

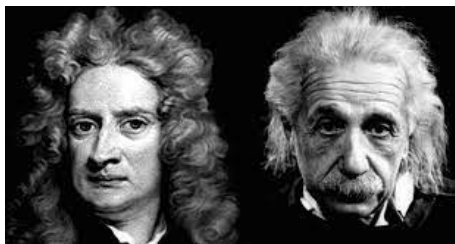
Particle physics: practice 1

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SPECIAL RELATIVITY

- Newton to Einstein: the problem of causality. Information can't be transferred instantaneously.



- “Special”: no acceleration. “General”: acceleration \equiv gravity.
- HEP can be thought of as the ultimate testbed for SR, since it's bread and butter for us (all particles relativistic).
- <https://pdg.lbl.gov/2019/reviews/rpp2019-rev-kinematics.pdf>

RELATIVISTIC KINEMATICS: MINKOWSKI SPACE

Relativistic theories are conveniently formulated in Minkowski space

Minkowski space = \mathbb{R}^4 + Minkowski (pseudo)metric

Euclidean space = \mathbb{R}^3 + Euclidean metric

Distance between points in E. space: $d(\vec{x}, \vec{y}) = (\vec{x} - \vec{y})^2 = (\vec{x} - \vec{y})_i (\vec{x} - \vec{y})_j \delta_{ij}$

Latin indices $1, \dots, 3$, sum over repeated indices understood

Invariant under translations $\vec{x} \rightarrow \vec{x} + \vec{a}$ and rotations $\vec{x} \rightarrow R\vec{x}$

Point in Minkowski space (= *event*): X^μ , $\mu = 0, 1, 2, 3$

$$X^\mu = (ct, \vec{x}) = (t, \vec{x})$$

In Minkowski space distances replaced by *interval*

$$\begin{aligned} \Delta s^2 &\equiv (X - Y)^2 \equiv (X - Y)^\mu (X - Y)^\nu g_{\mu\nu} \equiv (X - Y)^\mu (X - Y)_\mu \\ &= (X^0 - Y^0)^2 - (\vec{X} - \vec{Y})^2 \end{aligned}$$

Minkowski metric tensor: $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$

THE LIGHTCONE

Contravariant vectors: $X^\mu = (X^0, \vec{X})$

Covariant vectors: $X_\mu = g_{\mu\nu} X^\nu = (X^0, -\vec{X})$

Indices lowered by $g_{\mu\nu}$ and raised by $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$

$g^{\mu\nu}$ defined by $g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu$

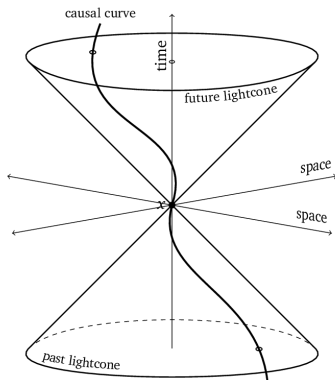
Minkowski scalar product $X \cdot Y \equiv X^\mu Y^\nu g_{\mu\nu} = X^\mu Y_\mu = X^0 Y^0 - \vec{X} \cdot \vec{Y}$

$\vec{X} \cdot \vec{Y}$: three-dimensional Euclidean scalar product

Interval is not a distance because it is not positive-definite:

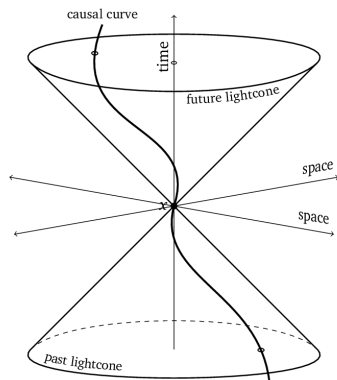
- $\Delta s^2 > 0$ *timelike* interval — $X^2 > 0$ timelike vector
- $\Delta s^2 < 0$ *spacelike* interval — $X^2 < 0$ spacelike vector
- $\Delta s^2 = 0$ *lightlike* or *null* interval — $X^2 = 0$ lightlike or null vector

THE LIGHTCONE (CONTD.)



- For a fixed event X
 - $(Y - X)^2 = 0, Y^0 - X^0 > 0$: *forward (future) lightcone* of X
 - $(Y - X)^2 = 0, Y^0 - X^0 < 0$: *backward (past) lightcone* of X
 - $(Y - X)^2 > 0, Y^0 - X^0 > 0$: *future of X (inside future lightcone)*
 - $(Y - X)^2 > 0, Y^0 - X^0 < 0$: *past of X (inside past lightcone)*

THE LIGHTCONE (CONTD.)



- Future lightcone of $X = 0$: events (Y^0, \vec{Y}) where \vec{Y} is reached at time Y^0 by light emitted by a point source at time $X^0 = 0$ from $\vec{x} = 0$
- Past lightcone of $X = 0$: events (Y^0, \vec{Y}) where a ray of light emitted at time Y^0 from \vec{Y} reaches $\vec{X} = 0$ at time $X^0 = 0$

LORENTZ TRANSFORMATIONS

Principles of special relativity:

- homogeneity and isotropy of space
- equivalence of all inertial reference frames
= travelling at a relative constant speed
- constancy of speed of light

⇒ equivalent frames are related by a Lorentz transformation $X' = \Lambda X$:
linear transformation that leaves every interval invariant

Analogous to rotations in 3D Euclidean space that leave the distance between points
invariant

$$\begin{aligned}(X' - Y')^2 &= (X - Y)^2 && \forall X, Y \\ \Rightarrow X'^2 + Y'^2 - 2X' \cdot Y' &= X^2 + Y^2 - 2X \cdot Y && \forall X, Y \\ \Rightarrow X' \cdot Y' &= X \cdot Y && \forall X, Y\end{aligned}$$

In components $X'^{\mu} = \Lambda^{\mu}_{\alpha} X^{\alpha}$

$$g_{\alpha\beta} X^{\alpha} Y^{\beta} = g_{\mu\nu} X'^{\mu} Y'^{\nu} = g_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} X^{\alpha} Y^{\beta} \quad \forall X, Y$$

LORENTZ TRANSFORMATIONS (CONTD.)

Using matrix notation $\Lambda_{\mu\alpha} = \Lambda^\mu{}_\alpha$, $\mathbf{g}_{\mu\nu} = g_{\mu\nu}$, $\mathbf{g}_{\mu\nu}^{-1} = g^{\mu\nu} = \mathbf{g}^{\mu\nu}$

$$\mathbf{g} = \Lambda^T \mathbf{g} \Lambda$$

$(\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1$, Λ invertible

- $\det \Lambda = 1$: *proper* transformations, leave orientation of space unchanged
- $\det \Lambda = -1$: *improper* transformations invert the orientation of space

$\Lambda^{-1} = \mathbf{g}^{-1} \Lambda^T \mathbf{g}$ still a Lorentz transformation

- $\mathbf{g} = [\Lambda \Lambda^{-1}]^T \mathbf{g} [\Lambda \Lambda^{-1}] = \Lambda^{-1 T} [\Lambda^T \mathbf{g} \Lambda] \Lambda^{-1} = \Lambda^{-1 T} \mathbf{g} \Lambda^{-1}$
- $\Lambda_{\alpha\beta}^{-1} = g^{\alpha\mu} \Lambda^\nu{}_\mu g_{\nu\beta} = \Lambda_\beta{}^\alpha$

From the $\alpha = 0$, $\beta = 0$ component of the defining relation

$$1 = \Lambda^0{}_0 \Lambda^0{}_0 - \Lambda^i{}_0 \Lambda^i{}_0 \implies \Lambda^0{}_0 \Lambda^0{}_0 = 1 + \Lambda^i{}_0 \Lambda^i{}_0 \geq 1$$

- $\Lambda^0{}_0 \geq 1$: *orthochronous* (does not change the sign of time)
- $\Lambda^0{}_0 \leq -1$: *non-orthochronous* (changes the sign of time)

PROPER ORTHOCHRONOUS LORENTZ GROUP

Proper orthochronous Lorentz transformations = three-dimensional rotations (the SO(3) group) and boosts

Most general transformation: rotation \times boost in x direction \times rotation

Boost along x :
$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \beta = \frac{v}{c} = v < 1$$
$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

Boost in general direction \vec{n} : rotate \vec{n} to x , boost, rotate back

Coordinates in the new frame:

$$\begin{aligned} ct' &= \gamma(ct + \beta x) & x' &= \gamma(x + \beta ct) \\ y' &= y & z' &= z \end{aligned}$$

\Rightarrow relates R to R' moving with speed β in the *negative* x direction

Nonrelativistic limit $\beta = v/c \ll 1 \Rightarrow$ Galilei transformations

$$ct' = ct \quad x' = x + vt$$

FULL LORENTZ GROUP

Most general Lorentz transformation = proper orthochronous transformation times P (*parity*), T (*time reversal*), or PT

$$P^\mu{}_\nu = \text{diag}(1, -1, -1, -1) \quad T^\mu{}_\nu = \text{diag}(-1, 1, 1, 1)$$

	$\det \Lambda = 1$		$\det \Lambda = -1$
$\Lambda^0{}_0 \geq 1$	proper orthochronous	\Rightarrow_P	improper orthochronous
	\Downarrow_T		\Downarrow_T
$\Lambda^0{}_0 \leq -1$	proper non-orthochronous	\Rightarrow_P	improper non-orthochronous

POINT PARTICLES: KINEMATICS

Trajectory $X^\mu(t)$ of point particle; over infinitesimal dt ,
 $X^\mu \rightarrow X^\mu + dX^\mu$

$$X^\mu(t) = (ct, \vec{x}(t)) = (t, \vec{x}(t))$$

$$dX^\mu(t) = (dt, d\vec{x}(t)) = dt(1, \frac{d\vec{x}}{dt}(t)) = dt(1, \vec{v}(t))$$

Empirical fact: for massive particles $\vec{v}^2 < 1$, for massless particles $\vec{v}^2 = 1$

$$(dX)^2 = dX^\mu dX_\mu = dt^2(1 - \vec{v}^2) \geq 0 \quad (\text{timelike})$$

$$\frac{dX^\mu}{dt}(t) = (1, \vec{v}(t))$$

$\frac{dX^\mu}{dt}$ not a Lorentz vector: $dX^\mu =$ vector, $dt \neq$ scalar

Massive particle $\vec{v}^2 < 1$: \exists reference frame in which $\vec{v} = 0$ (*rest frame*)

$$X_{\text{rest}}^\mu(\tau) = (\tau, \vec{0})$$

τ : *proper time* (time measured in the particle's rest frame)

$$(dX_{\text{rest}})^2 = d\tau^2 = (dX)^2 = dt^2(1 - \vec{v}^2) = \frac{dt^2}{\gamma^2}$$

POINT PARTICLES: KINEMATICS (CONTD.)

Proper time:

- $d\tau^2 = \frac{dt^2}{\gamma^2} \Rightarrow |dt| > |d\tau|$ (time-dilation effect)
- determine the elapsed proper time by going over to the instantaneous rest frame of the particle \Rightarrow twins' paradox

$$\tau = \int d\tau = \int_{t_0}^t dt' \sqrt{1 - \vec{v}^2(t')} \leq t - t_0$$

- true scalar $\Rightarrow \frac{d^n X^\mu}{d\tau^n}$ are true vectors

Four-velocity

$$u^\mu \equiv \frac{dX^\mu}{d\tau} = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau} \right) = (\gamma, \gamma \frac{d\vec{x}}{dt}) = (\gamma, \gamma \vec{v}) = (\gamma, \gamma \vec{\beta})$$

Four-momentum (vector u^μ times scalar m)

$$p^\mu \equiv m u^\mu = (\gamma m, \gamma m \vec{\beta})$$

$$p^0 = m\gamma = \frac{m}{\sqrt{1 - \vec{v}^2}} = E \quad p^i = m\gamma \vec{\beta}^i = \frac{m\vec{v}^i}{\sqrt{1 - \vec{v}^2}} = \vec{p}^i$$

POINT PARTICLES: FOUR-MOMENTUM IN THE NR LIMIT

Do E, \vec{p} match their non-relativistic definition when $\frac{|\vec{v}|}{c} \ll 1$?

Needs reinstating powers of c

$$p^0 = mc \frac{1}{\sqrt{1 - (\frac{\vec{v}}{c})^2}} = mc \left(1 + \frac{1}{2} \left(\frac{\vec{v}}{c} \right)^2 + \mathcal{O}\left(\left(\frac{v}{c}\right)^4\right) \right)$$

$$\vec{p} = mc \frac{\frac{\vec{v}}{c}}{\sqrt{1 - (\frac{\vec{v}}{c})^2}} = m\vec{v} \left(1 + \mathcal{O}\left(\left(\frac{v}{c}\right)^2\right) \right)$$

Second line ok, first line times c

$$p^0 c = mc^2 + \frac{1}{2} m \vec{v}^2 + \dots = E_0 + E_K^{\text{NR}} + \dots$$

\Rightarrow NR kinetic energy E_K^{NR} of a particle plus *rest energy* $E_0 = mc^2$

POINT PARTICLES: FOUR-MOMENTUM FOR $m \neq 0$ AND $m = 0$

Massive particles: $p^2 = m^2 > 0$

$$p^\mu = m \frac{dX^\mu}{d\tau} = \left(\frac{E}{c}, \vec{p} \right) \underset{c=1}{=} (E, \vec{p}) = (p^0, \vec{p})$$

Mass = relativistic invariant

$$p^2 = m^2 \gamma^2 (1 - \vec{\beta}^2) = m^2 > 0 \quad u^2 = \gamma^2 (1 - \vec{\beta}^2) = 1$$

Trajectory always inside the forward lightcone

Any constant would do, but m is *the* constant such that total momentum $\sum_i p_i = \sum_i m_i u_i$ of a system of particles is conserved

Also: correct NR limit of $p^\mu = mu^\mu$

Energy-momentum relation is called *dispersion relation*

$$E^2 = \vec{p}^2 + m^2$$

Massless particles: $p^2 = 0$

$$p^\mu = (\omega, \vec{k})$$

KINEMATICS OF 2-PARTICLE SCATTERING

Two particle \rightarrow two particle scattering process $ab \rightarrow cd$

Lab frame: one initial particle is at rest (= target)

$$\begin{aligned} p_a &= (E_L, \vec{p}_L) & p_b &= (m_b, 0) \\ p_c &= (E_c, \vec{p}_c) & p_d &= (E_d, \vec{p}_d) \end{aligned}$$

Scattering angle θ_L in the lab: angle between trajectories of c and a

$$\cos \theta_L = \frac{\vec{p}_L \cdot \vec{p}_c}{|\vec{p}_L| |\vec{p}_c|}$$

CM frame: vanishing total spatial momentum

$$\begin{aligned} p_a &= (E_a^*, \vec{p}^*) & p_b &= (E_b^*, -\vec{p}^*) \\ p_c &= (E_c^*, \vec{p}'^*) & p_d &= (E_d^*, -\vec{p}'^*) \end{aligned}$$

Scattering angle θ^* in the CM: angle formed by the trajectories of a and c

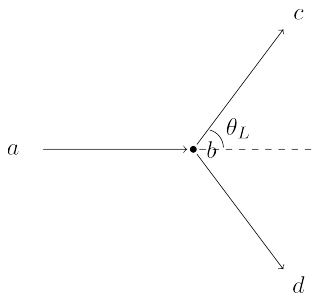
$$\cos \theta^* = \frac{\vec{p}^* \cdot \vec{p}'^*}{|\vec{p}^*| |\vec{p}'^*|}$$

Total center of mass energy $\sqrt{s} =$ Lorentz invariant

KINEMATICS OF 2-PARTICLE SCATTERING (CONTD.)

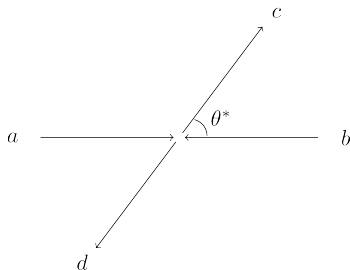
LAB

$$\begin{aligned} p_a &= (E_L, \vec{p}_L) & p_b &= (m_b, 0) \\ p_c &= (E_c, \vec{p}_c) & p_d &= (E_d, \vec{p}_d) \end{aligned}$$



CM

$$\begin{aligned} p_a &= (E_a^*, \vec{p}^*) & p_b &= (E_b^*, -\vec{p}^*) \\ p_c &= (E_c^*, \vec{p}'^*) & p_d &= (E_d^*, -\vec{p}'^*) \end{aligned}$$



KINEMATICS OF 2-PARTICLE SCATTERING (CONTD.)

$$p_a + p_b = p_c + p_d$$

- Four-momentum conservation implies $E_{c,d}^*$, $|\vec{p}_{c,d}^*| = |\vec{p}^*|$ determined uniquely in the CM, independent of θ^*
- $E_{c,d}$, $|\vec{p}_{c,d}|$ and θ_L in the lab by Lorentz transf., depend on θ^*

$$p_b = p_c + p_d - p_a$$

$$p_b^2 = (p_c + p_d)^2 + p_a^2 - 2p_a \cdot (p_c + p_d)$$

$$m_b^2 = s + m_a^2 - 2E_a^* \sqrt{s}$$

$$E_a^* = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} \quad \Rightarrow_{a \leftrightarrow b} \quad E_b^* = \frac{s + m_b^2 - m_a^2}{2\sqrt{s}}$$

- CM energy squared s Lorentz invariant $\Rightarrow E_a^*$ from E_L in the lab:

$$s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2p_a \cdot p_b = m_a^2 + m_b^2 + 2E_L m_b \Rightarrow E_L = \frac{s - m_a^2 - m_b^2}{2m_b}$$

- Exchanging $a, b \leftrightarrow c, d$

$$E_c^* = \frac{s + m_c^2 - m_d^2}{2\sqrt{s}} \quad E_d^* = \frac{s + m_d^2 - m_c^2}{2\sqrt{s}}$$

KINEMATICS OF 2-PARTICLE SCATTERING: CM

Center of mass energies:

$$E_a^* = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} \quad E_b^* = \frac{s + m_b^2 - m_a^2}{2\sqrt{s}}$$
$$E_c^* = \frac{s + m_c^2 - m_d^2}{2\sqrt{s}} \quad E_d^* = \frac{s + m_d^2 - m_c^2}{2\sqrt{s}}$$

Center of mass momentum magnitude $|\vec{p}^*|$:

$$|\vec{p}^*|^2 = E_a^{*2} - m_a^2 = \frac{(s+m_a^2-m_b^2)^2-4sm_a^2}{4s} = \frac{s^2+(m_a^2-m_b^2)^2-2s(m_a^2+m_b^2)}{4s}$$
$$= \frac{(s-m_a^2-m_b^2)^2-4m_a^2m_b^2}{4s} = \frac{[s-(m_a+m_b)^2][s-(m_a-m_b)^2]}{4s} = \frac{\lambda(s,m_a^2,m_b^2)}{4s}$$
$$|\vec{p}^*|^2 = E_c^{*2} - m_c^2 = \frac{(s+m_c^2-m_d^2)^2-4sm_c^2}{4s} = \frac{s^2+(m_c^2-m_d^2)^2-2s(m_c^2+m_d^2)}{4s}$$
$$= \frac{(s-m_c^2-m_d^2)^2-4m_c^2m_d^2}{4s} = \frac{[s-(m_c+m_d)^2][s-(m_c-m_d)^2]}{4s} = \frac{\lambda(s,m_c^2,m_d^2)}{4s}$$

Källén function: $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$

KINEMATICS OF 2-PARTICLE SCATTERING: LAB

Lab kinematics recovered from CM kinematics

Given $\vec{p}_{\text{lab,CM}}$, $E_{\text{lab,CM}}$ total spatial momentum/total energy in lab/CM

$$\begin{aligned} |\vec{p}_{\text{CM}}| = 0 &= \gamma_{\text{CM}}(|\vec{p}_{\text{lab}}| - \beta_{\text{CM}}E_{\text{lab}}) = \gamma_{\text{CM}}(|\vec{p}_L| - \beta_{\text{CM}}(m_b + E_L)) \\ \implies \beta_{\text{CM}} &= \frac{|\vec{p}_L|}{E_L + m_b} \end{aligned}$$

Inverse Lorentz transformation from CM to lab

$$\begin{aligned} E_{c,\text{lab}} &= \gamma_{\text{CM}}(E_c^* + \beta_{\text{CM}}|\vec{p}'^*| \cos \theta^*), \\ |\vec{p}_{c,\text{lab}}| \cos \theta_L &= \gamma_{\text{CM}}(|\vec{p}'^*| \cos \theta^* + \beta_{\text{CM}}E_c^*), \\ |\vec{p}_{c,\text{lab}}| \sin \theta_L &= |\vec{p}'^*| \sin \theta^*, \end{aligned}$$

Transverse directions unaffected by Lorentz transformation, azimuthal angle transforms trivially

EXAMPLE: PROTON-ANTIPROTON SCATTERING

For $p\bar{p}$ scattering in circular collider, $E_p = E_{\bar{p}} = 270$ GeV

$$\Rightarrow \sqrt{s} = 540 \text{ GeV}$$

Let now p be at rest in the lab.

Q. What should be the energy E_L of \bar{p} in the lab to obtain the same s ?

A. CM energy square s is a relativistic invariant, can be evaluated in any reference frame; in the lab

$$s = (p_p + p_{\bar{p}})^2 = 2(m_p^2 + E_L m_p) = 2m_p(m_p + E_L)$$

Solve for E_L and impose $\sqrt{s} = 540$ GeV ($\gg m_p$)

$$E_L = \frac{s - 2m_p^2}{2m_p} \simeq \frac{s}{2m_p} \simeq \frac{(540)^2}{2} \text{ GeV} \simeq \frac{30}{2} \cdot 10^4 \text{ GeV} = 150 \text{ TeV} \quad (!!!)$$

In general total CM energy $E_{\text{CM}} \simeq \sqrt{2m_p E_L}$

MANDELSTAM VARIABLES

Convenient set of relativistic invariant variables for $2 \rightarrow 2$ scattering

$$s \equiv (p_a + p_b)^2 = (p_c + p_d)^2$$

$$t \equiv (p_a - p_c)^2 = (p_b - p_d)^2$$

$$u \equiv (p_a - p_d)^2 = (p_b - p_c)^2$$

- s = total CM energy squared
- t = square of four-momentum transfer from a to c

$$t = p_a^2 + p_c^2 - 2p_a \cdot p_c = m_a^2 + m_c^2 - 2(E_a^* E_c^* - |\vec{p}^*| |\vec{p}'^*| \cos \theta^*)$$

- u = square of four-momentum transfer from a to d

$$u = p_a^2 + p_d^2 - 2p_a \cdot p_d = m_a^2 + m_d^2 - 2(E_a^* E_d^* + |\vec{p}^*| |\vec{p}'^*| \cos \theta^*)$$

u obtained from t after $m_c \rightarrow m_d$ and $\cos \theta^* \rightarrow -\cos \theta^*$

Energies and magnitudes of momenta entirely determined by s and particle masses $\Rightarrow t = t(s, \theta^*)$, or instead $\theta^* = \theta^*(s, t)$ and use s, t

MANDELSTAM VARIABLES (CONTD.)

Only two independent Mandelstam variables:

$$\begin{aligned}s + t + u &= (p_a + p_b)^2 + (p_a - p_c)^2 + (p_a - p_d)^2 \\ &= m_a^2 + m_b^2 + m_c^2 + m_d^2 + 2p_a \cdot (p_a + p_b - p_c - p_d) \\ &= m_a^2 + m_b^2 + m_c^2 + m_d^2\end{aligned}$$

Bounds on Mandelstam variables determine physical region for s, t, u

$$\begin{aligned}s &\geq \max((m_a + m_b)^2, (m_c + m_d)^2) \\ t &= (p_a - p_c)^2 = m_a^2 + m_c^2 - 2p_a \cdot p_c = 2(m_a^2 + m_c^2) - (p_a + p_c)^2 \\ &\leq 2(m_a^2 + m_c^2) - (m_a + m_c)^2 = (m_a - m_c)^2\end{aligned}$$

Similarly using p_b and p_d ; same approach for u

$$t \leq \min((m_a - m_c)^2, (m_b - m_d)^2) \quad u \leq \min((m_a - m_d)^2, (m_b - m_c)^2)$$

Lower bound from this and $t|u = m_a^2 + m_b^2 + m_c^2 + m_d^2 - s - u|t$

$$\begin{aligned}t &\geq \max(m_b^2 + m_c^2 + 2m_a m_d, m_a^2 + m_d^2 + 2m_b m_c) - s \\ u &\geq \max(m_a^2 + m_c^2 + 2m_b m_d, m_b^2 + m_d^2 + 2m_a m_c) - s\end{aligned}$$

MANDELSTAM VARIABLES (CONTD.)

Simplification if $m_a = m_b$, $m_c = m_d \Rightarrow E_a^* = E_b^* = E_c^* = E_d^* = \frac{\sqrt{s}}{2}$

$$t = m_a^2 + m_c^2 - \frac{s}{2} \left(1 - \cos \theta^* \sqrt{1 - \frac{4m_a^2}{s}} \sqrt{1 - \frac{4m_c^2}{s}} \right)$$

If also $m_a = m_c \equiv m$

$$t = 2m^2 - \frac{s}{2} \left(1 - \cos \theta^* \left(1 - \frac{4m^2}{s} \right) \right) = - (s - 4m^2) \sin^2 \frac{\theta^*}{2}$$

$$\boxed{s \geq 4m^2 \quad - (s - 4m^2) \leq t \leq 0}$$

- Upper limit: at threshold $s = 4m^2$ or when $\theta^* = 0$ (fwd scatter)
- Lower limit: when $\theta^* = \pi$ (backscattering)
- In this case $u(s, \theta^*) = t(s, \pi - \theta^*) \Rightarrow$ same bound applies to u ; role of $\theta^* = 0$ and $\theta^* = \pi$ exchanged

Relevant for

- elastic processes involving only one type of particles/antiparticles
- very high energy limit (masses negligible, particles \approx massless)

EXAMPLE: PROTON-PROTON SCATTERING

Elastic pp scattering, $\sqrt{s} = 53$ GeV

Differential cross section $\frac{d\sigma}{dt}(t)$ has a peak at $-t = t_0 = 1.81$ GeV²

E. Nagy *et al.*, Nucl. Phys. **B150** (1979) 221

Q. What is the corresponding scattering angle in the CM?

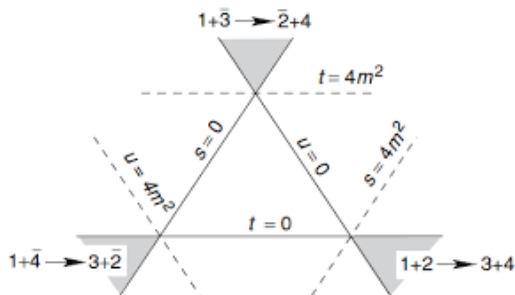
A. Elastic scattering of identical particles, $s/m_p^2 \gg 1$

$$-t = (s - 4m_p^2) \sin^2 \frac{\theta^*}{2} \simeq s \sin^2 \frac{\theta^*}{2}$$

$$\sin^2 \frac{\theta^*}{2} = -\frac{t}{s-4m_p^2} = \frac{1.81}{53^2-4\cdot 0.938^2} = \frac{1.81}{2805} = 6.45 \cdot 10^{-4}$$

$$\sin^2 \frac{\theta^*}{2} \simeq \frac{(\theta^*)^2}{4} \implies \theta^* \simeq 2\sqrt{5} \cdot 10^{-2} \simeq 5 \cdot 10^{-2}$$

MANDELSTAM PLANE



- Sides of eq. triangle: $s = 0$, $t = 0$ and $u = 0$ axes
- For appropriate side length $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$
- Physical region for the $a + b \rightarrow c + d$ process (equal masses) = wedge defined by the prolongation of the u and t axes

CROSSING SYMMETRY

QFT result: scattering amplitudes for $a + b \rightarrow c + d$, $a + \bar{c} \rightarrow \bar{b} + d$, $a + \bar{d} \rightarrow c + \bar{b}$ are part of a single analytic function extending beyond physical momenta, and related to each other

$$A_{ab \rightarrow cd}(p_a, p_b; p_c, p_d) = A_{a\bar{c} \rightarrow \bar{b}d}(p_a, -p_c; -p_b, p_d) = A_{a\bar{d} \rightarrow c\bar{b}}(p_a, -p_d; p_c, -p_b)$$

Use Mandelstam variables

$$a + b \rightarrow c + d \quad \mathcal{A}_s(s, t, u) = A_{ab \rightarrow cd}(p_a, p_b; p_c, p_d) \quad \text{s-channel}$$

$$a + \bar{c} \rightarrow \bar{b} + d \quad \mathcal{A}_t(s_t, t_t, u_t) = A_{a\bar{c} \rightarrow \bar{b}d}(p_a, p_{\bar{c}}; p_{\bar{b}}, p_d) \quad \text{t-channel}$$

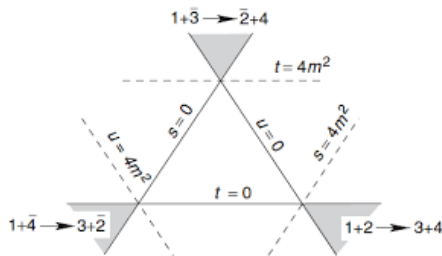
$$a + \bar{d} \rightarrow c + \bar{b} \quad \mathcal{A}_u(s_u, t_u, u_u) = A_{a\bar{d} \rightarrow c\bar{b}}(p_a, p_{\bar{d}}; p_c, p_{\bar{b}}) \quad \text{u-channel}$$

$$\begin{aligned} s &= (p_a + p_b)^2 & t &= (p_a - p_c)^2 & u &= (p_a - p_d)^2 \\ s_t &= (p_a + p_{\bar{c}})^2 & t_t &= (p_a - p_{\bar{b}})^2 & u_t &= (p_a - p_d)^2 \\ s_u &= (p_a + p_{\bar{d}})^2 & t_u &= (p_a - p_c)^2 & u_u &= (p_a - p_{\bar{b}})^2 \end{aligned}$$

Crossing-symmetry relations

$$\mathcal{A}_s(s, t, u) = \mathcal{A}_t(t, s, u) = \mathcal{A}_u(u, t, s)$$

CROSSING SYMMETRY (CONTD.)



$$\mathcal{A}_s(s, t, u) = \mathcal{A}_t(t, s, u) = \mathcal{A}_u(u, t, s)$$

- If s, t, u take physical values for the s -channel process $ab \rightarrow cd$, crossing relations involve \mathcal{A}_t and \mathcal{A}_u at unphysical values of their arguments

- Relations fully meaningful if \mathcal{A}_s can be analytically continued outside the physical domain
- For equal masses, physical regions of \mathcal{A}_t and \mathcal{A}_u are $s_t \geq 4m^2, t_t \leq 0$ and $s_u \geq 4m^2, t_u \leq 0$, but $t \leq 0$ and $s \geq 4m^2$

Physical regions = wedges outside Mandelstam triangle

INVARIANT PHASE SPACE

States of spinless particle, mass m are characterised by four-momenta p^μ with $p^2 = m^2$ and positive energy $p^0 \geq m > 0$

One-particle phase space:

$$\{p \in \mathbb{R}^4 | p^2 - m^2 = 0, p^0 > 0\} \subset \mathbb{R}^4$$

Measure of infinitesimal element of phase space

$$d\Phi^{(1)} = \frac{d^4 p}{(2\pi)^4} 2\pi \delta(p^2 - m^2) \theta(p^0)$$

- Manifestly invariant under orthochronous Lorentz transformations:
 p^2 invariant, $\text{sign}(p^0)$ invariant under orthochronous transformations
- Overall scale appropriate for relativistic normalisation of one-particle states: $\langle \vec{p}' | \vec{p} \rangle = (2\pi)^3 2p^0 \delta^{(3)}(\vec{p}' - \vec{p})$

INVARIANT PHASE SPACE (CONTD.)

Recast $d\Phi^{(1)}$ in more convenient form: for any f with simple zeros $\{x_n\}$

$$\delta(f(x)) = \sum_{x_n, f(x_n)=0} \frac{1}{|f'(x_n)|} \delta(x - x_n)$$

- multiply both sides by some function $h(x)$, integrate over \mathbb{R} , show that one gets the same result
- divide $\mathbb{R} = (-\infty, +\infty) = \cup_k I_k$ with $f(x)$ monotonic in I_k
 $\Rightarrow f$ invertible in I_k and vanishes at most once ($|f'| \neq 0$ there)
- set $y = f(x) \rightarrow x = f^{-1}(y)$ in each I_k

$$\begin{aligned} \int_{-\infty}^{+\infty} dx \delta(f(x)) h(x) &= \sum_k \int_{I_k} dx \delta(f(x)) h(x) \\ &= \sum_k \int_{f(I_k)} dy \frac{1}{|f'(f^{-1}(y))|} \delta(y) h(f^{-1}(y)) \\ &= \sum_k \int_{0 \in f(I_k)} dy \frac{1}{|f'(f^{-1}(0))|} \delta(y) h(f^{-1}(0)) = \sum_n \frac{1}{|f'(x_n)|} h(x_n) \end{aligned}$$

INVARIANT PHASE SPACE (CONTD.)

$$\begin{aligned}d\Phi^{(1)} &= \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2) \theta(p^0) = \frac{d^4p}{(2\pi)^3} \delta(p^{02} - \vec{p}^2 - m^2) \theta(p^0) \\ &= \frac{d^4p}{(2\pi)^3} \frac{1}{2|p^0|} [\delta(p^0 - \varepsilon(\vec{p})) + \delta(p^0 + \varepsilon(\vec{p}))] \theta(p^0) \\ &= \frac{d^4p}{(2\pi)^3} \frac{1}{2\varepsilon(\vec{p})} \delta(p^0 - \varepsilon(\vec{p})) \theta(p^0) = \frac{d^3p}{(2\pi)^3 2\varepsilon(\vec{p})} \equiv d\Omega_p \\ \varepsilon(\vec{p}) &\equiv \sqrt{\vec{p}^2 + m^2}\end{aligned}$$

n -particle phase space $\subset \mathbb{R}^{4n}$ corresponding to four-momenta of n particles subjected to a constraint on the total four-momentum
Measure of infinitesimal element:

$$d\Phi^{(n)} = \prod_{j=1}^n d\Omega_{p_j} (2\pi)^4 \delta^{(4)}(p_{\text{tot}} - \sum_{j=1}^n p_j)$$

Lorentz invariant: $d\Omega_{p_j}$ Lorentz invariant

$$\delta^{(4)}(\Lambda P) = |\det \Lambda|^{-1} \delta^{(4)}(P) = \delta^{(4)}(P)$$

INVARIANT PHASE SPACE: 2-PARTICLE CASE

Total momentum $p_{\text{tot}} = (E_{\text{tot}}, \vec{p}_{\text{tot}})$, particle energies

$$\varepsilon_i(\vec{p}) = \sqrt{\vec{p}^2 + m_i^2}$$

$$\begin{aligned} d\Phi^{(2)} &= \frac{d^3 p_1}{(2\pi)^3 2\varepsilon_1(\vec{p}_1)} \frac{d^3 p_2}{(2\pi)^3 2\varepsilon_2(\vec{p}_2)} (2\pi)^4 \delta^{(4)}(p_{\text{tot}} - p_1 - p_2) \\ &= \frac{1}{(2\pi)^2} \frac{d^3 p_1}{2\varepsilon_1(\vec{p}_1)} \frac{d^3 p_2}{2\varepsilon_2(\vec{p}_2)} \delta^{(3)}(\vec{p}_{\text{tot}} - \vec{p}_1 - \vec{p}_2) \delta(E_{\text{tot}} - \varepsilon_1(\vec{p}_1) - \varepsilon_2(\vec{p}_2)) \end{aligned}$$

Integrate trivially over \vec{p}_2 , setting it equal to $\vec{p}_2 = \vec{p}_{\text{tot}} - \vec{p}_1$

$$d\Phi^{(2)} = \frac{1}{(2\pi)^2} \frac{d^3 p_1}{2\varepsilon_1(\vec{p}_1)} \frac{1}{2\varepsilon_2(\vec{p}_{\text{tot}} - \vec{p}_1)} \delta(E_{\text{tot}} - \varepsilon_1(\vec{p}_1) - \varepsilon_2(\vec{p}_{\text{tot}} - \vec{p}_1))$$

To further integrate over $|\vec{p}_1|$ requires changing variables, most easily done working in the CM

$$\vec{p}_{\text{tot,CM}} = 0 \Rightarrow \vec{p}_{1\text{CM}} = -\vec{p}_{2\text{CM}}, \quad |\vec{p}_{1\text{CM}}| = |\vec{p}_{2\text{CM}}| = p$$

Dropping “CM” in the following

INVARIANT PHASE SPACE: 2-PARTICLE CASE (CONTD.)

Delta function depends on $E_{\text{tot}} - \varepsilon_1(p) - \varepsilon_2(p)$

Dropped vector sign on $\pm\vec{p}$

$$\left| \frac{\Delta E}{\Delta E p} [E_{\text{tot}} - \varepsilon_1(p) - \varepsilon_2(p)] \right| = \left[\frac{p}{\varepsilon_1(p)} + \frac{p}{\varepsilon_2(p)} \right] = \frac{p}{\varepsilon_1(p)\varepsilon_2(p)} [\varepsilon_1(p) + \varepsilon_2(p)]$$

Changing variables to $d^3p_1 = dp p^2 d \cos \theta^* d\phi^* = dp p^2 d\Omega^*$

$$\begin{aligned} d\Phi^{(2)} &= \frac{1}{(2\pi)^2} \frac{dp p^2 d\Omega^*}{2\varepsilon_1(p)} \frac{1}{2\varepsilon_2(p)} \underbrace{\frac{\varepsilon_1(p)\varepsilon_2(p)}{p} [\varepsilon_1(p) + \varepsilon_2(p)]^{-1} \delta(p - p^*)}_{\delta(E_{\text{tot}} - \varepsilon_1(p) - \varepsilon_2(p))} \\ &= \frac{d\Omega^*}{(2\pi)^2} \frac{p^*}{4(\varepsilon_1(p^*) + \varepsilon_2(p^*))} = \frac{d\Omega^*}{(2\pi)^2} \frac{p^*}{4E_{\text{tot}}^*} = \frac{d\Omega^*}{16\pi^2} \frac{p^*}{\sqrt{s}} \\ &= \frac{d\Omega^*}{32\pi^2} \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{s} \end{aligned}$$

For equal masses $\lambda(s, m^2, m^2) = s(s - 4m^2) \Rightarrow d\Phi^{(2)} = \frac{d\Omega^*}{32\pi^2} \sqrt{\frac{s - 4m^2}{s}}$