HOMEWORK N. 3

Path integral methods

Consider a theory of a neutral scalar field ϕ defined by the generating functional

$$Z[J] = \int \mathrm{D}\phi \, e^{iS[\phi]+iJ\cdot\phi} \,, \qquad J\cdot\phi = \int d^4x \, J(x)\phi(x) \,.$$

Let us denote correlation functions as

$$\langle \phi(x_1)\ldots\phi(x_n)\rangle = \frac{\int \mathrm{D}\phi \, e^{iS[\phi]}\phi(x_1)\ldots\phi(x_n)}{\int \mathrm{D}\phi \, e^{iS[\phi]}} = \frac{1}{i}\frac{\delta}{\delta J(x_1)} \cdots \frac{1}{i}\frac{\delta}{\delta J(x_n)} \frac{Z[J]}{Z[0]}\bigg|_{J=0} \,.$$

1. Exploit invariance of the measure of integration in the definition of $Z[J]$ under the change of variables

$$\phi(x) = \bar{\phi}(x) + c(x) \,,$$

for an arbitrary function $c(x)$, to derive the following relation,

$$\left\langle \left(\frac{\delta S}{\delta\phi(x)} + J(x) \right) e^{iJ\cdot\phi} \right\rangle = 0 \,. \qquad (*)$$

2. By taking functional derivatives with respect to J and then setting $J = 0$, show that

$$\left\langle \frac{\delta S}{\delta\phi(x)}\phi(x_1)\ldots\phi(x_n) \right\rangle = i\sum_{j=1}^{n} \delta(x-x_j)\langle \underbrace{\phi(x_1)\ldots\phi(x_n)}_{\phi(x_j)\ \text{omitted}} \rangle \,.$$

This relation shows that the insertion into a generic correlation function of the equations of motion, $\frac{\delta S}{\delta\phi(x)}$, gives zero up to contact terms (i.e., terms proportional to delta functions).

3. For a generic local action S, $\frac{\delta S}{\delta\phi(x)} = S'[\phi(x)]$ is a special type of functional of $\phi(x)$, one which is a function only of $\phi(x)$ and its derivatives at x. Show that for a general functional F,

$$F[\phi] = \sum_{n=0}^{\infty} \int d^4x_1 \ldots \int d^4x_n \, f_n(x_1,\ldots,x_n)\phi(x_1)\ldots\phi(x_n)$$

one can write

$$\langle F[\phi]\phi(x_1)\ldots\phi(x_n)e^{iJ\cdot\phi}\rangle = F\left[\tfrac{1}{i}\tfrac{\delta}{\delta J}\right]\frac{Z[J]}{Z[0]} \,.$$

Specialise to $F[\phi] = S'[\phi(x)]$ to convert equation $(*)$ into an equation for $Z[J]$. This is the Schwinger-Dyson equation for the generating functional.

4. Specialise now to the $\lambda\phi^4$ theory with action

$$S[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi + \frac{\lambda}{4!}\phi^4 \right) .$$

Using either point 2 or point 4 above, derive a differential equation for the two-point function, or propagator,

$$D(x, y) = \langle \phi(x)\phi(y) \rangle .$$