\vec{p}^{\,2} + m^2)|\tilde{\phi}(\vec{p})|^2] \end{aligned}$$

Collection of harmonic oscillators of mass $M = 1$, frequency $\omega_p^2 = \vec{p}^{\,2} + m^2$
Using explicit solution and $(p^0)^2 = \vec{p}^{\,2} + m^2$:

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3 2p^0}\, e^{i\vec{p}\cdot\vec{x}}(a(\vec{p}) + a(-\vec{p})^*) \qquad \Rightarrow \tilde{\phi}(\vec{p}) = \frac{1}{2p^0}(a(\vec{p}) + a(-\vec{p})^*)$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3 2p^0}\, e^{i\vec{p}\cdot\vec{x}}\frac{p^0}{i}(a(\vec{p}) - a(-\vec{p})^*) \qquad \Rightarrow \tilde{\pi}(\vec{p}) = \frac{1}{2i}(a(\vec{p}) - a(-\vec{p})^*)$$

# Quantisation of the scalar field (contd.)

$$\begin{aligned} H &= \tfrac{1}{2}\int \tfrac{d^3p}{(2\pi)^3}\left[|\tilde{\pi}(\vec{p})|^2 + (p^0)^2|\tilde{\phi}(\vec{p})|^2\right] \\ &= \tfrac{1}{2}\int \tfrac{d^3p}{(2\pi)^3}\tfrac{1}{2}\left[a(\vec{p})a(\vec{p})^* + a(-\vec{p})^* a(-\vec{p})\right] \\ &= \int d^3p\, p^0 \left(\frac{a(\vec{p})}{\sqrt{2p^0(2\pi)^3}}\right)^* \left(\frac{a(\vec{p})}{\sqrt{2p^0(2\pi)^3}}\right) = \int d^3p\, p^0 A(\vec{p})^* A(\vec{p}) \end{aligned}$$

Set of harmonic oscillators, frequency $p^0$, normal mode amplitude $A(\vec{p})$

Quantisation of harmonic oscillator:

- promote $A(\vec{p}), A(\vec{p})^*$ to operators $A(\vec{p}), A(\vec{p})^\dagger$
- impose commutation relations

$$[A(\vec{p}), A(\vec{q})^\dagger] = \delta^{(3)}(\vec{p}-\vec{q}) \qquad [A(\vec{p}), A(\vec{q})] = [A(\vec{p})^\dagger, A(\vec{q})^\dagger] = 0$$

$$\Rightarrow [a(\vec{p}), a(\vec{q})^\dagger] = 2p^0(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \qquad [a(\vec{p}), a(\vec{q})] = [a(\vec{p})^\dagger, a(\vec{q})^\dagger] = 0$$

Can be interpreted as annihilation/creation operators!

Quantised Hermitean scalar field: $\phi(x) = \int d\Omega_p \left(e^{-ip\cdot x}a(\vec{p}) + e^{ip\cdot x}a(\vec{p})^\dagger\right)$

# Properties of the scalar field

1. Field/particle relation

Quantised scalar field $\Rightarrow$ creation/annihilation operators $\Rightarrow$ Fock space (requires vacuum state $|0\rangle$) $\Rightarrow$ particles of mass $m$ (same as in KG eq.)

Particles of mass $m \Rightarrow$ Fock space $\Rightarrow$ creation/annihilation operators $\Rightarrow$ quantised scalar field obeying $(\Box + m^2)\phi(x) = 0$

2. Transformation properties

Unitary rep. of Lorentz transformations/translations on Fock space:

$$U(\Lambda)|\vec{p}_1, \ldots, \vec{p}_n\rangle = |\Lambda\vec{p}_1, \ldots, \Lambda\vec{p}_n\rangle \qquad U(\Lambda)|0\rangle = |0\rangle$$

$$U(a)|\vec{p}_1, \ldots, \vec{p}_n\rangle = e^{-i(\sum_j p_j)\cdot a}|\vec{p}_1, \ldots, \vec{p}_n\rangle \qquad U(a)|0\rangle = |0\rangle$$

Implies transformation properties of $a(\vec{p})$, $a(\vec{p})^\dagger$

$$U(\Lambda)^\dagger a(\vec{p})^\dagger U(\Lambda) = a(\Lambda^{-1}\vec{p})^\dagger \qquad U(a)^\dagger a(\vec{p})^\dagger U(a) = e^{ip\cdot a}a(\vec{p})^\dagger$$

Simple field transformation properties

$$U(\Lambda)^\dagger \phi(x) U(\Lambda) = \phi(\Lambda^{-1}x) \qquad U(a)^\dagger \phi(x) U(a) = \phi(x + a)$$

Easy to build Lorentz/translation invariant interactions

# Properties of the scalar field

3. Locality

Using commutation relations of creation/annihilation operators

$$[\phi(x), \phi(y)] = \int d\Omega_p \int d\Omega_q \left[e^{-i(p\cdot x - q\cdot y)} - e^{i(p\cdot x - q\cdot y)}\right] (2\pi)^3 2q^0 \delta^{(3)}(\vec{p} - \vec{q}\,)$$
$$= \int d\Omega_p \left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right] = \Delta(x-y) - \Delta(y-x)$$

$\Delta(x)$ Lorentz-invariant, symmetric for spacelike $x$

$$\Delta(x) = \int d\Omega_p \, e^{i\vec{p}\cdot\vec{x}\,'} = \int d\Omega_p \, e^{-i\vec{p}\cdot\vec{x}\,'} = \Delta(-x) \quad \text{if} \ \ x^2 < 0$$

Spacelike $x$ can always be Lorentz-transformed to $(0, \vec{x}\,')$

$$\Rightarrow [\phi(x), \phi(y)] = 0 \text{ if } (x-y)^2 < 0 \ (\textit{microcausality})$$

Field derivatives: $[\partial_\mu \phi(x), \phi(y)] = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon}[\phi(x + \epsilon\hat{\mu}) - \phi(x), \phi(y)]$

$$\Rightarrow [\partial_\mu \phi(x), \phi(y)] = 0 \text{ if } (x-y)^2 < 0$$

$(x-y)^2 < 0 \Rightarrow (x + \epsilon\hat{\mu} - y)^2 = (x-y)^2 + 2\epsilon(x-y)_\mu + \epsilon^2\hat{\mu}^2 < 0$ for sufficiently small $\epsilon$

Interactions/observables built out of $\phi(x)$ will commute at spacelike separations, as required by locality

# Antiparticles

Suppose particles have additive conserved real charge $q$

$$Q|\vec{p}\rangle = q|\vec{p}\rangle \qquad Q|\vec{p}_1,\ldots,\vec{p}_n\rangle = nq|\vec{p}_1,\ldots,\vec{p}_n\rangle \qquad [Q,H]=0$$

Commutation relations:

$$[Q,a(\vec{p})] = -qa(\vec{p}) \qquad [Q,a(\vec{p})^\dagger] = qa(\vec{p})^\dagger$$

$$[Q,\phi(x)] = -q\int d\Omega_p \, \left(a(\vec{p}\,)e^{-ip\cdot x} - a(\vec{p}\,)^\dagger e^{ip\cdot x}\right) = -q\bar{\phi}(x)$$

$[Q,H]=0 \Rightarrow H = H(\phi,\bar{\phi})$ must depend also on $\bar{\phi}(x)$

But $[\phi(x),\bar{\phi}(y)] = \Delta(x-y)+\Delta(y-x) \neq 0$ when $(x-y)^2<0$

$$\Rightarrow \text{locality/causality lost unless } q=0 \text{ (neutral particles)}$$

How to describe charged particles?

Dropping reality/Hermiticity requirement, general solution of KG

$$\phi(x) = \int d\Omega_p \, \left(e^{-ip\cdot x}a(\vec{p}) + e^{ip\cdot x}b(\vec{p})^*\right) \neq \phi(x)^* \qquad \text{(classical)}$$

$$\phi(x) = \int d\Omega_p \, \left(e^{-ip\cdot x}a(\vec{p}) + e^{ip\cdot x}b(\vec{p})^\dagger\right) \neq \phi(x)^\dagger \qquad \text{(quantum)}$$

# Antiparticles (contd.)

Repeat analysis $\Rightarrow$ *two* sets of harmonic oscillators, quantise as usual:

$$[a(\vec{p}), a(\vec{q})^\dagger] = 2p^0(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \qquad [b(\vec{p}), b(\vec{q})^\dagger] = 2p^0(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$$

$$[a(\vec{p}), a(\vec{q})] = [a(\vec{p})^\dagger, a(\vec{q})^\dagger] = [b(\vec{p}), b(\vec{q})] = [b(\vec{p})^\dagger, b(\vec{q})^\dagger]$$
$$= [a(\vec{p}), b(\vec{q})^\dagger] = [a(\vec{p}), b(\vec{q})] = [a(\vec{p})^\dagger, b(\vec{q})^\dagger] = 0$$

*Two* types of particles ($a$ and $b$) with the same mass $m$ ($p^0 = \sqrt{\vec{p}^{\,2} + m^2}$)
Field obeys microcausality requirement

$$[\phi(x), \phi(y)] = 0 \qquad [\phi(x), \phi(y)^\dagger] = 0 \quad \text{if } (x-y)^2 < 0$$

Commutators with charge $Q$: assign charges $q_{a,b}$ to $a, b$

$$[Q, a(\vec{p})] = -q_a a(\vec{p}) \qquad [Q, b(\vec{p})] = -q_b b(\vec{p})$$

$$[Q, \phi(x)] = -\int d\Omega_p \left(q_a a(\vec{p}\,)e^{-ip\cdot x} - q_b b(\vec{p}\,)^\dagger e^{ip\cdot x}\right) = -q_a\phi(x) \text{ if } q_b = -q_a$$

There are no problems with microcausality only if $b$-particles have charges opposite to those of $a$-particles

$\Rightarrow$ antiparticles: same mass, opposite charges

# Antiparticles (contd.)

Under parity: momenta $\vec{p} \to -\vec{p}$, coordinates $x = (x^0, \vec{x}) \to x_P = (x^0, -\vec{x})$

State transformation properties:

$$P|\vec{p}; a\rangle = \eta | -\vec{p}; a\rangle \qquad P|\vec{p}; b\rangle = \eta' | -\vec{p}; b\rangle$$

$\eta$, $\eta'$: intrinsic parities

For creation operators

$$P^\dagger a(\vec{p})^\dagger P = \eta a(-\vec{p})^\dagger \qquad P^\dagger b(\vec{p})^\dagger P = \eta' b(-\vec{p})^\dagger$$

Field transformation:

$$\begin{aligned} P^\dagger \phi(x) P &= \int d\Omega_p \left( \eta^* a(-\vec{p}\,) e^{-ip\cdot x} + \eta' b(-\vec{p}\,)^\dagger e^{ip\cdot x} \right) \\ &= \eta^* \int d\Omega_p \left( a(\vec{p}\,) e^{-ip\cdot x_P} + \eta\eta' b(\vec{p}\,)^\dagger e^{ip\cdot x_P} \right) \end{aligned}$$

Parity invariance + microcausality $\Rightarrow P^\dagger \phi(x) P \equiv \eta^* \phi(x_P)$

$$\boxed{\Rightarrow \eta\eta' = 1}$$

# Hamiltonian formalism

Quantisation procedure was heuristic, proceed more formally: operators are tricky (e.g., ordering problems)

Assumptions: field solves EOM and obeys the commutation relations inherited from those of creation/annihilation operators

In general: comm. rel. of creation/annihilation operators $\Leftrightarrow$ comm. rel. of fields

Consider non-Hermitean case for generality:

$$[\phi(x), \phi(y)] = 0 \qquad [\phi(x), \phi(y)^\dagger] = \Delta(x-y) - \Delta(y-x)$$

Equal times commutation relations of fields and derivatives suffice

Equivalent to commutation relations of creation/annihilation operators

$$[A(x), B(y)]_{\mathrm{ET}} \equiv [A(x^0, \vec{x}), B(y^0, \vec{y})]_{x^0=y^0}$$

Trivially (only commuting creation/annihilation operators appear)

$$[\phi(x), \phi(y)]_{\mathrm{ET}} = 0 \qquad [\phi(x), \partial_0\phi(y)]_{\mathrm{ET}} = 0 \qquad [\partial_0\phi(x), \partial_0\phi(y)]_{\mathrm{ET}} = 0$$

$$[\phi(x)^\dagger, \phi(y)^\dagger]_{\mathrm{ET}} = 0 \qquad [\phi(x)^\dagger, \partial_0\phi(y)^\dagger]_{\mathrm{ET}} = 0 \qquad [\partial_0\phi(x)^\dagger, \partial_0\phi(y)^\dagger]_{\mathrm{ET}} = 0$$

# Hamiltonian formalism (contd.)

Slightly less trivially: for all $\vec{x}$, $\vec{y}$

$$[\phi(x), \phi(y)^\dagger]_{\mathrm{ET}} = \int d\Omega_p \left[ e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right] = 0$$

Taking the derivative wrt $y^0$ and then setting $x^0 = y^0$

$$[\phi(x), \partial_0\phi(y)^\dagger]_{\mathrm{ET}} = \int d\Omega_p \, ip^0 \left[ e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right]$$

$$= i \int d\Omega_p 2p^0 e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = i \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = i\delta^{(3)}(\vec{x} - \vec{y}\,)$$

$$[\phi(x)^\dagger, \partial_0\phi(y)]_{\mathrm{ET}} = i\delta^{(3)}(\vec{x} - \vec{y}\,)$$

No contradiction: nonzero commutator for $\vec{x} = \vec{y}$ at equal times, lightlike separation

Taking the derivative wrt both $x^0$ and $y^0$ and then setting $x^0 = y^0$

$$[\partial_0\phi(x), \partial_0\phi(y)^\dagger]_{\mathrm{ET}} = \int d\Omega_p \, (p^0)^2 \left[ e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right]$$

$$= \int d\Omega_p \, (\vec{p}^{\,2} + m^2) \left[ e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right] = 0$$

All other commutators obtained from the above via spatial derivatives, vanish for spacelike separations

Two or more temporal derivatives reduce to none or one plus spatial derivatives using EOM

# Hamiltonian formalism (contd.)

Equal time commutation relations = canonical commutation relations for canonical coordinates $q$ and canonically conjugated momenta $p$

$$q = \phi(0,\vec{x}),\ \phi(0,\vec{x})^\dagger \qquad p = \pi(0,\vec{x}) = \partial_0\phi(0,\vec{x})^\dagger,\ \pi(0,\vec{x})^\dagger = \partial_0\phi(0,\vec{x})$$

Nonzero commutators:

$$[\phi(0,\vec{x}),\pi(0,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \qquad [\phi(0,\vec{x})^\dagger,\pi(0,\vec{y})^\dagger] = i\delta^{(3)}(\vec{x}-\vec{y})$$

Equations of motions (EOM)

$$\partial_0\phi(t,\vec{x}) = \pi(t,\vec{x})^\dagger \qquad \partial_0\pi(t,\vec{x}) = (\vec{\nabla}^2 - m^2)\phi(t,\vec{x})^\dagger$$
$$\partial_0\phi(t,\vec{x})^\dagger = \pi(t,\vec{x}) \qquad \partial_0\pi(t,\vec{x})^\dagger = (\vec{\nabla}^2 - m^2)\phi(t,\vec{x})$$

Can be derived from a Hamiltonian via Hamilton equations

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

Hamiltonian (time-independent)

$$H = \int d^3x \left\{\pi(x)\pi(x)^\dagger + [\vec{\nabla}\phi(x)]^\dagger \cdot [\vec{\nabla}\phi(x)] + m^2\phi(x)^\dagger\phi(x)\right\} + \mathrm{const.}$$

# Hamiltonian formalism (contd.)

Temporal evolution = Heisenberg equations:

$$[H,\phi(x)] = \int d^3y\,[\pi(y)\pi(y)^\dagger,\phi(x)]|_{y^0=x^0} = \int d^3y\,[\pi(y),\phi(x)]|_{y^0=x^0}\pi(y)^\dagger$$
$$= -i\int d^3y\,\pi(y)^\dagger\delta^{(3)}(\vec{x}-\vec{y}) = -i\pi(x)^\dagger = -i\partial_0\phi(x)$$

All good, but is this a system of bosons? Plug in explicit form of fields:

$$\int d^3x\{\pi(x)\pi(x)^\dagger + [\vec{\nabla}\phi(x)]^\dagger\cdot[\vec{\nabla}\phi(x)] + m^2\phi(x)^\dagger\phi(x)\}$$
$$= \int d^3x\int d\Omega_p\int d\Omega_q\Big\{(-ip_0)(iq_0)\big[a(\vec{p}\,)e^{-ip\cdot x} - b(\vec{p}\,)^\dagger e^{ip\cdot x}\big]\big[a(\vec{q}\,)^\dagger e^{iq\cdot x} - b(\vec{p}\,)e^{-iq\cdot x}\big]$$
$$+ (i\vec{p})\cdot(-i\vec{q})\big[a(\vec{p}\,)^\dagger e^{ip\cdot x} - b(\vec{p}\,)e^{-ip\cdot x}\big]\big[a(\vec{q}\,)e^{-iq\cdot x} - b(\vec{p}\,)^\dagger e^{iq\cdot x}\big]$$
$$+ m^2\big[a(\vec{p}\,)^\dagger e^{ip\cdot x} + b(\vec{p}\,)e^{-ip\cdot x}\big]\big[a(\vec{q}\,)e^{-iq\cdot x} + b(\vec{p}\,)^\dagger e^{iq\cdot x}\big]\Big\}$$
$$= \int d\Omega_p\,\tfrac{(p^0)^2}{2p^0}\Big\{\{a(\vec{p}\,)^\dagger, a(\vec{p}\,)\} + \{b(\vec{p}\,)^\dagger, b(\vec{p}\,)\} + \underbrace{([a(\vec{p}\,)^\dagger, b(\vec{p}\,)^\dagger]e^{i2p^0x^0} + \text{h.c.})}_{=0}\Big\}$$

# Hamiltonian formalism (contd.)

Almost good, but not fully satisfactory:

$$\{a(\vec{p}\,)^\dagger, a(\vec{p}\,)\} = 2a(\vec{p}\,)^\dagger a(\vec{p}\,) + [a(\vec{p}\,), a(\vec{p}\,)^\dagger] = 2a(\vec{p}\,)^\dagger a(\vec{p}\,) + (2\pi)^3 2p^0 \delta^{(3)}(0)$$

$\delta^{(3)}(0)$ ("$=$"$\infty$) is a constant and can be subtracted from $H$, but still...

$$H \text{ "="} \int d\Omega_p \, p^0 \big\{ a(\vec{p}\,)^\dagger a(\vec{p}\,) + b(\vec{p}\,)^\dagger b(\vec{p}\,) \big\}$$

Origin of the problem: fields are highly singular operators, product of fields at same $x$ is not well defined and requires regularisation

To fix the problem define *normal ordered product* of free fields: expanding in creation/annihilation operators and reorder strings placing all creation operators to the left of annihilation operators

Example:

$$\begin{aligned} :\phi(x)\phi(y)^\dagger: \; &= \int d\Omega_p \int d\Omega_q : \big[ a(\vec{p}\,) e^{-ip\cdot x} + b(\vec{p}\,)^\dagger e^{ip\cdot x} \big] \big[ a(\vec{q}\,)^\dagger e^{iq\cdot y} + b(\vec{p}\,) e^{-iq\cdot y} \big] : \\ &\equiv \int d\Omega_p \int d\Omega_q \big[ a(\vec{q}\,)^\dagger a(\vec{p}\,) e^{-ip\cdot x} e^{iq\cdot y} + a(\vec{p}\,) b(\vec{p}\,) e^{-ip\cdot x} e^{-iq\cdot y} \\ &\qquad + b(\vec{p}\,)^\dagger a(\vec{q}\,)^\dagger e^{ip\cdot x} e^{iq\cdot y} + b(\vec{p}\,)^\dagger b(\vec{p}\,) e^{ip\cdot x} e^{-iq\cdot y} \big] \end{aligned}$$

# Hamiltonian formalism (contd.)

In terms of *positive-frequency* and *negative-frequency* components

$$\phi_+(x) \equiv \int d\Omega_p\, a(\vec{p}\,)e^{-ip\cdot x} \qquad \phi_-(x) \equiv \int d\Omega_p\, b(\vec{p}\,)^\dagger e^{ip\cdot x}$$

$$\phi_+^*(x) \equiv \int d\Omega_p\, b(\vec{p}\,)e^{-ip\cdot x} \qquad \phi_-^*(x) \equiv \int d\Omega_p\, a(\vec{p}\,)^\dagger e^{ip\cdot x}$$

$$\phi(x) = \phi_+(x) + \phi_-(x) \qquad \phi(x)^\dagger = \phi_+^*(x) + \phi_-^*(x)$$

Normal-ordered (NO) product: positive-frequency components on the right, negative-frequency components on the left

$$:\phi(x)\phi(y)^\dagger: = \phi_+(x)\phi_+^*(x) + \phi_-^*(x)\phi_+(x) + \phi_-(x)\phi_+^*(x) + \phi_-(x)\phi_-^*(x)$$

$$\underbrace{\phi(x)\phi(y)^\dagger}_{\substack{\text{singular for}\\ x\to y}} = :\phi(x)\phi(y)^\dagger: + [\phi_+(x), \phi_-^*(x)] = :\phi(x)\phi(y)^\dagger: + \underbrace{\langle 0|\phi(x)\phi(y)^\dagger|0\rangle}_{\substack{\text{singular for}\\ x\to y}}$$

By construction: NO product symmetric, vacuum expectation value 0

$$:\phi(x)\phi(y)^\dagger: = :\phi(y)^\dagger\phi(x): \qquad \langle 0|:\phi(x)\phi(y)^\dagger:|0\rangle = 0$$

# Hamiltonian formalism (contd.)

Now we can fix the problem: define

$$H = \int d^3x \; : \left\{\pi(x)^\dagger \pi(x) + [\vec{\nabla}\phi(x)]^\dagger \cdot [\vec{\nabla}\phi(x)] + m^2 \phi(x)^\dagger \phi(x)\right\} :$$

Order of the product is irrelevant

Repeating the calculation

$$\begin{aligned} H &= \tfrac{1}{2} \int d\Omega_p \, p^0 : \left[ \{a(\vec{p}\,)^\dagger, a(\vec{p}\,)\} + \{b(\vec{p}\,)^\dagger, b(\vec{p}\,)\} \right] : \\ &= \int d\Omega_p \, p^0 \left[ a(\vec{p}\,)^\dagger a(\vec{p}\,) + b(\vec{p}\,)^\dagger b(\vec{p}\,) \right] \end{aligned}$$

- Time-independent
- Commutator with fields provides temporal evolution
- No divergences
- Describes a system of bosons

# Lagrangian formalism

Hamiltonian formalism

- not manifestly Lorentz-invariant
- how do we find pairs of conjugate variables?

Solution to both issues: Lagrangian formalism

Given Hamiltonian system, $H(q,p)$, solve for $p = p(q,\dot{q})$ using $\dot{q} = \frac{\partial H}{\partial p}$

$\to$ Lagrangian system $\qquad L(q,\dot{q}) = \dot{q}p - H(q,p)$

Given Lagrangian system, $L(q,\dot{q})$, define $p = \frac{\partial L}{\partial \dot{q}}$, solve $\dot{q} = \dot{q}(q,p)$

$\to$ Hamiltonian system $\qquad H(q,p) = \dot{q}p - L(q,\dot{q})$

Lagrangian formalism is manifestly invariant: using $\dot{\phi} = \pi^\dagger$

$$L = \int d^3x : \left\{2\dot{\phi}(x)^\dagger\dot{\phi}(x) - \dot{\phi}(x)^\dagger\dot{\phi}(x) - [\vec{\nabla}\phi(x)]^\dagger \cdot [\vec{\nabla}\phi(x)] - m^2\phi(x)^\dagger\phi(x)\right\} :$$
$$= \int d^3x : \left\{\partial_\mu\phi(x)^\dagger\partial^\mu\phi(x) - m^2\phi(x)^\dagger\phi(x)\right\} : = \int d^3x\, \mathcal{L}(\phi,\partial\phi)$$
$$S = \int dt\, L = \int d^4x\, \mathcal{L}(\phi,\partial\phi)$$

$\mathcal{L}$ is a density $\Rightarrow$ $S$ is Lorentz-invariant

# Lagrangian formalism (contd.)

*Canonical quantisation* procedure:

- take the *Lagrangian density*, e.g.,

$$\mathcal{L}(x) = :\partial_\mu \phi(x)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x):$$

- construct the *action* integrating over spacetime,

$$S = \int d^4x\, \mathcal{L}(x)$$

- derive the equations of motion via the variational principle $\delta S = 0$
- impose *canonical commutation relations* among fields and conjugate momenta $\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(x))}$

$$[\phi(x), \pi(y)]_{\mathrm{ET}} = i\delta^{(3)}(\vec{x}-\vec{y}) \qquad [\phi(x), \phi(y)]_{\mathrm{ET}} = [\pi(x), \pi(y)]_{\mathrm{ET}} = 0$$

# Lagrangian formalism (contd.)

Lagrangian density $\mathcal{L}(\phi_i, \partial\phi_i)$ (translation-invariant case)

More generally $\mathcal{L}(\phi_i, \partial\phi_i, x)$, but explicit $x$-dependence plays no role in EOM

Equations of motion obtained from a variational principle

Given spacetime region $D$, define action functional

$$S_D[\phi] = \int_D d^4x\, \mathcal{L}(\phi_i(x), \partial_\mu\phi_i(x))$$

Variational principle: $\forall D$ extremise $S_D$, variation $\delta S_D = 0$ of the action under variations $\delta\phi_i$ of the fields subject to $\delta\phi_i = 0$ on boundary $\partial D$

$$\begin{aligned}
\delta S_D[\phi] &= \int_D d^4x \left[\mathcal{L}(\phi_i(x) + \delta\phi_i(x), \partial_\mu\phi_i(x) + \partial_\mu\delta\phi_i(x)) - \mathcal{L}(\phi_i(x), \partial_\mu\phi_i(x))\right] \\
&= \int_D d^4x \textstyle\sum_i \left[\delta\phi_i(x) \frac{\partial\mathcal{L}}{\partial\phi_i}(\phi_i(x), \partial_\mu\phi_i(x)) + \partial_\mu\delta\phi_i(x) \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}(\phi_i(x), \partial_\mu\phi_i(x))\right] \\
&= \int_D d^4x \textstyle\sum_i \delta\phi_i(x) \left[\frac{\partial\mathcal{L}}{\partial\phi_i}(\phi_i(x), \partial_\mu\phi_i(x)) - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}(\phi_i(x), \partial_\mu\phi_i(x))\right] \\
&\quad + \textstyle\sum_i \int_{\partial D} d\Sigma_\mu\, \delta\phi_i(x) \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}(\phi_i(x), \partial_\mu\phi_i(x))
\end{aligned}$$

# Lagrangian formalism (contd.)

$$\delta S_D[\phi] = \int_D d^4x \sum_i \delta\phi_i(x) \left[ \tfrac{\partial \mathcal{L}}{\partial \phi_i}(\phi_i(x), \partial_\mu \phi_i(x)) - \partial_\mu \tfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}(\phi_i(x), \partial_\mu \phi_i(x)) \right]$$
$$+ \sum_i \int_{\partial D} d\Sigma_\mu \, \delta\phi_i(x) \tfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}(\phi_i(x), \partial_\mu \phi_i(x)) = 0$$

Boundary term vanishes, $D$ and variation of fields inside $D$ arbitrary
$\Rightarrow$ *Euler–Lagrange equations*:

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0$$

Conjugate momenta:

$$\pi_i \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)}$$

Quantisation: solve EOM and impose *equal-time commutation relations*

$$[\phi_i(x), \phi_j(y)]_{\mathrm{ET}} = [\phi_i(t, \vec{x}), \phi_j(t, \vec{y})] = 0$$
$$[\pi_i(x), \pi_j(y)]_{\mathrm{ET}} = [\pi_i(t, \vec{x}), \pi_j(t, \vec{y})] = 0$$
$$[\phi_i(x), \pi_j(y)]_{\mathrm{ET}} = [\phi_i(t, \vec{x}), \pi_j(t, \vec{y})] = i\delta_{ij}\delta^{(3)}(\vec{x} - \vec{y})$$

# Lagrangian formalism (contd.)

Construct Hamiltonian via Legendre transform

$$H = \int d^3x\, \mathcal{H}(\phi_i(x), \pi_i(x)) = \int d^3x \left\{ \left[ \textstyle\sum_i \partial_0\phi_i(x)\pi_i(x) \right] - \mathcal{L}(\phi_i(x), \partial_\mu\phi_i(x)) \right\}$$

$\mathcal{H}$: *Hamiltonian density*, $\partial_0\phi_i(x)$ expressed as a function of $\phi_i$ and $\pi_i$

$H$ gives temporal evolution of fields and momenta (Heisenberg eqs.)

$$\dot\phi(t,\vec{x}) = i[H, \phi(t,\vec{x})] \qquad \dot\pi(t,\vec{x}) = i[H, \pi(t,\vec{x})]$$
$$\Longrightarrow \phi(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt} \qquad \pi(t,\vec{x}) = e^{iHt}\pi(0,\vec{x})e^{-iHt}$$

This completes the canonical quantisation program

Should also provide representation of CCR algebra on some Hilbert space

Regularisation procedures (e.g., NO) needed for short-distance singularities

Why the fancy formalism?

- Interested in building an interacting theory, and the canonical quantisation program automatically provides a causal theory
- Lagrangian formalism makes manifest the symmetries of the theory and guarantees covariance of EOM
- Guarantees existence of conserved quantities associated to the symmetries of the theory (Noether's theorem)

# Noether's theorem

Consider generic Lagrangian and action (including also $x$-dependence)

$$S = \int_D d^4x \, \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x), x)$$

Consider mapping of fields and coordinates

$$\phi_i'(x') = \mathcal{M}_i(\phi_j(x), x) \qquad x' = X(x)$$

Transformation $(\mathcal{M}, X)$ maps from observer $\mathcal{O}$ to observer $\mathcal{O}'$

Define

$$S' = \int_{D'} d^4x' \, \mathcal{L}(\phi_i'(x'), \partial_\mu' \phi_i'(x'), x')$$

*Same* Lagrangian density but transformed fields and domain $D' = X(D)$

In general $S \neq S'$: same system in different reference frames generally requires different Lagrangian densities/different EOM

# Noether's theorem (contd.)

**Theorem 1:** If $S'$ and $S$ differ only by boundary terms

$$S' = S + \int_D d^4x \, \partial_\mu F^\mu(\phi_i(x), x)$$

then $\phi_i$ and $\phi'_i$ obey the same EOM equations of motion

**Proof:** let $\phi_i$ be solution of EOM, $\delta\phi_i(x)$ arbitrary infinitesimal variation vanishing on $\partial D$; $\phi'_i(x') = \mathcal{M}_i(\phi_j(x), x)$ correspondingly changes by $\delta\phi'_i(x') = \mathcal{M}_i(\phi_j(x) + \delta\phi_j(x), x) - \mathcal{M}_i(\phi_j(x), x)$ vanishing on $\partial D'$

$$\begin{aligned}
\delta S' &= \int_{D'} d^4x' \sum_k \delta\phi'_k(x') \left[ \tfrac{\partial \mathcal{L}}{\partial \phi'_k}(\phi'_i(x'), \partial'_\mu \phi'_i(x')) - \partial'_\mu \tfrac{\partial \mathcal{L}}{\partial(\partial'_\mu \phi'_k)}(\phi'_i(x'), \partial'_\mu \phi'_i(x')) \right] \\
&\quad + \int_{\partial D'} d\Sigma'_\mu \sum_k \delta\phi'_k(x') \tfrac{\partial \mathcal{L}}{\partial(\partial'_\mu \phi'_k)}(\phi'_i(x'), \partial'_\mu \phi'_i(x')) \\
&= \int_{D'} d^4x' \sum_k \delta\phi'_k(x') \left[ \tfrac{\partial \mathcal{L}}{\partial \phi'_k}(\phi'_i(x'), \partial'_\mu \phi_i(x')) - \partial'_\mu \tfrac{\partial \mathcal{L}}{\partial(\partial'_\mu \phi'_k)}(\phi'_i(x'), \partial'_\mu \phi'_i(x')) \right] \\
&= \delta S + \int_{\partial D} d\Sigma_\mu \sum_k \delta_k \phi(x) \tfrac{\partial F^\mu}{\partial \phi_k}(\phi_i(x), x) = \delta S = 0
\end{aligned}$$

$\delta\phi$ arbitrary $\Rightarrow$ $\delta\phi'$ arbitrary $\Rightarrow$ $\phi'_i$ obey same EOM as $\phi_i$

# Noether's theorem (contd.)

**Theorem 2:** If $S' - S =$ boundary terms for a continuous family of transformations, there is an associated conserved current, $\partial_\mu J^\mu = 0$

**Proof:** take an infinitesimal such transformation

$$\phi'_i(x') = \phi_i(x) + \delta\phi_i(x) = \phi_i(x) + \epsilon M_i(\phi, x)$$
$$x'^{\,\mu} = x^\mu + \delta x^\mu = x^\mu + \epsilon \mathcal{A}^\mu(x)$$

For future utility:

$$\phi'_i(x) = \phi'_i(x' - \delta x) = \phi'_i(x') - \delta x^\mu \partial_\mu \phi_i(x) = \phi_i(x) + \delta\phi_i(x) - \delta x^\mu \partial_\mu \phi_i(x)$$
$$\equiv \phi_i(x) + \overline{\delta\phi}_i(x) = \phi_i(x) + \epsilon[M_i(\phi, x) - \mathcal{A}^\mu(x)\partial_\mu\phi_i(x)]$$

Change variables in $S'$ back to $x$, then expand in $\epsilon$

$$S' = \int_D d^4x \left| \det_{\mu\nu} \tfrac{\partial x'^\mu}{\partial x^\nu} \right| \mathcal{L}(\phi'(x + \delta x), \partial_\mu \phi'(x + \delta x), x + \delta x)$$

To lowest order

$$\left| \det_{\mu\nu} \tfrac{\partial x'^\mu}{\partial x^\nu} \right| = \left| \det_{\mu\nu} \left(\delta^\mu{}_\nu + \tfrac{\partial \delta x^\mu}{\partial x^\nu}\right) \right| = \left| 1 + \mathrm{tr}\left(\tfrac{\partial \delta x^\mu}{\partial x^\nu}\right) \right| = \left|1 + \partial_\mu \delta x^\mu\right| = 1 + \partial_\mu \delta x^\mu$$

# Noether's theorem (contd.)

Expanding to lowest order in $\epsilon$

$$S' = \int_D d^4x \Big\{ \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) + \partial_\mu \Big[ \delta x^\mu \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) \Big] + \textstyle\sum_i \Big[ \overline{\delta\phi}_i(x) \frac{\partial}{\partial \phi_i} \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) + \partial_\mu (\overline{\delta\phi}_i(x)) \frac{\partial}{\partial(\partial_\mu \phi_i)} \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) \Big] \Big\}$$

Using $S' - S =$ b.t. and integration by parts (drop $x$-dep. of fields)

$$0 = \int_D d^4x \Big\{ \partial_\mu \Big[ \delta x^\mu \mathcal{L}(\phi, \partial_\mu \phi, x) + \textstyle\sum_i \overline{\delta\phi}_i \frac{\partial}{\partial(\partial_\mu \phi_i)} \mathcal{L}(\phi, \partial_\mu \phi, x) - \epsilon F^\mu(\phi, x) \Big] + \textstyle\sum_i \overline{\delta\phi}_i \Big[ \frac{\partial}{\partial \phi_i} \mathcal{L}(\phi, \partial_\mu \phi, x) - \partial_\mu \frac{\partial}{\partial(\partial_\mu \phi_i)} \mathcal{L}(\phi, \partial_\mu \phi, x) \Big] \Big\}$$

Boundary term must be infinitesimal too

Imposing EOM, since $D$ and $\epsilon$ arbitrary $\Rightarrow$ conserved current $\partial_\mu J^\mu = 0$

$J^\mu$: *Noether current*

$$J^\mu(\phi, \partial_\mu \phi, x) = \mathcal{A}^\mu \mathcal{L}(\phi, \partial_\mu \phi, x) + \textstyle\sum_i [M_i(\phi, x) - \mathcal{A}^\nu \partial_\nu \phi_i] \frac{\partial}{\partial(\partial_\mu \phi_i)} \mathcal{L}(\phi, \partial_\mu \phi, x) - F^\mu(\phi, x)$$

# Noether's theorem (contd.)

Expanding to lowest order in $\epsilon$

$$S' = \int_D d^4x \Big\{ \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) + \partial_\mu \Big[ \delta x^\mu \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) \Big]$$
$$+ \sum_i \Big[ \overline{\delta\phi}_i(x) \tfrac{\partial}{\partial \phi_i} \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) + \partial_\mu (\overline{\delta\phi}_i(x)) \tfrac{\partial}{\partial(\partial_\mu \phi_i)} \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) \Big] \Big\}$$

Using $S' - S =$ b.t. and integration by parts (drop $x$-dep. of fields)

$$0 = \int_D d^4x \Big\{ \partial_\mu \Big[ \delta x^\mu \mathcal{L}(\phi, \partial_\mu \phi, x) + \sum_i \overline{\delta\phi}_i \tfrac{\partial}{\partial(\partial_\mu \phi_i)} \mathcal{L}(\phi, \partial_\mu \phi, x) - \epsilon F^\mu(\phi, x) \Big]$$
$$+ \sum_i \overline{\delta\phi}_i \Big[ \tfrac{\partial}{\partial \phi_i} \mathcal{L}(\phi, \partial_\mu \phi, x) - \partial_\mu \tfrac{\partial}{\partial(\partial_\mu \phi_i)} \mathcal{L}(\phi, \partial_\mu \phi, x) \Big] \Big\}$$

Boundary term must be infinitesimal too

Imposing EOM, since $D$ and $\epsilon$ arbitrary $\Rightarrow$ conserved current $\partial_\mu J^\mu = 0$

$J^\mu$: *Noether current*

$$J^\mu(\phi, \partial_\mu \phi, x)$$
$$= \mathcal{A}^\mu \mathcal{L}(\phi, \partial_\mu \phi, x) + \sum_i [M_i(\phi, x) - \mathcal{A}^\nu \partial_\nu \phi_i] \tfrac{\partial}{\partial(\partial_\mu \phi_i)} \mathcal{L}(\phi, \partial_\mu \phi, x) - F^\mu(\phi, x)$$

# Noether's theorem (contd.)

$Q = \int d^3x \, J^0(x)$: *Noether charge*

$Q$ is conserved:

$$\frac{d}{dt} Q = \int d^3x \, \partial_0 J^0(x) = - \int d^3x \, \partial_k J^k(x) = - \lim_{V\to\infty} \int_V d^3x \, \partial_k J^k(x)$$
$$= - \lim_{V\to\infty} \int_{\partial V} dn_k \, J^k(x) = 0$$

Fields assumed sufficiently well behaved at spatial infinity, so that the flux of $J^k$ at infinity vanishes

Explicit expression:

$$Q = \int d^3x \Big[ \mathcal{A}^0 \mathcal{L} + \textstyle\sum_i [M_i - \mathcal{A}^\nu \partial_\nu \phi_i] \frac{\partial}{\partial(\partial_0 \phi_i)} \mathcal{L} - F^0 \Big]$$
$$= \int d^3x \Big[ \mathcal{A}^0 \Big( \mathcal{L} - \textstyle\sum_i \partial_0 \phi_i \pi_i \Big) + \textstyle\sum_i [M_i - \mathcal{A}^j \partial_j \phi_i] \pi_i - F^0 \Big]$$
$$= \int d^3x \Big[ \textstyle\sum_i [M_i - \mathcal{A}^j \partial_j \phi_i] \pi_i - \mathcal{A}^0 \mathcal{H} - F^0 \Big]$$

# Noether current: translations

$\mathcal{L}$ without explicit $x$ dependence $\Rightarrow$ $S$ invariant (no boundary terms)

Transformation law:

$$\phi'(x') = \phi(x) \qquad x' = x + a$$
$$\Longrightarrow \mathcal{A}^{\mu}_{(\nu)} = \delta^{\mu}{}_{\nu} \qquad M_i = 0$$

Four conserved currents $\Rightarrow$ *canonical energy-momentum tensor*

$$\Theta^{\mu}{}_{\nu} = -J^{\mu}_{(\nu)} = \sum_i \partial_\nu \phi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \delta^{\mu}{}_{\nu}\mathcal{L} \qquad \Theta^{\mu\nu} = \sum_i \partial^\nu \phi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \eta^{\mu\nu}\mathcal{L}$$

$\Theta^{0\nu}$ represent energy and momentum density of the system

$$\int d^3x\, \Theta^{00} = \int d^3x \left[\sum_i \partial^0 \phi_i \pi_i - \mathcal{L}\right] = \int d^3x\, \mathcal{H} = H = P^0$$
$$\int d^3x\, \Theta^{0k} = \int d^3x \sum_i \partial^k \phi_i \pi_i = -\int d^3x \sum_i \partial_k \phi_i \pi_i = P^k$$

$P^\mu$ (energy-momentum) are the *generators* of translations