HOMEWORK N. 2

LSZ reduction

Consider two types of massive spinless particles N and π of masses m_N and m_π, associated with interpolating Hermitean scalar fields $\phi_N(x)$ and $\phi_\pi(x)$, respectively.

1. Due to translation invariance, the scattering amplitude is proportional to a momentum-conserving delta function,

$$S_{N\to N'\pi} = {}_{\rm out}\langle N(p'_N)\pi(p_\pi)|N(p_N)\rangle_{\rm in} = (2\pi)^4\delta^{(4)}(p'_N + p_\pi - p_N)i\mathcal{M}(p_N, p'_N, p_\pi)\,. \qquad (*)$$

Show that for momenta satisfying the on-shell condition (i.e., $p_\alpha^2 = m_\alpha^2$) one can never satisfy the condition of momentum conservation.

2. One can use the LSZ reduction procedure to express $S_{N\to N'\pi}$ in terms of the interpolating fields. Perform explicitly, step by step, the LSZ reduction for the particle π, and write $S_{N\to N'\pi}$ in terms of the matrix element ${}_{\rm out}\langle N(p'_N)|\phi_\pi(z)|N(p_N)\rangle_{\rm in}$.

The expression for $S_{N\to N'\pi}$ in terms of the interpolating fields found at point 2 is well defined also for values of the four momentum p_π that are not on-shell. This provides a definition of the matrix element $S_{N\to N'\pi}$ off the mass shell. Again due to translation invariance, this quantity has the same form as in equation $(*)$, but momentum conservation can now be satisfied, and so in general $S^{\text{off-shell}}_{N\to N'\pi} \neq 0$. In particular, the on-shell limit of $\mathcal{M}(p_N, p'_N, p_\pi)$ starting from off-shell momenta is a well-defined and generally nonzero quantity.

3. Show that $p'_N + p_\pi - p_N = 0$ can be satisfied with p_N, p'_N on-shell if one takes p_π spacelike (hence unphysical).

4. Use translation invariance of the theory to factor out the delta function from $S^{\text{off-shell}}_{N\to N'\pi}$ and write $\mathcal{M}(p_N, p'_N, p_\pi)$ in terms of the matrix element ${}_{\rm out}\langle N(p'_N)|\phi_\pi(0)|N(p_N)\rangle_{\rm in}$. Using this result, show that the latter quantity has a pole at $(p'_N - p_N)^2 = m_\pi^2$. Is this pole in the region where p_N, p'_N are on-shell?