HOMEWORK N. 1

Källén–Lehmann representation of the two-point function

Consider a Hermitian scalar field $\phi(x)$. Assume that the spectrum of the theory contains only a single scalar particle of mass m: a complete set of asymptotic states contains the one particle states $|p\rangle$, with $p = (p^0, \vec{p})$ where the energy is given by $p^0(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$, and the many particles states $|p_1, \ldots, p_n\rangle_{\text{in,out}}$. Under translations and Lorentz transformations,

$$U(a)|p_1, \ldots, p_n\rangle = e^{-i\sum_j p_j \cdot a}|p_1, \ldots, p_n\rangle_{\text{in,out}} \,,$$
$$U(\Lambda)|p_1, \ldots, p_n\rangle = |\Lambda p_1, \ldots, \Lambda p_n\rangle_{\text{in,out}} \,.$$

Relativistic normalisation is assumed. Assume finally that

$$\langle 0|\phi(x)|0\rangle = 0 \,.$$

1. What is the minimal energy of the many-particle states?

2. Consider the two-point function

$$\langle 0|\phi(x)\phi(0)|0\rangle \,.$$

Inserting between the two operators a complete set of states,

$$\mathbf{1} = |0\rangle\langle 0| + \int d\Omega_p \, |p\,\rangle\langle p| + \tfrac{1}{2}\int d\Omega_{p_1} \int d\Omega_{p_2} \, |p_1 p_2\,\rangle\langle p_1 p_2| + \ldots \equiv \sum_n |n\rangle\langle n| \,,$$

(the last equality just defines the short-hand notation) with $d\Omega_p$ the usual invariant phase-space measure, and exploiting translation and Lorentz invariance show that

$$\langle 0|\phi(x)\phi(0)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \, e^{-ip\cdot x} f(p^2) \,,$$

where f depends only on p^2, has support only on $p^0 \geq m \geq 0$, and is positive,

$$f(p^2) = 2\pi\theta(p^0)\rho(p^2) \,, \qquad \rho(p^2) \geq 0 \,.$$

(The factor of 2π is conventional.)

3. Inserting the identity in the form

$$1 = \int_{-\infty}^{\infty} ds \, \delta(s - p^2)$$

and exchanging order of integration show that

$$\langle 0|\phi(x)\phi(0)|0\rangle = \int_{m^2}^{\infty} ds\,\rho(s)\int d\Omega_{p^{(s)}}\, e^{-ip^{(s)}\cdot x}\,,$$

where

$$d\Omega_{p^{(s)}} = \frac{d^3p}{(2\pi)^3 2p^{(s)0}}\,, \qquad p^{(s)} = (\sqrt{\vec{p}^2+s},\vec{p})$$

is the invariant phase-space measure for a particle of mass squared s.

4. Using translation invariance show that

$$\langle 0|\phi(0)\phi(x)|0\rangle = \int_{m^2}^{\infty} ds\,\rho(s)\int d\Omega_{p^{(s)}}\, e^{ip^{(s)}\cdot x}\,.$$

5. Use the results above to show that

$$\langle 0|[i\partial_0\phi(x),\phi(0)]|0\rangle = \int_{m^2}^{\infty} ds\,\rho(s)\int \frac{d^3p}{2(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\left(e^{-ip^{(s)0}x^0} + e^{ip^{(s)0}x^0}\right)\,,$$

and in particular that

$$\langle 0|[i\partial_0\phi(x),\phi(0)]|0\rangle\big|_{x^0=0} = \delta^{(3)}(\vec{x})\int_{m^2}^{\infty} ds\,\rho(s)\,.$$

6. Assuming now that the scalar field is obtained via canonical quantisation from a Lagrangian of the form

$$\mathcal{L} = \tfrac{1}{2}\mathcal{Z}\partial_\mu\phi\partial^\mu\phi + \mathcal{U}(\phi)\,,$$

show that one must have

$$\mathcal{Z}\int_{m^2}^{\infty} ds\,\rho(s) = 1\,.$$

Assuming that the integral is convergent, what does this imply for the sign of $\mathcal{Z}$?

7. Using points 3 and 4 it is trivial to show that

$$\langle 0|T\{\phi(x)\phi(0)\}|0\rangle = \int_{m^2}^{\infty} ds\,\rho(s)\int d\Omega_{p^{(s)}}\,\left(\theta(x^0)e^{-ip^{(s)}\cdot x} + \theta(-x^0)e^{ip^{(s)}\cdot x}\right)\,.$$

Using the residue theorem show that

$$\int \frac{dp^0}{2\pi}\frac{i}{(p^0)^2-\vec{p}^2-s+i\epsilon}e^{-ip^0x^0} = \frac{1}{2\sqrt{\vec{p}^2+s}}\left(\theta(x^0)e^{-ip^{(s)}\cdot x} + \theta(-x^0)e^{ip^{(s)}\cdot x}\right)\,.$$

Use this to conclude that

$$\langle 0|T\{\phi(x)\phi(0)\}|0\rangle = \int_{m^2}^{\infty} ds\,\rho(s)\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\frac{i}{p^2-s+i\epsilon} = \int_{m^2}^{\infty} ds\,\rho(s)\Delta_{\rm F}(x;s)\,,$$

where $\Delta_{\rm F}(x;s)$ denotes the Feynman propagator for a scalar particle of mass squared s. This is the **Källén–Lehmann representation** of the scalar propagator.

8. Using what you know about the spectrum, show that

$$\rho(s) = Z\delta(s - m^2) + \sigma(s)\theta(s - 4m^2)\,,$$

with positive Z and $\sigma(s)$, and so

$$\langle 0|T\{\phi(x)\phi(0)\}|0\rangle = Z\Delta_{\mathrm{F}}(x; m^2) + \int_{4m^2}^{\infty} ds\, \sigma(s)\Delta_{\mathrm{F}}(x; s)\,.$$

Using the result obtain above at point 6, show that for $\mathcal{Z} = 1$

$$Z + \int_{4m^2}^{\infty} ds\, \sigma(s) = 1\,,$$

and conclude that

$$0 \leq Z \leq 1\,.$$

Can you think of a case in which $Z = 1$?

9. Using the explicit form of $\rho(s)$ found in the previous point, show that the momentum-space propagator,

$$G_2(p) = \int d^4x\, e^{ip\cdot x} \langle 0|T\{\phi(x)\phi(0)\}|0\rangle\,,$$

has a pole at $p^2 = m^2$.