Advanced field theory

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Lecture 3 or: LSZ reduction formula

In/out states: energy/momentum eigenstates of the full theory

$$|p_1, \ldots, p_n\rangle_{\mathrm{in,out}}$$

Transform like free multiparticle states (but are fully interacting states!)

$$e^{-iP\cdot a}|p_1, \ldots, p_n\rangle_{\mathrm{in,out}} = e^{-i\sum_j p_j\cdot a}|p_1, \ldots, p_n\rangle_{\mathrm{in,out}}$$
$$U(\Lambda)|p_1, \ldots, p_n\rangle_{\mathrm{in,out}} = |\Lambda p_1, \ldots, \Lambda p_n\rangle_{\mathrm{in,out}}$$

Correspond to initial/final states of scattering experiments
(time-independent states of non-interacting particles)

Vacuum and one-particle states are stable

$$|p\rangle = |p\rangle_{\mathrm{in}} = |p\rangle_{\mathrm{out}} \qquad |0\rangle = |0\rangle_{\mathrm{in}} = |0\rangle_{\mathrm{out}}$$

- $\phi_{\mathrm{in,out}}|0\rangle_{\mathrm{in,out}} \in$ 1-particle subspace (see lecture 2)
- $|0\rangle_{\mathrm{in,out}} = |0\rangle$ since translation invariant, and such state is unique

Associated asymptotic fields

$$(\Box + m^2)\phi_{\rm in,out} = 0 \qquad \phi_{\rm in,out}(x) = e^{iP\cdot x}\phi_{\rm in,out}(0)e^{-iP\cdot x}$$

$$\phi_{\rm in,out}(x) = \int d\Omega_p \left\{ a_{\rm in,out}(p)e^{-ip\cdot x} + a_{\rm in,out}(p)^\dagger e^{ip\cdot x} \right\}$$

Full field assumed to interpolate between $\phi_{\rm in}$ and $\phi_{\rm out}$

$$\phi(x) \to \sqrt{Z}\phi_{\rm in,out}(x) \quad \text{as} \quad x^0 \to \mp\infty \quad \text{in the weak sense}$$

LSZ condition: for any normalisable $|\alpha\rangle, |\beta\rangle$ and f, $(\Box + m^2)f = 0$

$$\boxed{\begin{aligned} \lim_{t\to-\infty} \langle\alpha|\phi^f(t)|\beta\rangle &= \sqrt{Z}\langle\alpha|\phi^f_{\rm in}|\beta\rangle \\ \lim_{t\to+\infty} \langle\alpha|\phi^f(t)|\beta\rangle &= \sqrt{Z}\langle\alpha|\phi^f_{\rm out}|\beta\rangle \end{aligned}}$$

$$\phi^f(t) \equiv i\int d^3x\, f(x)^* \overleftrightarrow{\partial}_0 \phi(t,\vec{x}) \qquad \phi^f_{\rm in,out} \equiv i\int d^3x\, f(x)^* \overleftrightarrow{\partial}_0 \phi_{\rm in,out}(t,\vec{x})$$

Field normalisation constant:

$$\langle 0|\phi(x)|p\rangle = \sqrt{Z}e^{-ip\cdot x} = \sqrt{Z}\langle 0|\phi_{\rm in}(x)|p\rangle = \sqrt{Z}\langle 0|\phi_{\rm out}(x)|p\rangle$$

S-matrix: transition amplitude from initial to final multiparticle state

$$S_{\beta\alpha} = {}_{\rm out}\langle p'_1, \ldots, p'_{n'} | p_1, \ldots, p_n \rangle_{\rm in} = {}_{\rm out}\langle \beta | \alpha \rangle_{\rm in}$$

α, β should be taken normalisable first, then take limit of momentum eigenstates

Main properties of the S-matrix

- in/out states assumed complete $\Rightarrow$ $S_{\beta\alpha}$ unitary matrix

$$ {}_{\rm out}\langle \beta | = \sum_\alpha (S_{\beta\alpha})\, {}_{\rm in}\langle \alpha | \equiv {}_{\rm in}\langle \beta | S $$

$$\begin{aligned}
\delta_{\beta\alpha} &= {}_{\rm out}\langle \beta | \alpha \rangle_{\rm out} = {}_{\rm in}\langle \beta | S S^\dagger | \alpha \rangle_{\rm in} && \Rightarrow S S^\dagger = \mathbf{1} \\
{}_{\rm out}\langle \beta | S^\dagger S | \alpha \rangle_{\rm out} &= {}_{\rm in}\langle \beta | S S^\dagger S | \alpha \rangle_{\rm out} = {}_{\rm in}\langle \beta | S | \alpha \rangle_{\rm out} \\
&= {}_{\rm out}\langle \beta | \alpha \rangle_{\rm out} = \delta_{\beta\alpha} && \Rightarrow S^\dagger S = \mathbf{1}
\end{aligned}$$

- Poincaré invariant $U(a,\Lambda)^\dagger S U(a,\Lambda) = S$

$$S_{(\Lambda\beta)(\Lambda\alpha)} = {}_{\rm out}\langle\Lambda\beta|\Lambda\alpha\rangle_{\rm in} = {}_{\rm out}\langle\beta|U(\Lambda)^\dagger U(\Lambda)|\alpha\rangle_{\rm in} = S_{\beta\alpha}$$
$$S_{(a\beta)(a\alpha)} = {}_{\rm out}\langle\beta|e^{i\sum_{j'} p'_{j'}a}e^{-i\sum_j p_j a}|\alpha\rangle_{\rm in} = {}_{\rm out}\langle\beta|U(a)^\dagger U(a)|\alpha\rangle_{\rm in} = S_{\beta\alpha}$$

- stability of vacuum and 1-particle states

$$S_{00} = {}_{\rm out}\langle 0|0\rangle_{\rm in} = \langle 0|0\rangle = 1$$
$$S_{p'p} = {}_{\rm out}\langle p'|p\rangle_{\rm in} = \langle p'|p\rangle = (2\pi)^3 2p^0\delta^{(3)}(\vec{p}\,' - \vec{p})$$

- S connects in and out fields: since $|\beta\rangle_{\rm out}$ complete

$$\begin{aligned} {}_{\rm out}\langle\beta p| &= {}_{\rm out}\langle\beta|a_{\rm out}(p) = {}_{\rm in}\langle\beta p|S = {}_{\rm in}\langle\beta|a_{\rm in}(p)S \\ &= {}_{\rm in}\langle\beta|SS^\dagger a_{\rm in}(p)S = {}_{\rm out}\langle\beta|S^\dagger a_{\rm in}(p)S \\ &\Rightarrow S^\dagger a_{\rm in}(p)S = a_{\rm out}(p) \\ &\Rightarrow S^\dagger \phi_{\rm in}(x)S = \phi_{\rm out}(x) \end{aligned}$$

Reduction formula

$$
\begin{aligned}
{}_{\rm out}\langle\beta|\alpha p\rangle_{\rm in} &= {}_{\rm out}\langle\beta|a_{\rm in}(p)^\dagger|\alpha\rangle_{\rm in} \\
&= {}_{\rm out}\langle\beta|a_{\rm out}(p)^\dagger|\alpha\rangle_{\rm in} + {}_{\rm out}\langle\beta|a_{\rm in}(p)^\dagger - a_{\rm out}(p)^\dagger|\alpha\rangle_{\rm in} \\
&= {}_{\rm out}\langle\beta - p|\alpha\rangle_{\rm in} - i\int d^3x\, e^{-ip\cdot x}\,\overleftrightarrow{\partial}_0\;{}_{\rm out}\langle\beta|\phi_{\rm in}(x) - \phi_{\rm out}(x)|\alpha\rangle_{\rm in}
\end{aligned}
$$

$$
\begin{aligned}
&{}_{\rm out}\langle\beta|\alpha p\rangle_{\rm in} - {}_{\rm out}\langle\beta - p|\alpha\rangle_{\rm in} \\
\text{use } t\text{-independence}\quad &= -i\lim_{x_0\to-\infty}\int d^3x\, e^{-ip\cdot x}\,\overleftrightarrow{\partial}_0\;{}_{\rm out}\langle\beta|\phi_{\rm in}(x)|\alpha\rangle_{\rm in} \\
&\quad + i\lim_{x_0\to+\infty}\int d^3x\, e^{-ip\cdot x}\,\overleftrightarrow{\partial}_0\;{}_{\rm out}\langle\beta|\phi_{\rm out}(x)|\alpha\rangle_{\rm in} \\
\text{use LSZ condition}\quad &= -\frac{i}{\sqrt{Z}}\lim_{x_0\to-\infty}\int d^3x\, e^{-ip\cdot x}\,\overleftrightarrow{\partial}_0\;{}_{\rm out}\langle\beta|\phi(x)|\alpha\rangle_{\rm in} \\
&\quad + \frac{i}{\sqrt{Z}}\lim_{x_0\to+\infty}\int d^3x\, e^{-ip\cdot x}\,\overleftrightarrow{\partial}_0\;{}_{\rm out}\langle\beta|\phi(x)|\alpha\rangle_{\rm in} \\
&= \frac{i}{\sqrt{Z}}\int d^4x\,\partial_0\left\{e^{-ip\cdot x}\,\overleftrightarrow{\partial}_0\;{}_{\rm out}\langle\beta|\phi(x)|\alpha\rangle_{\rm in}\right\}
\end{aligned}
$$

$$
\begin{aligned}
{}_{\rm out}\langle\beta|\alpha p\rangle_{\rm in} - {}_{\rm out}\langle\beta - p|\alpha\rangle_{\rm in} \\
&= \frac{i}{\sqrt{Z}}\int d^4x\,\Big\{e^{-ip\cdot x}\,\partial_0^2\;{}_{\rm out}\langle\beta|\phi(x)|\alpha\rangle_{\rm in} \\
&\qquad\qquad - {}_{\rm out}\langle\beta|\phi(x)|\alpha\rangle_{\rm in}\partial_0^2 e^{-ip\cdot x}\Big\} \\
&= \frac{i}{\sqrt{Z}}\int d^4x\,\Big\{e^{-ip\cdot x}\,\partial_0^2\;{}_{\rm out}\langle\beta|\phi(x)|\alpha\rangle_{\rm in} \\
&\qquad\qquad - {}_{\rm out}\langle\beta|\phi(x)|\alpha\rangle_{\rm in}(\vec{\nabla}^2 - m^2)e^{-ip\cdot x}\Big\}
\end{aligned}
$$

use $(\Box + m^2)e^{-ip\cdot x} = 0$

integration by parts

$$
\begin{aligned}
&= \frac{i}{\sqrt{Z}}\int d^4x\,e^{-ip\cdot x}\,(\partial_0^2 - \vec{\nabla}^2 + m^2)\;{}_{\rm out}\langle\beta|\phi(x)|\alpha\rangle_{\rm in} \\
&= \frac{i}{\sqrt{Z}}\int d^4x\,e^{-ip\cdot x}\,(\Box + m^2)\;{}_{\rm out}\langle\beta|\phi(x)|\alpha\rangle_{\rm in}
\end{aligned}
$$

Same for out particle

$$
\begin{aligned}
{}_{\text{out}}\langle\beta|\phi(x)|\alpha\rangle_{\text{in}} &= {}_{\text{out}}\langle\gamma p'|\phi(x)|\alpha\rangle_{\text{in}} = {}_{\text{out}}\langle\gamma|a_{\text{out}}(p')\phi(x)|\alpha\rangle_{\text{in}} \\
&= {}_{\text{out}}\langle\gamma|a_{\text{out}}(p')\phi(x) - \phi(x)a_{\text{in}}(p')|\alpha\rangle_{\text{in}} + {}_{\text{out}}\langle\gamma|\phi(x)a_{\text{in}}(p')|\alpha\rangle_{\text{in}} \\
&= {}_{\text{out}}\langle\gamma|a_{\text{out}}(p')\phi(x) - \phi(x)a_{\text{in}}(p')|\alpha\rangle_{\text{in}} + {}_{\text{out}}\langle\gamma|\phi(x)|\alpha - p'\rangle_{\text{in}}
\end{aligned}
$$

$$
\begin{aligned}
{}_{\text{out}}\langle\beta|\phi(x)|\alpha\rangle_{\text{in}} - {}_{\text{out}}\langle\gamma|\phi(x)|\alpha - p'\rangle_{\text{in}} \\
&= i\int d^3y\, e^{ip\cdot y}\,\overleftrightarrow{\partial}_0{}^{y}\ {}_{\text{out}}\langle\gamma|\phi_{\text{out}}(y)\phi(x) - \phi(x)\phi_{\text{in}}(y)|\alpha\rangle_{\text{in}} \\
&= \lim_{y^0\to+\infty} i\int d^3y\, e^{ip\cdot y}\,\overleftrightarrow{\partial}_0{}^{y}\ {}_{\text{out}}\langle\gamma|\phi_{\text{out}}(y)\phi(x)|\alpha\rangle_{\text{in}} \\
&\quad - \lim_{y^0\to-\infty} i\int d^3y\, e^{ip\cdot y}\,\overleftrightarrow{\partial}_0{}^{y}\ {}_{\text{out}}\langle\gamma|\phi(x)\phi_{\text{in}}(y)|\alpha\rangle_{\text{in}} \\
&= \lim_{y^0\to+\infty} \frac{i}{\sqrt{Z}}\int d^3y\, e^{ip\cdot y}\,\overleftrightarrow{\partial}_0{}^{y}\ {}_{\text{out}}\langle\gamma|\phi(y)\phi(x)|\alpha\rangle_{\text{in}} \\
&\quad - \lim_{y^0\to-\infty} \frac{i}{\sqrt{Z}}\int d^3y\, e^{ip\cdot y}\,\overleftrightarrow{\partial}_0{}^{y}\ {}_{\text{out}}\langle\gamma|\phi(x)\phi(y)|\alpha\rangle_{\text{in}} \\
\text{T-ordered} \Rightarrow \text{OK} \quad &= \frac{i}{\sqrt{Z}}\int d^4y\, \partial_0{}^{y}\left\{e^{ip\cdot y}\,\overleftrightarrow{\partial}_0{}^{y}\ {}_{\text{out}}\langle\gamma|\mathrm{T}\{\phi(y)\phi(x)\}|\alpha\rangle_{\text{in}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&{}_{\rm out}\langle\beta|\phi(x)|\alpha\rangle_{\rm in} - {}_{\rm out}\langle\gamma|\phi(x)|\alpha-p'\rangle_{\rm in} \\
&\qquad = \frac{i}{\sqrt{Z}}\int d^4y\, \partial_0{}^y \left\{ e^{ip'\cdot y}\overleftrightarrow{\partial_0}{}^y\ {}_{\rm out}\langle\gamma|\mathrm{T}\{\phi(y)\phi(x)\}|\alpha\rangle_{\rm in}\right\} \\
&\qquad = \frac{i}{\sqrt{Z}}\int d^4y\, \left\{ e^{ip'\cdot y}(\partial_0{}^y)^2\ {}_{\rm out}\langle\gamma|\mathrm{T}\{\phi(y)\phi(x)\}|\alpha\rangle_{\rm in}\right. \\
&\qquad\qquad\qquad \left. - {}_{\rm out}\langle\gamma|\mathrm{T}\{\phi(y)\phi(x)\}|\alpha\rangle_{\rm in}(\vec{\nabla}_y^2 - m^2)e^{ip'\cdot y}\right\} \\
&\qquad = \frac{i}{\sqrt{Z}}\int d^4y\, e^{ip'\cdot y}(\Box_y + m^2)_{\rm out}\langle\gamma|\mathrm{T}\{\phi(y)\phi(x)\}|\alpha\rangle_{\rm in}
\end{aligned}
$$

For $\alpha p \cap \gamma p' = \emptyset$

$$
\begin{aligned}
{}_{\rm out}\langle\gamma p'|\alpha p\rangle_{\rm in} &= \left(\frac{i}{\sqrt{Z}}\right)^2 \int d^4x\, e^{-ip\cdot x}\int d^4y\, e^{ip'\cdot y} \\
&\qquad \times (\Box_x + m^2)(\Box_y + m^2)\ {}_{\rm out}\langle\gamma|\mathrm{T}\{\phi(y)\phi(x)\}|\alpha\rangle_{\rm in}
\end{aligned}
$$

Repeat for all particles

LSZ reduction formula

$$
{}_{\rm out}\langle p_1', \ldots, p_{n'}' | p_1, \ldots, p_n \rangle_{\rm in}
= \prod_{j=1}^{n} \int d^4 x_j \, e^{-ip_j \cdot x_j} \left(\frac{\Box_{x_j} + m^2}{-i\sqrt{Z}} \right) \prod_{j'=1}^{n'} \int d^4 y_{j'} \, e^{ip_{j'}' \cdot y_{j'}} \left(\frac{\Box_{y_{j'}} + m^2}{-i\sqrt{Z}} \right)
\times \ \langle 0 | \mathrm{T}\{\phi(y_1) \ldots \phi(y_{n'}) \phi(x_1) \ldots \phi(x_n)\} | 0 \rangle
$$

if $\{p_1', \ldots, p_{n'}'\} \cap \{p_1, \ldots, p_n\} = \emptyset$
otherwise also disconnected pieces

- S-matrix from Green's functions of the interacting fields
- Green's functions symmetric $\Rightarrow$ manifest crossing symmetry
- invariant under rescaling of $\phi \to \sqrt{z}\phi$ since also $\sqrt{Z} \to \sqrt{zZ}$
- any local operator $\mathcal{O}(x)$ with $\langle 0 | \mathcal{O}(0) | p \rangle \neq 0$ can be used

Momentum space

$$G_{n+n'}(y,x) = \langle 0|\mathrm{T}\{\phi(y_1)\ldots\phi(y_{n'})\phi(x_1)\ldots\phi(x_n)\}|0\rangle$$

$$(2\pi)^4\delta^{(4)}(p'-p)\tilde{G}_{n+n'}(-p',p) = \prod_{j=1}^{n}\int d^4x_j e^{-ip_j\cdot x_j}\prod_{j'=1}^{n'}\int d^4y_{j'} e^{ip'_{j'}\cdot y_{j'}}\, G_{n+n'}(y,x)$$

$${}_{\mathrm{out}}\langle p'_1,\ldots,p'_{n'}|p_1,\ldots,p_n\rangle_{\mathrm{in}}$$

$$= (2\pi)^4\delta^{(4)}(p'-p)\lim_{p^2,p'^2\to m^2}\prod_{j=1}^{n}\left(\frac{p_j^2-m^2}{i\sqrt{Z}}\right)\prod_{j'=1}^{n'}\left(\frac{p'^2_{j'}-m^2}{i\sqrt{Z}}\right)\tilde{G}_{n+n'}(-p',p)$$

- S-matrix = residues of the multi-poles of momentum-space Green's functions at $p_j^2, p'^2_{j'} \to m^2$ (on-shell) (physical mass)

$$\Updownarrow$$

remove a $\sqrt{Z}\times$ free propagator $\overset{\text{on shell}}{=}$ full propagator for each particle

How do we compute G_n? Perturbation theory
$\phi \to \sqrt{Z}\phi_{\rm in,out}$ (weakly), ϕ and $\phi_{\rm in,out}$ obey CCR
Look for $U(t)$ connecting (interacting) ϕ and (free) $\phi_{\rm in}$

$$\phi(t) = U(t)^\dagger \phi_{\rm in}(t) U(t)$$
$$\pi(t) = U(t)^\dagger \pi_{\rm in}(t) U(t)$$

Free canonical field

$$\begin{cases} \dot\phi_{\rm in}(x) = i[H_{\rm in}(\phi_{\rm in},\pi_{\rm in}),\phi_{\rm in}(x)] & (= i[H(\phi,\pi),\phi_{\rm in}(x)] \neq i[H(\phi_{\rm in},\pi_{\rm in}),\phi_{\rm in}(x)]) \\ \dot\pi_{\rm in}(x) = i[H_{\rm in}(\phi_{\rm in},\pi_{\rm in}),\pi_{\rm in}(x)] & (= i[H(\phi,\pi),\pi_{\rm in}(x)] \neq i[H(\phi_{\rm in},\pi_{\rm in}),\pi_{\rm in}(x)]) \end{cases}$$

Interacting (Heisenberg) field

$$\begin{cases} \dot\phi(x) = i[H(\phi,\pi),\phi(x)] \\ \dot\pi(x) = i[H(\phi,\pi),\pi(x)] \end{cases}$$

$$H(\phi,\pi) = H_{\rm in}(\phi,\pi) + [H(\phi,\pi) - H_{\rm in}(\phi,\pi)] \equiv H_{\rm in}(\phi,\pi) + H_I(\phi,\pi)$$

$$\begin{aligned}
\dot{\phi}_{\rm in}(t) &= \tfrac{\partial}{\partial t}[U(t)\phi(t)U(t)^\dagger] \\
&= \dot{U}(t)\phi(t)U(t)^\dagger - U(t)\phi(t)U(t)^\dagger\dot{U}(t)U(t)^\dagger + U(t)\dot{\phi}(t)U(t)^\dagger \\
&= [\dot{U}(t)U(t)^\dagger, \phi_{\rm in}(t)] + iU(t)[H(\phi(t),\pi(t)),\phi(t)]U(t)^\dagger \\
&= [\dot{U}(t)U(t)^\dagger, \phi_{\rm in}(t)] + i[H(\phi_{\rm in}(t),\pi_{\rm in}(t)),\phi_{\rm in}(t)] \\
&= i[H_{\rm in}(\phi_{\rm in},\pi_{\rm in}),\phi_{\rm in}(t)] + [\dot{U}(t)U(t)^\dagger + iH_I(\phi_{\rm in}(t),\pi_{\rm in}(t)),\phi_{\rm in}(t)] \\
&= \dot{\phi}_{\rm in}(t) + [\dot{U}(t)U(t)^\dagger + iH_I(\phi_{\rm in}(t),\pi_{\rm in}(t)),\phi_{\rm in}(t)]
\end{aligned}$$

$$\dot{\pi}_{\rm in}(t) = \dot{\pi}_{\rm in}(t) + [\dot{U}(t)U(t)^\dagger + iH_I(\phi_{\rm in}(t),\pi_{\rm in}(t)),\pi_{\rm in}(t)]$$

$$\Rightarrow \frac{\delta}{\delta(\phi_{\rm in}(t),\pi_{\rm in}(t))}[\dot{U}(t)U(t)^\dagger + iH_I(\phi_{\rm in}(t),\pi_{\rm in}(t))] = 0$$

$$\begin{aligned}
\Rightarrow i\dot{U}(t)U(t)^\dagger = H_I(\phi_{\rm in}(t),\pi_{\rm in}(t)) + \mathcal{E}_0(t) &\equiv H_I'(\phi_{\rm in}(t),\pi_{\rm in}(t)) \\
\equiv V_I(t) + \mathcal{E}_0(t) &\equiv V_I'(t)
\end{aligned}$$

Differential equation

$$\dot{U}(t) = -iV_I'(t)U(t) \qquad U(0) = \mathbf{1}$$

Set $\mathcal{U}(t,t') = U(t)U(t')^\dagger$, with $\mathcal{U}(t,t) = \mathbf{1}$

$$\partial_t \mathcal{U}(t,t') = -iV_I'(t)\mathcal{U}(t,t')$$

We already know the solution

$$\mathcal{U}(t,t') = U(t)U(t')^\dagger = \mathrm{Texp}\left\{-i\int_{t'}^{t} d\tau\, V_I'(\tau)\right\}$$

so

$$U(t) = \mathcal{U}(t,0) = \mathrm{Texp}\left\{-i\int_0^t d\tau\, V_I'(\tau)\right\}$$
$$= \mathrm{Texp}\left\{-i\int_0^t d\tau\, V_I(\tau)\right\} e^{-i\int_0^t d\tau \mathcal{E}_0(\tau)}$$
$$U(t)^\dagger = \mathcal{U}(0,t) = \mathrm{Texp}\left\{-i\int_t^0 d\tau\, V_I'(\tau)\right\}$$

Plug into Green's function ($t \geq \max_i t_i \geq -t$)

$$
\begin{aligned}
&\langle 0|\mathrm{T}\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\}|0\rangle \\
&= \langle 0|\mathrm{T}\{U(t_1)^\dagger\phi_{\rm in}(x_1)U(t_1)U(t_2)^\dagger\phi_{\rm in}(x_2)U(t_2)\ldots U(t_n)^\dagger\phi_{\rm in}(x_n)U(t_n)\}|0\rangle \\
&= \langle 0|U(t)^\dagger\mathrm{T}\{\mathcal{U}(t,t_1)\phi_{\rm in}(x_1)\mathcal{U}(t_1,t_2)\phi_{\rm in}(x_2)\ldots \\
&\qquad\qquad\qquad \ldots\times\mathcal{U}(t_{n-1},t_n)^\dagger\phi_{\rm in}(x_n)\mathcal{U}(t_n,-t))\}U(-t)|0\rangle \\
&= \langle 0|U(t)^\dagger\mathrm{T}\{\phi_{\rm in}(x_1)\ldots\phi_{\rm in}(x_n)\exp\left\{-i\int_{-t}^{t}d\tau\, V_I'(\tau)\right\}\}U(-t)|0\rangle \\
&\underset{t\to\infty}{\to}\langle 0|\mathrm{T}\{\phi_{\rm in}(x_1)\ldots\phi_{\rm in}(x_n)\exp\left\{-i\int_{-t}^{t}d\tau\, V_I'(\tau)\right\}\}|0\rangle \\
&\qquad \times\langle 0|U(t)^\dagger|0\rangle\langle 0|U(-t)|0\rangle
\end{aligned}
$$

$$
\begin{aligned}
{}_{\rm in}\langle p\alpha|U(-t)|0\rangle &= -i\int d^3x\, e^{ip\cdot x}\overleftrightarrow{\partial}_0\, {}_{\rm in}\langle\alpha|\phi_{\rm in}(x)U(-t)|0\rangle \\
&= -i\int d^3x\, e^{ip\cdot x}\overleftrightarrow{\partial}_0\, {}_{\rm in}\langle\alpha|U(x^0)\phi(x)U(x^0)^\dagger U(-t)|0\rangle \\
&= -i\int d^3x\, {}_{\rm in}\langle\alpha|U(x^0)[e^{ip\cdot x}\overleftrightarrow{\partial}_0\phi(x)]U(x^0)^\dagger U(-t)|0\rangle \\
&\quad - i\int d^3x\, e^{ip\cdot x}{}_{\rm in}\langle\alpha|[\dot{U}(x^0)U(x^0)^\dagger, U(x^0)\phi(x)U(x^0)^\dagger]U(-t)|0\rangle \\
&\overset{x^0=-t}{=} -i\int d^3x\, {}_{\rm in}\langle\alpha|U(-t)[e^{ip\cdot x}\overleftrightarrow{\partial}_0\phi(x)]|0\rangle \\
&\qquad - i\int d^3x\, e^{ip\cdot x}\, {}_{\rm in}\langle\alpha|\underbrace{[-iV_I(-t),\phi_{\rm in}(x)]}_{=0}U(-t)|0\rangle \\
&\underset{t\to\infty}{\rightarrow} -i\int d^3x\, {}_{\rm in}\langle\alpha|U(-t)[e^{ip\cdot x}\overleftrightarrow{\partial}_0\sqrt{Z}\phi_{\rm in}(x)]|0\rangle = \sqrt{Z}\, {}_{\rm in}\langle\alpha|U(-t)a(p)_{\rm in}|0\rangle = 0
\end{aligned}
$$

$$
\Longrightarrow \quad \lim_{t\to\infty} U(\pm t)|0\rangle = \lambda_\pm|0\rangle
$$

$$\lim_{t\to\infty}\langle 0|U(t)^\dagger|0\rangle\langle 0|U(-t)|0\rangle = \lim_{t\to\infty}\langle 0|U(-t)|0\rangle\langle 0|U(t)^\dagger|0\rangle$$

$$= \lim_{t\to\infty}\langle 0|U(-t)U(t)^\dagger|0\rangle = \lim_{t\to\infty}\langle 0|U(t)U(-t)^\dagger|0\rangle^*$$

$$= \lim_{t\to\infty}\langle 0|\mathrm{Texp}\left\{-i\int_{-t}^{t} d\tau\, V_I'(\tau)\right\}|0\rangle^*$$

Gell-Mann–Low formula

$$\langle 0|\mathrm{T}\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\}|0\rangle$$

$$= \lim_{t\to\infty}\frac{\langle 0|\mathrm{T}\{\phi_{\mathrm{in}}(x_1)\ldots\phi_{\mathrm{in}}(x_n)\mathrm{exp}\left\{-i\int_{-t}^{t} d\tau\, V_I'(\tau)\right\}\}|0\rangle}{\langle 0|\mathrm{Texp}\left\{-i\int_{-t}^{t} d\tau\, V_I'(\tau)\right\}|0\rangle}$$

$$= \frac{\langle 0|\mathrm{T}\{\phi_{\mathrm{in}}(x_1)\ldots\phi_{\mathrm{in}}(x_n)\mathrm{exp}\left\{-i\int_{-\infty}^{+\infty} d\tau\, V_I'(\tau)\right\}\}|0\rangle}{\langle 0|\mathrm{Texp}\left\{-i\int_{-\infty}^{+\infty} d\tau\, V_I'(\tau)\right\}|0\rangle}$$

Starting point for perturbation theory (drop $\mathcal{E}_0$ since it cancels out)

$$\langle 0|\mathrm{T}\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\}|0\rangle$$

$$= \frac{\langle 0|\mathrm{T}\{\phi_{\rm in}(x_1)\ldots\phi_{\rm in}(x_n)\exp\left\{-i\int_{-\infty}^{+\infty}d\tau\, V_I(\tau)\right\}\}|0\rangle}{\langle 0|\mathrm{T}\exp\left\{-i\int_{-\infty}^{+\infty}d\tau\, V_I(\tau)\right\}|0\rangle}$$

$$= \frac{\sum_N \frac{(-i)^N}{N!}\langle 0|\mathrm{T}\{\phi_{\rm in}(x_1)\ldots\phi_{\rm in}(x_n)\int_{-\infty}^{+\infty}d\tau_1\, V_I(\tau_1)\ldots\int_{-\infty}^{+\infty}d\tau_N\, V_I(\tau_N)\}|0\rangle}{\sum_M \frac{(-i)^M}{M!}\langle 0|\mathrm{T}\{\int_{-\infty}^{+\infty}d\tau_1\, V_I(\tau_1)\ldots\int_{-\infty}^{+\infty}d\tau_M\, V_I(\tau_M)\}|0\rangle}$$

- Apply usual machinery of Feynman diagrams
- Numerator = all diagrams (connected and disconnected) with N external sources (= vertex $J\phi$)
- Denominator = vacuum-to-vacuum diagrams ⇒ cancel out disconnected vacuum bubbles in the numerator

Putting together LSZ + Gell-Mann–Low ($\{p'_1,\ldots,p'_{n'}\}\cap\{p_1,\ldots,p_n\}=\emptyset$)

$$
{}_{\rm out}\langle p'_1,\ldots,p'_{n'}|p_1,\ldots,p_n\rangle_{\rm in}
$$

$$
=(2\pi)^4\delta^{(4)}(p'-p)\lim_{p^2,p'^2\to m^2}\prod_{j=1}^{n}\left(\frac{p_j^2-m^2}{i\sqrt{Z}}\right)\prod_{j'=1}^{n'}\left(\frac{p'^2_{j'}-m^2}{i\sqrt{Z}}\right)
\times\frac{\{\sum \text{diagrams with } n'+n \text{ legs } (-p',p)\}}{\{\sum \text{vacuum-to-vacuum diagrams}\}}
$$

$$
=(2\pi)^4\delta^{(4)}(p'-p)\frac{\{\sum \text{amputated diagrams with } n'+n \text{ legs } (-p',p)\}}{Z^{\frac{n+n'}{2}}\{\sum \text{vacuum-to-vacuum diagrams}\}}
$$

Renormalisation:

- renormalised (physical) mass $m_B = Z_m m \Rightarrow$ leaves residue Z in prop.
- renormalised coupling $\lambda_B = Z_\lambda \lambda$

$$
Z_\lambda^V Z^{-\frac{E}{2}} = (Z_\lambda Z^{\frac{nV}{2}})^V Z^{-\frac{nV}{2}V} Z^{-\frac{E}{2}} = (Z_\lambda Z^{\frac{nV}{2}})^V \underbrace{Z^{-(I+E)}}_{\text{makes all dressed propagators finite}}
$$

- tune $Z_\lambda Z^{\frac{nV}{2}}$ to reabsorb remaining divergences

References:

- The treatment of LSZ reduction and derivation of Gell-Mann–Low formula follows

  J.D. Bjorken, S.D. Drell, "Relativistic quantum fields"