# Particle physics

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# Dyson's formula

$$S = \Omega_-^\dagger \Omega_+ = \lim_{t_2\to+\infty} \lim_{t_1\to-\infty} e^{iH_0 t_2} e^{-iHt_2} e^{iHt_1} e^{-iH_0 t_1} = \lim_{t_2\to+\infty} \lim_{t_1\to-\infty} \mathcal{U}(t_2, t_1)$$

To find $S$:

1. write differential equation for unitary operator $\mathcal{U}(t_2, t_1)$
2. solve it with "initial condition" $\mathcal{U}(t, t) = \mathbf{1}$
3. take limits

$$\begin{aligned} \frac{\partial}{\partial t_2}\mathcal{U}(t_2, t_1) &= e^{iH_0 t_2} i(H_0 - H) e^{-iH_0 t_2} \mathcal{U}(t_2, t_1) \\ &= -i e^{iH_0 t_2} V e^{-iH_0 t_2} \mathcal{U}(t_2, t_1) = -iV_I(t_2)\mathcal{U}(t_2, t_1) \\ V_I(t) &\equiv e^{iH_0 t} V e^{-iH_0 t} \end{aligned}$$

$\mathcal{U}(t_2, t_1)^\dagger = \mathcal{U}(t_1, t_2)$, nothing new from $\partial/\partial t_1$

$V_I(t)$: interaction Hamiltonian in *interaction picture*

| | | |
|---|---|---|
| Schrödinger picture: | $\vert\psi(t)\rangle_S = e^{-iHt}\vert\psi(0)\rangle_S$ | $\mathcal{O}_S$ |
| Heisenberg picture: | $\vert\psi\rangle_H = \vert\psi(0)\rangle_S$ | $\mathcal{O}_H(t) = e^{iHt}\mathcal{O}_S e^{-iHt}$ |
| Dirac (interaction) picture: | $\vert\psi(t)\rangle_I = e^{iH_0 t} e^{-iHt}\vert\psi(0)\rangle_S$ | $\mathcal{O}_I(t) = e^{iH_0 t}\mathcal{O}_S e^{-iH_0 t}$ |

$${}_S\langle\phi(t)|\mathcal{O}_S|\psi(t)\rangle_S = {}_H\langle\phi|\mathcal{O}_H(t)|\psi\rangle_H = {}_I\langle\phi(t)|\mathcal{O}_I(t)|\psi(t)\rangle_I$$

# Dyson's formula (contd.)

Solution:

$$\mathcal{U}(t_2,t_1) = \mathrm{Texp}\left\{-i\int_{t_1}^{t_2} dt\, V_I(t)\right\}$$
$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_1}^{t_2} d\tau_1 \ldots \int_{t_1}^{t_2} d\tau_n T\left\{V_I(\tau_1)\ldots V_I(\tau_n)\right\}$$

$T$: *time-ordering* symbol, places operators in descending time order

$$T\{A_1(t_1)A_2(t_2)\} = \theta(t_1-t_2)A_1(t_1)A_2(t_2) + \theta(t_2-t_1)A_2(t_2)A_1(t_1)$$
$$T\{A_1(t_1)\ldots A_n(t_n)\} = \sum_P \theta(t_{P(1)}-t_{P(2)})\ldots\theta(t_{P(n-1)}-t_{P(n)})A_{P(1)}(t_{P(1)})\ldots A_{P(n)}(t_{P(n)})$$

Sum over permutations $P$ of $\{1,\ldots,n\}$

Check solution: (obviously $\mathcal{U}(t,t) = \mathbf{1}$)

$$\mathcal{U}(t_2,t_1) = \sum_{n=0}^{\infty}(-i)^n \int_{t_1}^{t_2} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2 \ldots \int_{t_1}^{\tau_{n-1}} d\tau_n\, V_I(\tau_1)\ldots V_I(\tau_n)$$
$$\frac{\partial}{\partial t_2}\mathcal{U}(t_2,t_1) = -iV_I(t_2)\sum_{n=1}^{\infty}(-i)^{n-1}\int_{t_1}^{t_2} d\tau_2 \ldots \int_{t_1}^{\tau_{n-1}} d\tau_n\, V_I(\tau_2)\ldots V_I(\tau_n)$$
$$= -iV_I(t_2)\sum_{n=0}^{\infty}(-i)^n\int_{t_1}^{t_2} d\tau_1 \ldots \int_{t_1}^{\tau_{n-1}} d\tau_n\, V_I(\tau_1)\ldots V_I(\tau_n) = -iV_I(t_2)\mathcal{U}(t_2,t_1)$$

$$\boxed{S = \mathcal{U}(+\infty,-\infty) = \mathrm{Texp}\left\{-i\int_{-\infty}^{+\infty} dt\, V_I(t)\right\}} \qquad \textit{(Dyson's formula)}$$

# Cross sections - reprise

Rate of scattering events (n. events per unit time) for a beam of flux $\Phi$ (particles per unit area per unit time) on a target with $N_t$ particles:

$$\frac{\Delta N_{\text{events}}}{\Delta t} = \sigma N_t \Phi$$

1. Count all scattering events $\Rightarrow$ measure *total cross section*

$$\sigma = \frac{\Delta N_{\text{events}}}{\Delta t N_t \Phi}$$

2. Classify scattering events $\Rightarrow$ measure *differential cross section*

$$\Delta\sigma_\alpha(\xi) = \frac{\Delta N_{\text{events}}(\alpha,\xi)}{\Delta t \Delta\xi N_t \Phi}\Delta\xi \underset{\Delta t,\Delta\xi\to 0}{\Longrightarrow} \frac{d\sigma_\alpha}{d\xi}(\xi) = \frac{dN_{\text{events}}(\alpha,\xi)}{dt d\xi N_t \Phi}$$

- $\alpha$: discrete variables (e.g., number/type of particles, $s_z$), essentially label different processes
- $\xi$: continuous variables (e.g., momenta)
- $\Delta N_{\text{events}}(\alpha_0,\xi_0)$: scattering events w/ $\alpha = \alpha_0$, $\xi \in [\xi_0 - \frac{\Delta\xi}{2}, \xi_0 + \frac{\Delta\xi}{2}]$

$\frac{d\sigma}{d\xi_1 d\xi_2 \ldots}$ are *not* derivatives

# Cross sections - reprise (contd.)

Transition probability $P_{fi}$ from initial state $|\phi_i\rangle$ to final state $|\phi_f\rangle$ not directly measurable due to practical limitations

- initial state in one single given scattering process not known with arbitrary accuracy
- particle states used in scattering experiments obtained through practical processes (e.g. acceleration of particles, preparation of beams) affected by inherent uncertainties
- accurate determination of the actual wave function of the state impossible

What can be measured is the transition probability averaged over many experiments, corresponding to many slightly different initial states $\Rightarrow$ cross section related to averaged $P_{fi}$

Ideally: repeat $N$ times experiment with one scatterer in target, one particle in beam $\Rightarrow N_{\text{events}}/N = P_{fi}$

# Cross sections - reprise (contd.)

Differential cross section in ideal case (and $N = 1$):

$$\Delta\sigma = \frac{\Delta N_{\text{events}}}{N_t \frac{N_b}{A_b}} = \frac{P_{fi}}{\Delta t \Phi(N_b = 1)}$$

**Q.**: What $P_{fi}$ should we use since we do not known $\phi_{i,f}$ accurately?

**A.**: for initial/final states well peaked around definite particle momenta details do not matter, relevant $P_{fi}$ is transition probability between idealised initial/final momentum eigenstates

- Can be shown using wave-packet description of initial/final states and careful consideration of how scattering experiments are carried out
- Alternatively: quantisation in a periodic box (equivalent results & much simpler)

# Cross section from the $S$ matrix

Transition probability :

$$P_{fi} = \frac{|\langle \phi_f | S | \phi_i \rangle|^2}{\langle \phi_f | \phi_f \rangle \langle \phi_i | \phi_i \rangle}$$

Energy-momentum conservation implies

$$S = \mathbf{1} + i(2\pi)^4 \delta^{(4)}(P_f - P_i)\mathcal{M}$$

$\mathbf{1}$: no-scattering term

$P_{fi}$ not well defined for momentum eigenstates

Non-normalisable states, square of $\delta^{(4)}(P_f - P_i)$

Trick: regularise expressions by putting system in a finite $T \times V = T \times L^3$ four-dimensional box with periodic boundary conditions

- allowed momenta become discrete, $p_j = \frac{2\pi k_j}{L}$, $k_j \in \mathbb{Z}$
- corresponding eigenstates are normalisable
- four-momentum conserving Dirac-delta replaced by Kronecker delta, can be squared without problems

# Cross section from the $S$ matrix (contd.)

Make sure correct normalisation obtained in infinite-volume limit:

- infinite-volume momentum eigenstates: relativistic normalisation

$$\langle \vec{p}\,'|\vec{p}\rangle = (2\pi)^3 2p^0 \delta^{(3)}(\vec{p}\,' - \vec{p})$$

- finite-volume momentum eigenstates: matching normalisation

$${}_V\langle \vec{p}\,'|\vec{p}\rangle_V = 2p^0 V \delta^{(3)}_{\vec{p}\,',\vec{p}}$$

One admissible momentum in a cube of volume $\frac{V}{(2\pi)^3} \Rightarrow$ density of modes $= \frac{(2\pi)^3}{V}$
$\Rightarrow \sum_{\vec{p},V} \to \frac{V}{(2\pi)^3} \int d^3p$ in the infinite-volume limit

- finite-volume $S$-matrix:

$$S_{T,V} = \mathbf{1}_{T,V} + iTV\delta^{(4)}_{P_f,P_i}\mathcal{M}_{T,V}$$

with $TV\delta^{(4)}_{P_f,P_i} \to (2\pi)^4\delta^{(4)}(P_f - P_i)$ and $\mathcal{M}_{T,V} \to \mathcal{M}$ as $T, V \to \infty$

# Cross section from the $S$ matrix (contd.)

Derivation:

- $\Delta P_{fi}$: transition probability from initial state of two particles with momenta $\vec{p}_{1,2}$ to final state of particles with prescribed momenta $\vec{p}_i'$
- Consider only $f \neq i$, drop no-scattering term
- Denote with $\Delta^3 p_i' = \frac{(2\pi)^3}{V}$ the size of the "unit cell" in the phase space of final particle $i$

$$\Delta P_{fi} = \underbrace{\frac{1}{4p_1^0 p_2^0 V^2}}_{\text{initial state norm.}} \underbrace{(TV)^2 \delta^{(4)}_{P_f,P_i} |\mathcal{M}_{T,V}|^2}_{|S_{fi}|^2} \underbrace{\prod_j \frac{1}{2p_j'^0 V} \frac{\Delta^3 p_i' V}{(2\pi)^3}}_{\text{final state norm.}}$$

For large $V$, $T$

- replace $\mathcal{M}_{T,V}$ with infinite-volume matrix element $\mathcal{M}_{fi}$
- combine one factor $VT$ with Kronecker delta to obtain Dirac delta

$$\Delta P_{fi} \longrightarrow \frac{T}{V} \frac{|\mathcal{M}_{fi}|^2}{4p_1^0 p_2^0} (2\pi)^4 \delta^{(4)}(P_f - P_i) \prod_j \frac{1}{2p_j'^0} \frac{\Delta^3 p_i'}{(2\pi)^3}$$

# Cross section from the $S$ matrix (contd.)

Elementary process involving only two particles, $N_t = 1$, $N_b = 1$
Beam flux: $\Phi = \frac{v}{V}$ (speed $v \times$ density $1/V$), duration of the process: $T$
"Number of scattering events" $=$ probability $\Delta P_{fi}$

Differential cross section:

$$\Delta\sigma = \frac{\Delta P_{fi}}{T N_t \Phi} = \frac{V}{T}\frac{\Delta P_{fi}}{v} = \frac{|\mathcal{M}_{fi}|^2}{4p_1^0 p_2^0 v}(2\pi)^4\delta^{(4)}(P_f - P_i)\prod_j \frac{\Delta^3 p_i'}{2p_j'^0(2\pi)^3}$$

All factors of $V, T$ cancel, can take the limit $T, V \to \infty$

$$d\sigma = \frac{|\mathcal{M}_{fi}|^2}{4p_1^0 p_2^0 v}(2\pi)^4\delta^{(4)}(P_f - P_i)\prod_j \frac{d^3 p_i'}{2p_j'^0(2\pi)^3} = \frac{|\mathcal{M}_{fi}|^2}{4p_1^0 p_2^0 v} d\Phi^{(n)}$$

$d\Phi^{(n)}$: infinitesimal invariant-volume element of $n$-particle phase space

$$d\Phi^{(n)} = \prod_j \frac{d^3 p_i'}{2p_j'^0(2\pi)^3}(2\pi)^4\delta^{(4)}(\sum_j p_j' - P_i)$$

# Cross section from the $S$ matrix (contd.)

Derivation done in the lab frame with one of the particles at rest, but we would rather have a Lorentz-invariant definition of the cross section

- $|\mathcal{M}|^2$ is Lorentz-invariant if
  - $S$ is Lorentz invariant (it should be!)
  - momentum eigenstates obey relativistic normalisation
- phase-space measure is Lorentz-invariant
- need the Lorentz-invariant expression for $p_1^0 p_2^0 v$

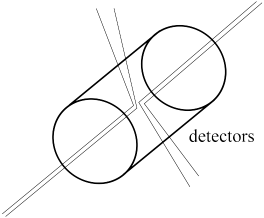
In the lab frame

$$p_1^0 p_2^0 v = E_1 v m_2 = |\vec{p}_1| m_2 = \sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}$$

Manifestly invariant expression for differential cross section:

$$\frac{d\sigma}{d\Phi^{(n)}} = \frac{|\mathcal{M}_{fi}|^2}{4\sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}}$$

# Collider experiments




- Two beams in opposite directions on same circular trajectory (assume same cross-sectional area $A_b$, velocity $v$)
- Beams of $N_{B1,2}$ bunches with $N_{b1,2}$ particles per bunch

- Take one bunch per beam, over period $T$ to go around whole circle:
  - two crossings $\Rightarrow$ collision frequency $2/T$
  - $N_{b1}N_{b2}$ possible pairwise particle interactions $\Rightarrow N_{\text{events}} = 2\frac{\sigma}{A_b}N_{b1}N_{b2}$
- Number of events per unit time:

$$\frac{\Delta N_{\text{events}}}{\Delta t} = \frac{2}{T}\frac{N_{B1}N_{B2}N_{b1}N_{b2}}{A_b}\sigma = \mathcal{L}\sigma \Longrightarrow \sigma = \frac{1}{\mathcal{L}}\frac{\Delta N_{\text{events}}}{\Delta t}$$

$\mathcal{L}$: *luminosity* delivered by the collider

To make further progress, we need a detailed relativistic quantum theory to compute $\mathcal{M}_{fi} \Rightarrow$ Quantum Field Theory

# Introduction to Quantum Field Theory

Fact of Nature: particles can be created or destroyed

- decays, e.g., $\pi^+ \to \mu^+ \nu_\mu$
- particle creation in collisions, e.g., $e^- e^+ \to X$

Formalism of quantum mechanics inadequate for such processes: to each particle in the game corresponds a wave function, how can that just appear or disappear?

- Not really a problem in the non-relativistic, low-energy regime: $p\,\bar{p} \to p\,\bar{p}\,e^-\,e^+$ requires $E_p^{\mathrm{CM}} - m_p \geq m_e \Rightarrow \beta \gtrsim 1/10$
- Serious problem at high energies when particle production and annihilation become important

Appropriate formalism must take into account

- principles of special relativity (Lorentz covariance, locality)
- principles of quantum mechanics (superposition principle, uncertainty principle)

Locality better dealt with using fields, uncertainty corresponds to non commutativity of measurements $\Rightarrow$ use field *operators*

# Introduction to Quantum Field Theory (contd.)

Fields are better suited to describe local interactions (= no action at a distance):

- fields: entities $\phi(x)$ defined everywhere in spacetime
- dynamics automatically local if field interactions depend only on value of fields at a point and in an infinitesimal neighbourhood (= $\phi(x)$, $\partial_\mu \phi(x)$, ...)
- automatically Lorentz-covariant dynamics easily constructed using fields with simple Lorentz transformation properties

Particles are better suited to describe experiments:

- straightforward construction of state superpositions
- multiparticle kinematics easy to describe in terms of Fock space: direct sum of $n$-particle Hilbert spaces, any number of particles allowed

How does one connect the two? Theory of quantised fields

# Fock space

Consider a system of non-interacting spinless bosons of mass $m$

Most general state: linear superposition of states with arbitrary number $n$ of particles with definite momenta (momentum operator eigenstates)

Basis of Hilbert space: $\{|\vec{p}_1, \ldots, \vec{p}_n\rangle\}_{n=0,1,\ldots,\infty}$
$n = 0$ state $|0\rangle$: *vacuum state* (no particle is present)

Experimental fact: particles of the same type are indistinguishable

Quantum state vector of a system of bosons invariant under any permutation $\mathrm{P}$ of the particle labels (*Bose-Einstein statistics*)

$$|\vec{p}_{\mathrm{P}(1)}, \ldots, \vec{p}_{\mathrm{P}(n)}\rangle = |\vec{p}_1, \ldots, \vec{p}_n\rangle$$

Formally: take one-particle states $|\vec{p}\,\rangle$ and fully symmetrise their $n$-fold tensor product

$$|\vec{p}_1, \ldots, \vec{p}_n\rangle = \tfrac{1}{n!} \textstyle\sum_{\mathrm{P}} |\vec{p}_{\mathrm{P}(1)}\rangle \otimes \ldots \otimes |\vec{p}_{\mathrm{P}(n)}\rangle$$

Space generated by $\{|\vec{p}_1, \ldots, \vec{p}_n\rangle\}_{n=0,1,\ldots,\infty} =$ *Fock space*

# Fock space (contd.)

Energy and momentum related by dispersion relation $E^2 = \vec{p}^{\,2} + m^2 \Rightarrow$ one-particle vectors are eigenvectors of four-momentum operator $P^\mu$

$$P^\mu|\vec{p}\,\rangle = p^\mu|\vec{p}\,\rangle \qquad p^0 = \sqrt{\vec{p}^{\,2} + m^2}$$

Momenta additive $\Rightarrow$ also $n$-particle states are $P^\mu$ eigenvectors

Relativistic normalisation:

$$\langle\vec{p}\,'|\vec{p}\rangle = (2\pi)^3 2p_0\delta^{(3)}(\vec{p}\,' - \vec{p})$$

$$\langle\vec{p}_1', \ldots, \vec{p}_{n'}'|\vec{p}_1, \ldots, \vec{p}_n\rangle = \delta_{n'n}\sum_{\mathrm{P}}\prod_{j=1}^n(2\pi)^3 2p_j^0\delta^{(3)}(\vec{p}_{\mathrm{P}(j)}' - \vec{p}_j)$$

$$\langle 0|0\rangle = 1$$

Invariant measure: $$d\Omega_p \equiv \frac{d^3p}{(2\pi)^3 2p^0}$$

Rotations: $p_0\delta^{(3)}(\vec{p}\,' - \vec{p})$ manifestly invariant, boosts in 1-direction: $(\Lambda p)^{0,1} = \gamma(p^{0,1} - \beta p^{1,0})$,

$$\delta((\Lambda p)'^1 - (\Lambda p)^1) = \delta(\gamma[(p'^1 - p^1) - \beta(p'^0 - p^0)]) = |\tfrac{\partial\gamma(p^1 - \beta p^0)}{\partial p^1}|^{-1}\delta(p'^1 - p^1)$$

$$= p^0|\gamma(p^0 - \beta p^1)|^{-1}\delta(p'^1 - p^1) = \tfrac{p^0}{(\Lambda p)^0}\delta(p'^1 - p^1)$$

# Creation/annihilation operators

*Creation operators* $a(\vec{p})^\dagger$:

$$a(\vec{p})^\dagger|\vec{p}_1,\ldots,\vec{p}_n\rangle \equiv |\vec{p},\vec{p}_1,\ldots,\vec{p}_n\rangle$$

*Annihilation operators* $a(\vec{p})$: adjoint of $a(\vec{p})^\dagger$

$$a(\vec{p})|\vec{p}_1,\ldots,\vec{p}_n\rangle = \sum_{j=1}^n (2\pi)^3 2p_j^0 \delta^{(3)}(\vec{p}_j - \vec{p}\,)|\vec{p}_1,\ldots,\vec{p}_{j-1},\vec{p}_{j+1},\ldots,\vec{p}_n\rangle$$

Creation/annihilation operators allow us to change as we please the particle content of a state

Since we cannot remove particles from the vacuum $\Rightarrow a(\vec{p}\,)|0\rangle = 0\ \forall\vec{p}$

The vacuum $|0\rangle$ is the only state annihilated by all $a(\vec{p}\,)$, any other state can be obtained from $|0\rangle$ by repeated application of $a(\vec{p})^\dagger$

$$|\vec{p}_1,\ldots,\vec{p}_n\rangle \equiv a(\vec{p}_1)^\dagger \ldots a(\vec{p}_n)^\dagger|0\rangle$$

Commutation relations follow from definition:

$$[a(\vec{p}\,),a(\vec{q}\,)] = [a(\vec{p}\,)^\dagger,a(\vec{q}\,)^\dagger] = 0 \qquad [a(\vec{p}\,),a(\vec{q}\,)^\dagger] = (2\pi)^3 2p^0 \delta^{(3)}(\vec{p}-\vec{q}\,)$$

# Creation/annihilation operators (contd.)

Number/number density operators

$$N = \int \frac{d^3p}{(2\pi)^3 2p^0}\, a(\vec{p}\,)^\dagger a(\vec{p}\,) = \int \frac{d^3p}{(2\pi)^3 2p^0}\, \nu(\vec{p}\,)$$

Commutation relations:

$$[\nu(\vec{p}\,), a(\vec{q}\,)] = [a(\vec{p}\,)^\dagger, a(\vec{q}\,)]a(\vec{p}\,) = -(2\pi)^3 2p^0 \delta^{(3)}(\vec{q} - \vec{p}\,)a(\vec{p}\,)$$
$$[\nu(\vec{p}\,), a(\vec{q}\,)^\dagger] = a(\vec{p}\,)^\dagger[a(\vec{p}\,), a(\vec{q}\,)^\dagger] = (2\pi)^3 2p^0 \delta^{(3)}(\vec{q} - \vec{p}\,)a(\vec{p}\,)^\dagger$$

$\Rightarrow$ $N$ diagonal operator, eigenvalue = number of particles

$$N|\vec{p}_1, \ldots, \vec{p}_n\rangle = n|\vec{p}_1, \ldots, \vec{p}_n\rangle$$

For any additive diagonal quantum number $f(\vec{p})$ (e.g., $P^\mu$), corresponding operator reads

$$F = \int \frac{d^3p}{(2\pi)^3 2p^0}\, f(\vec{p}\,)a(\vec{p}\,)^\dagger a(\vec{p}\,) \qquad F|\vec{p}_1, \ldots, \vec{p}_n\rangle = \left(\textstyle\sum_{j=1}^n f(\vec{p}_j\,)\right)|\vec{p}_1, \ldots, \vec{p}_n\rangle$$

Can we reproduce this with field operators?

# Fock space: generalisations

- Straightforward extension to several types of bosons:
  - introducing creation/annihilation operators for each type
  - impose standard commutation relations among operators of the same type
  - impose that operators of different type commute with each other
- Extension to fermions must take into account the different particle statistics: exchanging labels of any two particles must change the state vector by a minus sign (*Fermi-Dirac statistics*)
  - introduce creation and annihilation operators
  - impose *anticommutation relations*

$$\{a(\vec{p}\,), a(\vec{q}\,)\} = \{a(\vec{p}\,)^\dagger, a(\vec{q}\,)^\dagger\} = 0$$
$$\{a(\vec{p}\,), a(\vec{q}\,)^\dagger\} = (2\pi)^3 2p^0 \delta^{(3)}(\vec{p} - \vec{q}\,)$$

  - impose that operators corresponding to fermions of different type anticommute with each other
  - impose that operators corresponding to fermions and bosons commute with each other

# Quantisation of the scalar field

Can one replace the Schrödinger equation with a relativistic one? Sure: *Klein-Gordon* equation

$$(\Box + m^2)\phi(x) = 0$$

Can one obtain a sensible quantum mechanical theory from this, with $\phi(x)$ treated as a wave function? Nope:

- second-order in time as well as in space (as required by invariance) $\Rightarrow$ admits negative-energy solutions (leads to instability)
- only probability current with the right symmetry properties under Lorentz transformations leads to a probability density which is not positive-definite
- fundamental problem is that we cannot describe multiparticle interactions in terms of single-particle wave functions

Classically, Klein-Gordon equation describes a free scalar field:
can we quantise it?

Can be interpreted as quantisation of the wave function (*second quantisation*)

# Quantisation of the scalar field (contd.)

Solution for the classical field: use Fourier transform to momentum space

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\tilde{\phi}(p)$$

$$(p^2 - m^2)\tilde{\phi}(p) = 0 \Rightarrow \tilde{\phi}(p) = 2\pi\delta(p^2 - m^2)f(p^0, \vec{p})$$

Solution:

$$\tilde{\phi}(p) = \frac{2\pi}{2\varepsilon(\vec{p}\,)}\left\{\delta(p^0 - \varepsilon(\vec{p}\,))f(\varepsilon(\vec{p}\,), \vec{p}\,) + \delta(p^0 + \varepsilon(\vec{p}\,))f(-\varepsilon(\vec{p}\,), \vec{p}\,)\right\}$$

$$\varepsilon(\vec{p}\,) \equiv \sqrt{\vec{p}^{\,2} + m^2}$$

Back to coordinate space:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2\varepsilon(\vec{p}\,)}\left\{e^{-i(\varepsilon(\vec{p}\,)x^0 - \vec{p}\cdot\vec{x})}f(\varepsilon(\vec{p}\,), \vec{p}\,) + e^{i(\varepsilon(\vec{p}\,)x^0 + \vec{p}\cdot\vec{x})}f(-\varepsilon(\vec{p}\,), \vec{p}\,)\right\}$$

$$= \int \frac{d^3p}{(2\pi)^3 2\varepsilon(\vec{p}\,)}\left\{e^{-i(\varepsilon(\vec{p}\,)x^0 - \vec{p}\cdot\vec{x})}f(\varepsilon(\vec{p}\,), \vec{p}\,) + e^{i(\varepsilon(\vec{p}\,)x^0 - \vec{p}\cdot\vec{x})}f(-\varepsilon(\vec{p}\,), -\vec{p}\,)\right\}$$

# Quantisation of the scalar field (contd.)

Set $d\Omega_p = \frac{d^3p}{(2\pi)^3 2p^0}$ with $p^0 = \varepsilon(\vec{p}\,)$, denote

$$a(\vec{p}\,) \equiv f(\varepsilon(\vec{p}\,), \vec{p}\,) \qquad b(\vec{p}\,)^* \equiv f(-\varepsilon(\vec{p}\,), -\vec{p}\,)$$

Most general solution of Klein-Gordon equation:

$$\phi(x) = \int d\Omega_p \, \left\{ a(\vec{p}\,) e^{-ip\cdot x} + b(\vec{p}\,)^* e^{ip\cdot x} \right\}$$

Real field $\Rightarrow b(\vec{p}\,) = a(\vec{p}\,)$

What is the energy stored in the field? From $0 = (\Box + m^2)\phi = \ddot{\phi} - \Delta\phi + m^2\phi$

$$0 = \int d^3x \, \dot{\phi}[\ddot{\phi} - \Delta\phi + m^2\phi] = \int d^3x \, [\dot{\phi}\ddot{\phi} + \vec{\nabla}\dot{\phi}\cdot\vec{\nabla}\phi + m^2\dot{\phi}\phi]$$

$$0 = \frac{d}{dt}\frac{1}{2}\int d^3x \, [\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2]$$

$$H = \frac{1}{2}\int d^3x \, [\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2] = \text{const.}$$

Setting $\pi(x) = \partial^0\phi(x)$

$$H = \frac{1}{2}\int d^3x \, [\pi(\vec{x})^2 + (\vec{\nabla}\phi(\vec{x}))^2 + m^2\phi(\vec{x})^2]$$

$\pi(\vec{x}) = \pi(0, \vec{x})$, $\phi(\vec{x}) = \phi(0, \vec{x})$