HOMEWORK N. 4

Gauge theories

1. The *Wilson line* $W_{\mathcal{C}}$ associated with a path $\mathcal{C}$ connecting x_0 and x_1,

$$\mathcal{C} : [0, T] \to \mathbb{R}^4 \,, \qquad s \to x^\mu(s) \,,$$

$x(0) = x_0$ and $x(T) = x_1$, is defined as follows

$$W_{\mathcal{C}} \equiv \text{Texp}\left\{-ig \int_{\mathcal{C}} A_\mu(X) dX^\mu\right\} = \text{Texp}\left\{-ig \int_0^T ds\, A_\mu(x(s))\dot{x}^\mu(s)\right\} \,,$$

where $A_\mu = A_\mu^a t^a$ with $t^a = t^{a\dagger}$ Hermitean matrices providing a representation of the Lie algebra associated to the gauge group and A_μ^a the gauge fields; $\dot{x}^\mu = dx^\mu/ds$ the four-vector tangent to the path, and Texp the time-ordered exponential,

$$\text{Texp}\left\{\int_{s_0}^{s_1} ds\, f(s)\right\} = \sum_{n=0}^{\infty} \int_{s_0}^{s_1} ds_1 \int_{s_0}^{s_1} ds_2 \ldots \int_{s_0}^{s_1} ds_{n-1} \int_{s_0}^{s_1} ds_n$$
$$\times\, \theta(s_1 - s_2) \ldots \theta(s_{n-1} - s_n) f(s_1) f(s_2) \ldots f(s_{n-1}) f(s_n) \,.$$

where it is understood that the $n = 0$ term is just $\mathbf{1}$ (the identity matrix in the appropriate number of dimensions).

1.1. Let $W_{\mathcal{C}}(t)$ denote the Wilson line along $\mathcal{C}$ but only up to $x(t)$, $t < T$,

$$W_{\mathcal{C}}(t) \equiv \text{Texp}\left\{-ig \int_0^t ds\, A_\mu(x(s))\dot{x}^\mu(s)\right\} \,.$$

Show that $W_{\mathcal{C}}(t)$ obeys the following differential equation,

$$\frac{dW_{\mathcal{C}}(t)}{dt} = -igA_\mu(x(t))\dot{x}^\mu(t)W_{\mathcal{C}}(t) \,, \qquad (*)$$

with initial condition $W_{\mathcal{C}}(0) = \mathbf{1}$. (Hint: use the theta funtions in the definition of the time-ordered exponential to explicitly limit the integration ranges over $s_1, \ldots, s_n$.)

1.2. From Equation $(*)$ obtain a differential equation for $W_{\mathcal{C}}(t)^\dagger$. From the relation $W_{\mathcal{C}}(t)W_{\mathcal{C}}(t)^{-1} = \mathbf{1}$ derive a differential equation for $W_{\mathcal{C}}(t)^{-1}$. Compare the two equations and the corresponding initial conditions, and use uniqueness of the solution of the Cauchy problem to conclude that $W_{\mathcal{C}}(t)$ is a unitary matrix.

1.3. Under a finite gauge transformation $U(x) = e^{i\alpha_a(x)t^a}$, the gauge field $A_\mu(x)$ transforms as

$$A_\mu^U(x) = U(x)A_\mu(x)U(x)^\dagger + \frac{1}{ig}U(x)\partial_\mu U(x)^\dagger \,.$$

Imposing the axial gauge condition on A_μ^U, i.e., $A_3^U = 0$, derive a differential equation for $U(x)$, and write its solution in terms of a suitable (family of) Wilson lines.

2. Consider Faddeev-Popov quantisation of a gauge theory using the gauge-fixing functional

$$f^a[A;x] = n^\mu A_\mu^a \,.$$

2.1. Derive the Faddeev-Popov determinant corresponding to the above choice of $f^a[A;x]$, and its representation as a Grassmann integral over ghost and antighost fields.

2.2. Show that for the choice $B[f] = \prod_x \delta(f[A;x])$ for the functional implementing the gauge condition in the path integral, the ghosts decouple from the gauge fields.

2.3. Derive the gauge-field propagator in momentum space for the choice $B[f] = e^{\frac{i}{2\xi}\int d^4x\, f^a f^a}$. (Hint: the kernel of the quadratic part of the Lagrangean leaves invariant both the subspace spanned by the momentum p^μ and by n^μ and its orthogonal complement, and so must do its inverse. What does this tell you about the most general form of the propagator?)