Advanced field theory

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Lecture 2 or: Elements of non-perturbative quantum field theory

Perturbative quantisation:

- Lagrangian $\mathcal{L}(\Phi, \partial\Phi; m_B, g_B)$, quantise canonically $\Rightarrow$ Hamiltonian H
- regularise in UV (momentum cutoff Λ) and IR (volume cutoff V)
- split $H(t) = H(0) = H_0(0) + H_I(0)$, go over to interaction picture

$$\Phi(t) = e^{iHt} e^{-iH_0 t} \phi(t) e^{iH_0 t} e^{-iHt}$$
$$V(t) = e^{iH_0 t} H_I(0) e^{-iH_0 t} = V[\phi(t)]$$

Finite number of DOF, no problem with Haag's theorem

- S-matrix via Dyson's formula

$$S = \mathrm{Texp}\left\{-i \int dt\, V(t)\right\}$$

- expand in powers of V, use Wick's theorem

$$\langle f|S|i\rangle \to \int dt_1 \ldots \int dt_n \langle f| T\left\{\phi(t_1)\ldots\phi(t_n)\right\}|i\rangle$$

$\to$ sum of Feynman diagrams

- divergences appear, can be cured by tuning $g_B = g_B(\Lambda)$, $m_B = m_B(\Lambda)$

In momentum space

$$G_n(p_1, \ldots, p_{n-1}) : \text{ Fourier transf. of } \langle 0| T\{\Phi(t_1)\ldots\Phi(t_n)\}|0\rangle$$

Structure of divergence determined by locality and dimensionality

$$\begin{aligned} G_2(p; g_B(\Lambda), m_B(\Lambda)) &= \frac{i}{p^2 - m_B(\Lambda)^2 - \Sigma_B(p^2) + i\epsilon} \\ &= \frac{iZ_\phi(\Lambda)}{p^2 - m^2_{\rm phys} - \Sigma(p^2) + i\epsilon} = Z_\phi(\Lambda) G_2^R(p; g_R, m_R) \end{aligned}$$

$$G_n(p_1, \ldots, p_{n-1}; g_B(\Lambda), m_B(\Lambda)) = Z_\phi(\Lambda)^{\frac{n}{2}} G_n^R(p_1, \ldots, p_{n-1}; g_R, m_R)$$

S-matrix elements in terms of Green's functions $\rightarrow$ finite

$$\begin{aligned} \langle f|S|i\rangle &= G_n(p_1, \ldots, p_{n-1}) \left[Z_\phi^{-\frac{1}{2}} G_2(p_1)\ldots Z_\phi^{-\frac{1}{2}} G_2(p_n)\right]^{-1} u(p_1)\ldots u(p_n) \\ &= \left[G_n(p_1, \ldots, p_{n-1}) Z_\phi^{-\frac{n}{2}}\right] \left[G_2^R(p_1)\ldots G_2^R(p_n)\right]^{-1} u(p_1)\ldots u(p_n) \\ &= G_n^R(p_1, \ldots, p_{n-1}) \left[G_2^R(p_1)\ldots G_2^R(p_n)\right]^{-1} u(p_1)\ldots u(p_n) \end{aligned}$$

$Z_\phi^{-\frac{1}{2}} G_2(p)$ removes external propagator to be replaced by $u(p)$

$Z_\phi^{-\frac{1}{2}}$ required by different normalisation of fields

Perturbative quantisation works but theoretically not satisfactory

Appealing features:

- describes physics of interacting particles
- manifest Poincaré symmetry
- manifest locality

What would a quantum field theory look like under these assumptions?

- find general properties independent of quantisation procedure
- clarify field/particle connection
- axiomatic approaches developed by Lehmann-Symanzik-Zimmermann; Wightman; Haag; ...
- do not help with practical calculations, but help understanding QFT

What do we want from a quantum field theory? Wightman's axioms

- translation symmetry implemented by unitary $U(a) = e^{-iP\cdot a}$, $P_\mu = P_\mu^\dagger$
- unique normalisable vacuum state $|0\rangle$ translation invariant

$$P_\mu|0\rangle = 0 \qquad \langle 0|0\rangle = 1$$

- fields build all states out of the vacuum, $\{P[\phi]|0\rangle\} = \mathcal{H}$

dense set of states is enough

- energy-momentum spectrum: $p^2 \geq 0$, $p_0 \geq 0 \Rightarrow |0\rangle$ has minimal p_0

p_μ: eigenvalues of P_μ

- Lorentz symmetry implemented by unitary $U(\Lambda) \to$ full Poincaré symmetry implemented by $U(a, \Lambda) \Rightarrow |0\rangle$ Lorentz invariant

$$U(a)U(\Lambda)|0\rangle = U(\Lambda)U(\Lambda^{-1}a)|0\rangle = U(\Lambda)|0\rangle = |0\rangle \text{ by uniqueness}$$

- fields transform covariantly under Poincaré symmetry (technical)

$$U(\Lambda)^\dagger U(a)^\dagger \phi_i(x) U(a) U(\Lambda) = S_{ij}(\Lambda)\phi_j(\Lambda^{-1}x + a)$$

- microcausality (locality)

$$[\phi_i(x), \phi_j(y)]_\pm = 0 \text{ for } (x-y)^2 < 0$$

Physically motivated axioms, allow to derive many appealing properties

- cluster property

$$\begin{aligned} &\langle 0|\phi(x_1)\ldots\phi(x_n)\phi(y_1+\lambda a)\ldots\phi(y_m+\lambda a)|0\rangle \\ \underset{\lambda\to\infty}{\to}\; &\langle 0|\phi(x_1)\ldots\phi(x_n)|0\rangle\langle 0|\phi(y_1+\lambda a)\ldots\phi(y_m+\lambda a)|0\rangle \\ = \; &\langle 0|\phi(x_1)\ldots\phi(x_n)|0\rangle\langle 0|\phi(y_1)\ldots\phi(y_m)|0\rangle \end{aligned}$$

- Wightman's reconstruction theorem: full field theory reconstructed from vacuum expectation values $\langle 0|\phi(x_1)\ldots\phi(x_n)|0\rangle$
- existence of asymptotic fields, LSZ asymptotic condition (see below)
- spin/statistics theorem
- CPT theorem
- ...

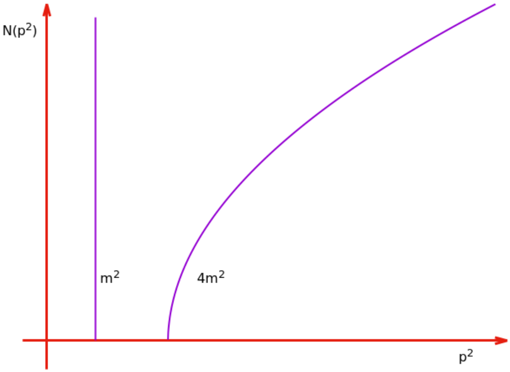
For simplicity we will make further, more detailed assumptions (some can actually be derived from the axioms)

Further assumptions on the spectrum:

- existence of single-particle states $|p;\alpha\rangle$ with $p_\mu p^\mu = m_\alpha^2$

$$\langle p';\alpha'|p;\alpha\rangle = \delta_{\alpha'\alpha} 2p^0 (2\pi)^3 \delta^{(3)}(\vec{p}\,' - \vec{p})$$

- vacuum and single-particle states form a discrete spectrum separated from the rest, and $\min_\alpha m_\alpha > 0$ (mass gap)



Spectral (Källén-Lehmann) representation of the propagator ($\phi = \phi^\dagger$)

$$D(x) \equiv \langle 0|T\{\phi(x)\phi(0)\}|0\rangle = \theta(x^0)\langle 0|\phi(x)\phi(0)|0\rangle + \theta(-x^0)\langle 0|\phi(0)\phi(x)|0\rangle$$

Insert complete set of P_μ eigenstates (including single-particle states)

$$\begin{aligned}\langle 0|\phi(x)\phi(0)|0\rangle &= \sum_n \langle 0|\phi(x)|n\rangle\langle n|\phi(0)|0\rangle \\ &= \sum_n \langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|n\rangle\langle n|\phi(0)|0\rangle \qquad \text{by translation invariance} \\ &= \sum_n e^{-ip_n\cdot x}|\langle 0|\phi(0)|n\rangle|^2 \qquad P_\mu \text{ eigenstates}\end{aligned}$$

Plug back in

$$\begin{aligned}D(x) &= \sum_n \left[\theta(x^0)e^{-ip_n\cdot x} + \theta(-x^0)e^{ip_n\cdot x}\right]|\langle 0|\phi(0)|n\rangle|^2 \\ &= \int \frac{d^4p}{(2\pi)^4}\left[\theta(x^0)e^{-ip\cdot x} + \theta(-x^0)e^{ip\cdot x}\right]\varrho(p)\end{aligned}$$

$$\varrho(p) \equiv \sum_n (2\pi)^4\delta^{(4)}(p_n - p)|\langle 0|\phi(0)|n\rangle|^2$$

Use Lorentz invariance and $U(\Lambda)^\dagger \phi(x) U(\Lambda) = \phi(\Lambda^{-1}x)$

$$\begin{aligned}
\varrho(\Lambda p) &= \sum_n (2\pi)^4 \delta^{(4)}(p_n - \Lambda p) |\langle 0|\phi(0)|n\rangle|^2 \\
&= \sum_{n_\Lambda} (2\pi)^4 \delta^{(4)}(p_{n_\Lambda} - \Lambda p) |\langle 0|\phi(0)|n_\Lambda\rangle|^2 \\
&= \sum_n (2\pi)^4 \delta^{(4)}(\Lambda p_n - \Lambda p) |\langle 0|\phi(0) U(\Lambda)|n\rangle|^2 \\
&= \sum_n (2\pi)^4 \delta^{(4)}(p_n - p) |\langle 0|\phi(0)|n\rangle|^2 = \varrho(p)
\end{aligned}$$

Lorentz invariant function of p + positive spectrum

$$\varrho(p) = 2\pi\theta(p^0)\rho(p^2)$$

For single-particle states

$$\langle 0|\phi(0)|p\rangle = \langle 0|\phi(0)|\Lambda p\rangle = f(p^2) = f(m^2) = \text{constant} \equiv \sqrt{Z}$$

$$D(x) = \int \frac{d^4p}{(2\pi)^4} \left[\theta(x^0)e^{-ip\cdot x} + \theta(-x^0)e^{ip\cdot x}\right] 2\pi\theta(p^0)\rho(p^2)$$

$$= \int_0^\infty ds\, \rho(s) \underbrace{\int \frac{d^4p}{(2\pi)^4} \left[\theta(x^0)e^{-ip\cdot x} + \theta(-x^0)e^{ip\cdot x}\right] 2\pi\theta(p^0)\delta(p^2 - s)}_{=\Delta_F(x;s) \text{ — free causal propagator of mass squared } s}$$

$$= \int_0^\infty ds\, \rho(s)\Delta_F(x;s) = \int_0^\infty ds\, \rho(s) \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x} \frac{i}{p^2 - s + i\epsilon}$$

Use assumption on spectrum: isolated single-particle states at $p^2 = m^2$, continuous spectrum starting at $p^2 = 4m^2$ ("two-particle threshold")

$$\rho(s) = Z\delta(s - m^2) + \theta(s - 4m^2)\sigma(s)$$

$$\tilde{D}(p) \equiv \int d^4x\, e^{ip\cdot x} D(x) = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{4m^2}^\infty ds\, \sigma(s) \frac{i}{p^2 - s + i\epsilon}$$

$\Rightarrow \tilde{\Delta}_F(p)$ has pole at physical particle mass with residue $Z > 0$

Exercise:

$$\langle 0|[\phi(x),\phi(0)]|0\rangle = \int_0^\infty ds\, \rho(s)\Delta(x;s)$$

$$\Delta(x;s) = \int \frac{d^4p}{(2\pi)^4}\left[e^{-ip\cdot x} - e^{ip\cdot x}\right] 2\pi\theta(p^0)\delta(p^2 - s)$$

For canonically quantised fields take $\partial/\partial x^0$ followed by $x^0 \to 0$, and show

$$\int_0^\infty ds\, \rho(s) = 1$$

and so $0 \le Z \le 1$

This results holds only for the regularised theory (if Lorentz invariance has not been broken), no idea what happens as $\Lambda_{\mathrm{UV}} \to \infty$

At $t \to \mp\infty$ states of our system should look like set of free particles

$$e^{-iHt}|\Psi\rangle \underset{t\to-\infty}{\to} \int d\Omega_{p_1} \ldots \int d\Omega_{p_n} f_1^{(+)}(p_1) \ldots f_n^{(+)}(p_n)|p_1, \ldots, p_n\rangle_{\rm in}$$

$$e^{-iHt}|\Psi\rangle \underset{t\to+\infty}{\to} \int d\Omega_{p_1} \ldots \int d\Omega_{p_n} f_1^{(-)}(p_1) \ldots f_n^{(-)}(p_n)|p_1, \ldots, p_n\rangle_{\rm out}$$

In/out states $|p_1, \ldots, p_n\rangle_{\rm in,out}$: under Lorentz/translations

$$U(\Lambda)U(a)|p_1, \ldots, p_n\rangle_{\rm in,out} = e^{-i(\sum_j p_j)\cdot a}|\Lambda p_1, \ldots, \Lambda p_n\rangle_{\rm in,out} \qquad p_j^2 = m^2$$

- same transformation properties as free multiparticle state, but using the representation $U(\Lambda)$, $U(a)$ of the interacting theory
- p_i identified as particle four-momenta

Can we get this out of our theory? Yes: Haag-Ruelle theory of scattering (assuming Wightman's axioms)

We make further assumptions and simplify the argument

Assume exist $\phi_{\text{in,out}}$ satisfying

1. covariance: $U(\Lambda)^\dagger U(a)^\dagger \phi_{\text{in,out}}(x) U(a) U(\Lambda) = \phi_{\text{in,out}}(\Lambda^{-1}x + a)$
2. free EOM: $(\Box + m^2)\phi_{\text{in,out}}(x) = 0$
3. free field CR: $[\phi_{\text{in,out}}(x), \phi_{\text{in,out}}(y)] = \Delta(x - y)$
4. asymptotics: $\phi(x) \underset{x^0 \to \mp\infty}{\rightsquigarrow} \phi_{\text{in,out}}(x)$ (in some sense)

- same Us as for $\phi \Rightarrow$ temporal evolution governed by H
- satisfy free Klein-Gordon equation with physical particle mass (instead of whatever nonlinear equations define the interacting theory)
- under Wightman's axioms, free field CR actually follow from obeying free Klein-Gordon (Jost-Schroer theorem)

From covariance:

$$\partial_\mu \phi_{\rm in}(x) = \partial_\mu e^{iP\cdot x}\phi_{\rm in}(0)e^{-iP\cdot x} = e^{iP\cdot x}[iP_\mu, \phi_{\rm in}(0)]e^{-iP\cdot x} = [iP_\mu, \phi_{\rm in}(x)]$$

From this and EOM

$$-\Box\phi_{\rm in}(x) = [P_\mu, [P^\mu, \phi_{\rm in}(x)]] = m^2\phi_{\rm in}(x)$$

For any P_μ eigenstate $P_\mu|\bar{p}\rangle = \bar{p}_\mu|\bar{p}\rangle$

$$\begin{aligned} m^2\langle 0|\phi_{\rm in}(x)|\bar{p}\rangle &= \langle 0|[P_\mu, [P^\mu, \phi_{\rm in}(x)]]|\bar{p}\rangle = -\langle 0|[P^\mu, \phi_{\rm in}(x)]P_\mu|\bar{p}\rangle \\ &= \langle 0|\phi_{\rm in}(x)P^\mu P_\mu|\bar{p}\rangle = \bar{p}^2\langle 0|\phi_{\rm in}(x)|\bar{p}\rangle \end{aligned}$$

$\Rightarrow \langle 0|\phi_{\rm in}(x)|\bar{p}\rangle \neq 0$ only if $|\bar{p}\rangle$ is a single-particle state

$\Rightarrow \phi_{\rm in}(x)$ creates single-particle states out of the vacuum

$\phi_{\rm in}(x)$ free field $\Rightarrow$ decompose in usual creation/annihilation operators

$$\phi_{\rm in}(x) = \int d\Omega_p \left\{ a_{\rm in}(p) e^{-ip\cdot x} + a_{\rm in}(p)^\dagger e^{ip\cdot x} \right\}$$

$$d\Omega_p = \frac{d^3 p}{(2\pi)^3 2p^0} \qquad p^0 = \sqrt{m^2 + \vec{p}^2}$$

Commutation relations

$$[a_{\rm in}(p), a_{\rm in}(q)^\dagger] = 2p^0 (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \qquad [a_{\rm in}(p), a_{\rm in}(q)] = 0$$

Extract creation operators from field

$$a_{\rm in}(p)^\dagger = -i \int d^3x \, e^{-ip\cdot x} \overset{\leftrightarrow}{\partial}_0 \phi_{\rm in}(x) \equiv < e^{-ip\cdot x}, \phi_{\rm in}(x) >$$

$< f(x), g(x) >$: Lorentz-invariant scalar product, time-independent for solutions of Klein-Gordon equation

Single-particle states

$$
\begin{aligned}
a_{\rm in}(p)^\dagger|0\rangle &\equiv |p\rangle_{\rm in} && \text{by definition}\\
&= |p\rangle && \text{identical to single-particle states of the theory}
\end{aligned}
$$

$\phi_{\rm in}(x)|0\rangle$ has non-zero projection only on 1-particle subspace

$$
\begin{aligned}
|p\rangle_{\rm in} &= a_{\rm in}(p)^\dagger|0\rangle =< e^{-ip\cdot x}, \phi_{\rm in}(x) > |0\rangle =< e^{-ip\cdot x}, e^{iP\cdot x} > \phi_{\rm in}(0)|0\rangle \\
&= \int d\Omega_q |q\rangle\langle q| < e^{-ip\cdot x}, e^{iP\cdot x} > \phi_{\rm in}(0)|0\rangle \\
&= \int d\Omega_q |q\rangle < e^{-ip\cdot x}, e^{iq\cdot x} > \langle q|\phi_{\rm in}(0)|0\rangle \\
&= \int d\Omega_q |q\rangle 2q^0 (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \langle q|\phi_{\rm in}(0)|0\rangle \\
&= |p\rangle\langle p|\phi_{\rm in}(0)|0\rangle
\end{aligned}
$$

$$
{}_{\rm in}\langle p'|p\rangle_{\rm in} = \langle p'|p\rangle \langle p'|\phi_{\rm in}(0)|0\rangle^* \langle p|\phi_{\rm in}(0)|0\rangle = \langle p'|p\rangle |\langle p|\phi_{\rm in}(0)|0\rangle|^2
$$

$$
\Rightarrow |\langle p|\phi_{\rm in}(0)|0\rangle|^2 = 1 \Rightarrow |p\rangle_{\rm in} = |p\rangle \qquad \text{(choose phases wisely)}
$$

Multiparticle states:

$$a_{\rm in}(p_1)^\dagger \dots a_{\rm in}(p_n)^\dagger |0\rangle \equiv |p_1, \dots, p_n\rangle_{\rm in}$$

Transformation properties:

$$\begin{aligned} U(\Lambda)U(a)a_{\rm in}(p)^\dagger &= < e^{-ip\cdot x}, U(\Lambda)U(a)\phi_{\rm in}(x) > \\ &= < e^{-ip\cdot x}, \phi_{\rm in}(\Lambda(x-a)) > U(\Lambda)U(a) \\ &= < e^{-ip\cdot \Lambda^{-1}x}, \phi_{\rm in}(x-\Lambda a) > U(\Lambda)U(a) \\ &= < e^{-i\Lambda p\cdot x}, \phi_{\rm in}(x-\Lambda a) > U(\Lambda)U(a) \\ &= e^{-ip\cdot a} < e^{-i\Lambda p\cdot x}, \phi_{\rm in}(x) > U(\Lambda)U(a) \\ &= e^{-ip\cdot a} a_{\rm in}(\Lambda p)^\dagger U(\Lambda)U(a) \end{aligned}$$

temporal translation justified since $< \cdot, \cdot >$ is t-indep.

$$\Rightarrow U(\Lambda)U(a)|p_1, \dots, p_n\rangle_{\rm in} = e^{-i(\sum_j p_j)\cdot a}|\Lambda p_1, \dots, \Lambda p_n\rangle_{\rm in}$$

These are the states that we wanted

In the same way:

$$\phi_{\rm out}(x) = \int d\Omega_p \, \left\{ a_{\rm out}(p) e^{-ip\cdot x} + a_{\rm out}(p)^\dagger e^{ip\cdot x} \right\}$$

Generally $a_{\rm out}(p) \neq a_{\rm in}(p)$

$$a_{\rm out}(p_1)^\dagger \dots a_{\rm out}(p_n)^\dagger |0\rangle \equiv |p_1, \dots, p_n\rangle_{\rm out}$$
$$|p\rangle_{\rm out} = |p\rangle = |p\rangle_{\rm in}$$

Correct transformation properties (same proofs)

In principle it could be that not all states are particle states at large times – we assume they are

Asymptotic condition: in and out states form two (generally different) complete sets of states in the Hilbert space

How are $\phi_{\rm in,out}$ related to ϕ?

$$\phi(x) \to \phi_{\rm in}(x) \quad \text{as} \quad x^0 \to -\infty \ ?$$

$\phi_{\rm in}$ creates only 1-particle states from the vacuum, ϕ other states as well

$$\phi(x) \to \sqrt{Z}\phi_{\rm in}(x) \quad \text{as} \quad x^0 \to -\infty \quad \text{as an operator?}$$

leads to contradictions

$$[\phi(x), \partial_0\phi(y)]_{\rm ET} \underset{x^0\to-\infty}{\to} Z[\phi_{\rm in}(x), \partial_0\phi_{\rm in}(y)]_{\rm ET} = iZ\delta^{(3)}(\vec{x}-\vec{y})$$

$$e^{iHx^0}[\phi(0,\vec{x}), \partial_0\phi(0,\vec{y})]e^{-iHx^0} \underset{x^0\to-\infty}{\to} iZ\delta^{(3)}(\vec{x}-\vec{y})$$

$$\langle 0|e^{iHx^0}[\phi(0,\vec{x}), \partial_0\phi(0,\vec{y})]e^{-iHx^0}|0\rangle \underset{x^0\to-\infty}{\to} iZ\delta^{(3)}(\vec{x}-\vec{y})\langle 0|0\rangle$$

$$\langle 0|[\phi(0,\vec{x}), \partial_0\phi(0,\vec{y})]|0\rangle \underset{x^0\to-\infty}{\to} iZ\delta^{(3)}(\vec{x}-\vec{y})$$

but LHS t-independent $\Rightarrow$ ϕ is a free field up to normalisation

For normalisable solutions $f(x)$ of Klein-Gordon equation, $(\Box + m^2)f = 0$

$$\phi^f(t) \equiv i \int d^3x\, f(x)^* \overleftrightarrow{\partial}_0 \phi(t,\vec{x}) \qquad \phi^f_{\mathrm{in,out}} \equiv i \int d^3x\, f(x)^* \overleftrightarrow{\partial}_0 \phi_{\mathrm{in,out}}(t,\vec{x})$$

Smears the fields in space, t-indepedent for in/out fields

<u>LSZ condition</u>: for any normalisable $|\alpha\rangle, |\beta\rangle$ and f we assume

$$\lim_{t\to-\infty} \langle\alpha|\phi^f(t)|\beta\rangle = \sqrt{Z}\langle\alpha|\phi^f_{\mathrm{in}}|\beta\rangle$$
$$\lim_{t\to+\infty} \langle\alpha|\phi^f(t)|\beta\rangle = \sqrt{Z}\langle\alpha|\phi^f_{\mathrm{out}}|\beta\rangle$$

$\sqrt{Z}$ must be the same for in/out fields

$$\int d\Omega_p\, \tilde{g}(p)\langle p|\phi^f(t)|0\rangle = i \int d\Omega_p\, \langle p|\phi(0)|0\rangle \tilde{g}(p) \int d^3x\, f(x)^* \overleftrightarrow{\partial}_0 e^{ip\cdot x}$$
$$= i\sqrt{Z} \int d^3x\, f(x)^* \overleftrightarrow{\partial}_0 \int d\Omega_p\, \tilde{g}(p) e^{ip\cdot x} = i\sqrt{Z} \underbrace{\int d^3x\, f(x)^* \overleftrightarrow{\partial}_0 g(x)}_{t\text{-indep.}}$$
$$= \sqrt{Z_{\mathrm{in,out}}} \int d\Omega_p\, \tilde{g}(p)\langle p|\phi^f_{\mathrm{in,out}}|0\rangle$$

$\exists$ of $\phi_{\mathrm{in,out}}$ and validity of LSZ condition derived in axiomatic approach

References:

- Wightman's axioms and its consequences are nicely described in
F. Strocchi, "An introduction to the non-perturbative foundations of quantum field theory"
- The treatment of in and out fields is adapted from
J.D. Bjorken, S.D. Drell, "Relativistic quantum fields"

Addendum: Lorentz invariance of $< f, g >$

$$f(x)\overleftrightarrow{\partial}_0 g(x) \equiv f(x)[\partial_0 g(x)] - [\partial_0 f(x)]g(x)$$

Boost in direction 1

$$\begin{cases} x'^0 = \gamma(x^0 - \beta x^1) \\ x'^1 = \gamma(x^1 - \beta x^0) \end{cases} \qquad d^3x = \frac{1}{\gamma} d^3x' \qquad \frac{\partial}{\partial x^0} = \frac{\partial x'^0}{\partial x^0}\frac{\partial}{\partial x'^0} = \gamma \frac{\partial}{\partial x'^0}$$

Changing integration variables $\vec{x} \to \vec{x}'$

$$\begin{aligned} \int d^3x\, f(\Lambda x)\overleftrightarrow{\partial}_0 g(\Lambda x) &= \int d^3x\, f(\Lambda x)\frac{\overleftrightarrow{\partial}}{\partial x^0} g(\Lambda x) \\ &= \int d^3x'\, \frac{1}{\gamma} f(x')\gamma \frac{\overleftrightarrow{\partial}}{\partial x'^0} g(x') \\ &= \int d^3x\, f(x)\overleftrightarrow{\partial}_0 g(x) \end{aligned}$$

Addendum: time independence of $< f, g >$ for solutions of $(\Box + m^2)f = 0$

$$\partial_0 \int d^3x\, f(x)\overleftrightarrow{\partial}_0 g(x) = \int d^3x \,\{f(x)[\partial_0^2 g(x)] - [\partial_0^2 f(x)]g(x)\}$$

$$= \int d^3x\, f(x)[(\vec{\nabla}^{\,2} - m^2)g(x)] - [(\vec{\nabla}^{\,2} - m^2)f(x)]g(x)$$

$$= \int d^3x\, \vec{\nabla} \cdot \left\{f(x)[\vec{\nabla} g(x)] - [\vec{\nabla} f(x)]g(x)\right\}$$

$$= \lim_{R\to\infty} \int_{\partial B_R} d^2\vec{\Sigma} \cdot \left\{f(x)[\vec{\nabla} g(x)] - [\vec{\nabla} f(x)]g(x)\right\} = 0$$

f, g assumed to vanish at infinity, B_R ball of radius R