Advanced field theory

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Lecture 1 or: How I learnt to stop worrying and love renormalisation

Why **quantum field theory**?

- want a quantum-mechanical description of processes at relativistic energies
- need to take into account the principles of both special relativity (SR) and quantum mechanics (QM)
  - SR: locality and Poincaré invariance
  - QM: superposition principle, uncertainty principle

Use **fields** $\phi(x)$, objects associated with spacetime points x

- use local and Poincaré-invariant field interactions $\Rightarrow$ SR satisfied (easier if fields transform in a simple way)
- make fields generally non-commuting linear operators $\Rightarrow$ QM satisfied

How do we build a quantum field theory? E.g., canonical quantisation

1. take classical Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + V(\phi)$
2. solve Euler-Lagrange EOM $\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$
3. identify conjugate momenta $\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$
4. impose canonical commutation relations (CCR) $\Rightarrow$ field operators

$$[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \qquad [\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})] = [\hat{\pi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = 0$$

What do we gain?

- CCR imply locality: observables commute at spacelike separation
- Noether's theorem $\Rightarrow$ conserved charges that generate unitary representations of Poincaré symmetry, and of other symmetries of the classical Lagrangian

Conditions may apply: symmetries can be spontaneously broken or anomalous

Small practical obstacle: cannot generally solve EOM, proceed by approximations - e.g., **perturbation theory**

Interaction picture: relate interacting (canonical) field $\phi(t)$ to free (canonical) field $\phi_{\rm IP}(t)$ by unitary transformation $U_{\rm IP}(t) = e^{iH_0 t} e^{-iHt}$

$$H[\phi,\pi] = \int d^3x \left(\pi(x)\partial_0\phi(x) - \mathcal{L}(\phi, \vec{\nabla}\phi, \partial_0\phi(\phi,\pi))\right)$$

$$H[\phi(t),\pi(t)] = H[\phi(0),\pi(0)] = H_0[\phi(0),\pi(0)] + V_{\rm I}[\phi(0),\pi(0)]$$

$$\phi(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt} \qquad \pi(t,\vec{x}) = e^{iHt}\pi(0,\vec{x})e^{-iHt}$$

$$\phi_{\rm IP}(t,\vec{x}) = e^{iH_0 t}\phi(0,\vec{x})e^{-iH_0 t} \qquad \pi_{\rm IP}(t,\vec{x}) = e^{iH_0 t}\pi(0,\vec{x})e^{-iH_0 t}$$

$$\phi(x) = U_{\rm IP}(t)^\dagger \phi_{\rm IP}(x) U_{\rm IP}(t) \qquad \pi(x) = U_{\rm IP}(t)^\dagger \pi_{\rm IP}(x) U_{\rm IP}(t)$$

Now solve the theory iteratively in powers of the interaction

$$H\Psi = E\Psi \Rightarrow (H_0 + V_I)(\Psi_0 + \Psi_1 + \ldots) = (E_0 + E_1 + \ldots)(\Psi_0 + \Psi_1 + \ldots)$$

$$H_0\Psi_0 = E_0\Psi_0$$

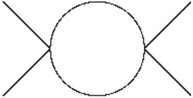
$$V_I\Psi_0 + H_0\Psi_1 = E_1\Psi_0 + E_0\Psi_1$$

$$\ldots$$

Big practical obstacle: divergences!

- S-matrix, Green's functions (= time-ordered correlation functions) divergent beyond lowest perturbative order

$$\langle 0| T\left\{\hat{\phi}(x_1)\dots\hat{\phi}(x_n)\right\}|0\rangle$$

ϕ^4 theory:  $+$  $+ \dots$

0-loop $\propto g$

$$\text{1-loop} \propto g^2 \int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2-i\epsilon}\frac{1}{(p-q)^2+m^2-i\epsilon} = \infty$$

- require renormalisation of field $\phi = Z_\phi \phi_R$, mass $m = Z_m m_R$, and coupling $g = Z_g g_R$ to get finite quantities when removing cutoff $\Rightarrow$ renormalised field is not canonical anymore

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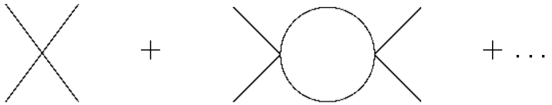

0-loop $\propto g$

$$\text{1-loop} \propto g^2 \int^{\Lambda} \frac{d^4q}{(2\pi)^4}\, \frac{1}{q^2+m^2-i\epsilon}\, \frac{1}{(p-q)^2+m^2-i\epsilon} \qquad \propto g^2 \log\frac{\Lambda}{\mu}$$

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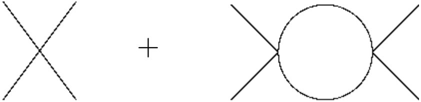
ϕ^4 theory:  $+ \ldots$

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$$g = Z_g g_R \equiv g_R - g_R^2 \log\frac{\Lambda}{\mu}$$

$$\text{0-loop} + \text{1-loop} = g + g^2 \log\frac{\Lambda}{\mu} = g_R - g_R^2 \log\frac{\Lambda}{\mu} + g_R^2 \log\frac{\Lambda}{\mu} + O(g_R^3)$$

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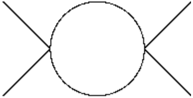
- S-matrix, Green's functions (= time-ordered correlation functions) divergent beyond lowest perturbative order

$$\langle 0 | T\left\{ \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \right\} | 0 \rangle$$

ϕ^4 theory:  $+$  $+ \ldots$

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Is this a problem?

- canonical procedure not written in stone — if it need to be supplemented by renormalisation, so be it
- canonical procedure useful to enforce locality and Poincaré symmetry, not spoiled by renormalisation if we do it right
- our real purpose is to obtain finite Green's functions with suitable locality and symmetry properties and build the S-matrix, how we do that is irrelevant — after all we still need to check against experiment

Field operators can be reconstructed from their Green's functions (Wightman's theorem)

Need for renormalisation is actually a feature if we are to build an interacting theory

- Haag's theorem: if unitary transformation to interaction picture exists then the interacting field is actually a free field. . .
- . . . but renormalised field ϕ_R is not unitarily related to ϕ_{IP} ($Z_\phi \neq 1$), theorem evaded

Quantisation procedure perturbatively equivalent to canonical procedure: path integral quantisation

Generating functional

$$Z[J] = \int \mathrm{D}\phi\, e^{i\int d^4x \mathcal{L}[\phi] + i\int d^4x\, J\phi} = \int \mathrm{D}\phi\, e^{iS[\phi]+iJ\cdot\phi} \qquad \mathrm{D}\phi = \prod_x d\phi(x)$$

Green's function obtained by functional derivatives

$$-i\frac{\delta \log Z[J]}{\delta J(x)}\bigg|_{J=0} = \langle\phi(x)\rangle = \langle 0|\hat{\phi}(x)|0\rangle$$

$$(-i)^2\frac{\delta^2 \log Z[J]}{\delta J(x)\delta J(y)}\bigg|_{J=0} = \langle\phi(x)\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle$$

$$= \langle 0|T\{\hat{\phi}(x)\hat{\phi}(y)\}|0\rangle - \langle 0|\hat{\phi}(x)|0\rangle\langle 0|\hat{\phi}(y)|0\rangle$$

$$\ldots$$

- [−] Path-integral ill-defined (what is the measure?)
- [=] Perturbative expansion needs regularisation and renormalisation as in canonical procedure
- [+] More intuitive, allows for non-perturbative approaches (lattice)

Renormalisation conceptually independent of divergences

- start with $Z = Z[J; m, g]$ and regularise by some UV cutoff Λ (momentum cutoff, inverse lattice spacing,...)
  $\Rightarrow Z = Z[J; m, g; \Lambda]$, finite and adequate for $p \ll \Lambda$
- m, g are thought of as mass and coupling — but are they?

We want to describe the collision of particles initially far away from each other ($\approx$ free), so ϕ must describe free particles in some suitable limit

- at $t = \mp\infty$ Green's functions should describe free particles, we need

Fields should be smeared over small regions in time and space

$$\langle \phi(t_1, \vec{x}_1)\phi(t_2, \vec{x}_2)\rangle \underset{t_{1,2}\to\mp\infty}{\to} Z_\phi^2 D_{\rm free}(t_1 - t_2, \vec{x}_1 - \vec{x}_2)$$

$$\tilde{D}_{\rm free}(p) = \frac{i}{p^2 - m_{\rm phys}^2 + i\epsilon}$$

- $m_{\rm phys}$ must be matched to the particles we want to describe
- Z_ϕ accounts for interacting field creating also multiparticle states

Is $m_{\rm phys} = m$? Generally no: what particles are described at asymptotic times is for the theory to decide after interactions are taken into account

Exercise: compute the two-point function exactly for the interaction Lagrangian $\mathcal{L}_I = K\phi^2$ by resumming diagrams — and also the other way

For g: $\sigma_{2\to2}^{\rm elastic} \propto |\mathcal{M}_{2\to2}|^2$, from $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ (LSZ formula)

$$i\mathcal{M}_{2\to2} = \frac{1}{(2\pi)^6 Z_\phi^4}\int d^4x_1\int d^4x_2\int d^4x_3\int d^4x_4\, e^{ip_1'\cdot x_1}e^{ip_2'\cdot x_2}e^{-ip_1\cdot x_3}e^{-ip_2\cdot x_4}$$
$$\times(\Box_{x_1}+m_{\rm phys}^2)(\Box_{x_2}+m_{\rm phys}^2)(\Box_{x_3}+m_{\rm phys}^2)(\Box_{x_4}+m_{\rm phys}^2)$$
$$\times\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$$

Define $g_{\rm phys}$ from $\mathcal{M}_{2\to2}(\vec{p}_i = 0) = g_{\rm phys}$ (arbitrary, but reasonable)

$\sigma_{2\to2}^{\rm elastic}(\vec{p}_i \to 0) \propto g_{\rm phys}^2$

m and g must be tuned so that $m_{\rm phys}$, $g_{\rm phys}$ match experiments

$$\begin{cases} m_{\rm phys} = f_m(m,g) \\ g_{\rm phys} = f_g(m,g)\end{cases} \Longrightarrow \begin{cases} m = F_m(m_{\rm phys},g_{\rm phys}) \\ g = F_g(m_{\rm phys},g_{\rm phys})\end{cases}$$

Divergences complicate the picture technically, not conceptually:

in the regulated theory $\begin{cases} m_{\rm phys} = f_m(m, g; \Lambda) \\ g_{\rm phys} = f_g(m, g; \Lambda) \end{cases}$, limits $\Lambda \to \infty$ do not exist

If the theory is renormalisable

$$\begin{cases} m_{\rm phys} = f_m(m, g; \Lambda) = f_m(Z_m(\Lambda)m_R, Z_g(\Lambda)g_R; \Lambda) \underset{\Lambda\to\infty}{\to} f_m^{(R)}(m_R, g_R) \\ g_{\rm phys} = f_g(m, g; \Lambda) = f_g(Z_m(\Lambda)m_R, Z_g(\Lambda)g_R; \Lambda) \underset{\Lambda\to\infty}{\to} f_g^{(R)}(m_R, g_R) \end{cases}$$

have finite limits $\Lambda \to \infty$ at m_R, g_R fixed for suitable $Z_{m,g}$

$$\begin{cases} m_R = F_m^{(R)}(m_{\rm phys}, g_{\rm phys}) \\ g_R = F_g^{(R)}(m_{\rm phys}, g_{\rm phys}) \end{cases}$$

— so tune

$$\begin{cases} m = m(\Lambda) = Z_m(\Lambda)F_m^{(R)}(m_{\rm phys}, g_{\rm phys}) = \bar{Z}_m(\Lambda)m_{\rm phys} \\ g = g(\Lambda) = Z_g(\Lambda)F_g^{(R)}(m_{\rm phys}, g_{\rm phys}) = \bar{Z}_g(\Lambda)g_{\rm phys} \end{cases}$$

last step is a finite renormalisation

Last step not necessary: m_R, g_R need not be identified with $m_{\rm phys}, g_{\rm phys}$, can be chosen arbitrarily

- most physical choice: $(m_R, g_R) = (m_{\rm phys}, g_{\rm phys})$

$$\tilde{D}(p) = \frac{i}{p^2 - m^2 - \Sigma(p^2) + i\epsilon} \underset{p^2 \to m^2_{\rm phys}}{\to} \frac{iZ_\phi^2}{p^2 - m^2_{\rm phys} + i\epsilon}$$

$$\mathcal{M}_{2\to 2}(\vec{p}_i = 0) = g_{\rm phys}$$

- in general we can put as much finite part as we want with divergences

$$p^2 - m^2 - \Sigma_1^{\rm div} p^2 - \Sigma_2^{\rm div} m^2 - \Sigma_3^{\rm div} - \Sigma^{\rm fin}(p^2)$$
$$= Z_\phi^{-2}(p^2 - m_R^2) - \Sigma_R^{\rm fin}(p^2)$$
$$Z_\phi^{-2} = 1 - \Sigma_1^{\rm div} - C_1 \qquad Z_\phi^{-2} m_R^2 = (1 + \Sigma_2^{\rm div}) Z_m^2 m_R^2 + C_0 + \Sigma_3^{\rm div}$$
$$\Sigma_R^{\rm fin}(p^2) = \Sigma^{\rm fin}(p^2) - C_0 - C_1 p^2$$

- C_0, C_1 arbitrary, fixed by conditions on $\Sigma_R^{\rm fin}(p^2)$ at $p^2 = \mu^2$ for some renormalisation scale μ — similarly with g
- renormalised quantities depend on μ: "running" m_R, g_R

Renormalisation is not bad — but still a big nuisance for symmetries

- classical Lagrangian has a set of symmetries
- regulator required for quantisation may break one or more of them
  - momentum cutoff $\rightarrow$ breaks Lorentz symmetry, gauge symmetries
  - lattice $\rightarrow$ breaks Lorentz and translation symmetry, gauge symmetries can be preserved

    discrete subgroups of Poincaré symmetry are preserved
  - dimensional regularisation $\rightarrow$ all spacetime symmetries are fine, gauge symmetries are fine

    but internal chiral symmetry of massless fermions is spoiled
- must make sure that desired symmetries spoiled by regulator are recovered after renormalisation but hard
  — better not to spoil them in the first place

Perturbatively most convenient renormalisation scheme:
dimensional regularisation + minimal subtraction (MS) scheme

- 4 dimensions $\rightarrow d$ dimensions, dimensionless "cutoff" $\varepsilon = 4 - d$
- divergences = poles in ε, renormalised by subtracting them only

$\overline{\mathrm{MS}}$ scheme includes also a fixed constant

- generally, divergences must be polynomial in momenta and masses, and polynomial or logarithmic in (physical) cutoff
- in dimensional regularisation, only logarithmic divergences appear (= poles in ε), independent of masses

$$g = \mu^{c\varepsilon} Z_g(g_R, \varepsilon) g_R \qquad m = Z_m(g_R, \varepsilon) m_R$$

$[g] = 0$ in $d = 4 \Rightarrow [g] = c\varepsilon$ in dimension d
mass scale μ required to account for this

- at fixed physical $g_{\mathrm{phys}}, m_{\mathrm{phys}}$, bare g, m depend on ε but not on μ

$$g = g(\varepsilon; g_{\mathrm{phys}}, m_{\mathrm{phys}}) \qquad m = m(\varepsilon; g_{\mathrm{phys}}, m_{\mathrm{phys}})$$

$\Rightarrow$ running $g_R(\mu), m_R(\mu)$, with μ dependence determined by

$$\mu \frac{dg}{d\mu} = 0 \qquad \mu \frac{dm}{d\mu} = 0$$

When is a theory renormalisable?

Take

$$S = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + \sum_i V_i(\phi, \partial\phi)$$

with vertices $V_i = g_i \partial_\mu^{k_i}\phi^{n_i}$, quantise perturbatively

- redefining $\phi = Z_\phi \phi_R$, $m = Z_m m_R$, $g_i = Z_{g_i} g_{iR}$

  $$S(\phi; m, g) = S(\phi_R; m_R, g_R) + \delta S(\phi_R; m_R, g_R)$$

  with δS containing $Z_\phi^2 - 1$, $Z_m^2 - 1$, $Z_{g_i} - 1$
- at every g_i order new divergences are polynomial in m and momenta

  Example: $$I = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p+q)^2 - m^2 + i\epsilon} \frac{1}{p^2 - m^2 + i\epsilon} \sim \log\Lambda$$

  $$\frac{dI}{dq_\mu} = \int \frac{d^4p}{(2\pi)^4} \frac{-2(p+q)_\mu}{[(p+q)^2 - m^2 + i\epsilon]^2} \frac{1}{p^2 - m^2 + i\epsilon} \to \text{ convergent}$$

  $\Rightarrow$ equivalent to contribution of local vertices V_i with (divergent) coefficient δZ_i; **if not contained in S, they must be added**
- <u>choose</u> $Z_{g_i} - 1 = \delta Z_i \Rightarrow$ contributions from δS cancel the divergences, finite result

General case: divergences appear if the integral is not convergent for large momenta

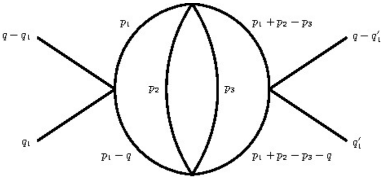
- overall divergence: when scaling $p_i \to \kappa p_i$, $\kappa \to \infty$
- subdivergences: when $p_i \to \kappa p_i$, $\kappa \to \infty$ with certain linear combination Δp of momenta kept fixed
- other possible large-momentum limits (e.g., rescaling $p_i \to \kappa_i p_i$) can be reduced to the two above
- if no Δp is fixed, taking sufficiently many derivatives w.r.t. masses and/or external momenta the integrand is made better behaved

$$\text{integrand} = P(p) \prod_i \frac{1}{(p_i + q_i)^2 - m^2 + i\epsilon} \prod_j \frac{1}{(p_j + \Delta q_j)^2 - m^2 + i\epsilon}$$

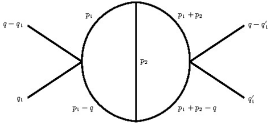
$$\frac{\partial\, \text{integrand}}{\partial q_\mu} \sim \frac{-2p_\mu\, \text{integrand}}{p^2} \qquad \frac{\partial\, \text{integrand}}{\partial m} \sim \frac{-2m\, \text{integrand}}{p^2}$$

$\Rightarrow$ overall divergence is local ($=$ polynomial in $q_\mu \to \partial_\mu$ and m)

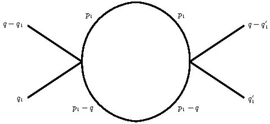
Example: 3-loop diagram



- no constraint $\sim$ new overall divergence
- one of p_1, p_2, p_3, $p_1 + p_2 - p_3$ fixed $\sim$ same div. as 2-loop diagram (contains subdivs. $\to$ 1-loop diagram)



- two of them fixed $\sim$ same div. as 1-loop diagram



What can possibly go wrong?

- combinatorics of terms and counterterms always match, not a problem
- overall divergences always local, not a problem if enough terms are present in the Lagrangian. . .

Subdivergences are taken care of by lower-order counterterms

- . . . but number of required terms may be increasing with the perturbative order!

Power counting: given diagram G

- internal bosonic/fermionic line $\to p^{-d_B}, p^{-d_F}$, integral $d^4 p$ (usually $d_B = 2$, $d_F = 1$)
- vertex $\to \delta(\sum p)$, one used for overall momentum conservation
- i-th type of vertex (schematic): $\partial_\mu^{k_i} \phi^{n_{Bi}} \bar{\psi}^{\bar{o}_{Fi}} \psi^{n_{Fi} - \bar{o}_{Fi}} \to p^{k_i}$

$$D_G = (4 - d_B) I_B + (4 - d_F) I_F + 4 - 4 \sum_i V_i + \sum_i V_i k_i$$

Degree of overall divergence of G is $\omega_G \leq D_G$ (cancellations may happen)

Dimension of i-th coupling $d_{g_i} = 4 - k_i - n_{Bi} - \frac{3}{2} n_{Fi} < 4$

Topological relations: $E_{B,F} + 2I_{B,F} = \sum_i V_i n_{B,Fi}$

$$
\begin{aligned}
D_G &= \frac{4-d_B}{2}\left(\sum_i V_i n_{Bi} - E_B\right) + \frac{4-d_F}{2}\left(\sum_i V_i n_{Fi} - E_F\right) \\
&\quad + 4 - 4\sum_i V_i + \sum_i V_i k_i \\
&= 4 - \frac{4-d_B}{2} E_B - \frac{4-d_F}{2} E_F \\
&\quad + \sum_i V_i \left(k_i - 4 + \frac{4-d_B}{2} n_{Bi} + \frac{4-d_F}{2} n_{Fi}\right) \\
&= f(E_B, E_F) + \sum_i V_i \left(k_i - f(n_{Bi}, n_{Fi})\right) \\
f(n_B, n_F) &= 4 - \frac{4-d_B}{2} n_B - \frac{4-d_F}{2} n_F
\end{aligned}
$$

$$
\boxed{\text{If } k_i \le f(n_{Bi}, n_{Fi})\ \forall i \Rightarrow \omega_G \le f(E_B, E_F)}
$$

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If all the interactions satisfying $k_i \leq f(n_{Bi}, n_{Fi})$ are included in the action then the required counterterm must be of the same form of one of them
$\Rightarrow$ all divergences can be cancelled by renormalisation as outlined above

$$k \leq 4 - \frac{4 - d_B}{2} n_B - \frac{4 - d_F}{2} n_F$$

$$\frac{2 - d_B}{2} n_B + \frac{1 - d_F}{2} n_F \leq 4 - k - n_B - \frac{3}{2} n_F = d_g < 4$$

$$\boxed{\text{In the standard case if } d_B = 2,\ d_F = 1 \Rightarrow d_g \geq 0 \text{ is required}}$$

$\Rightarrow$ renormalisable theory (by power counting)

If $k_{\bar{\imath}} > f(n_{B\bar{\imath}}, n_{F\bar{\imath}})$, increasing $V_{\bar{\imath}}$ any ω_G is possible, and in general new types of vertices are required at each perturbative order
$\Rightarrow$ non-renormalisable theory

Example 1: real scalar field ϕ

$$V(\phi) = \sum g_n \phi^n + \sum_{k\geq 1, n\geq 1} g_{k,n} (\partial_\mu \phi \partial^\mu \phi)^k \phi^n$$

$$d_{g_n} = 4 - n \geq 0 \qquad \text{if } n \leq 4$$
$$d_{g_{k,n}} = 4 - 4k - n < 0 \qquad \text{if both } n, k > 1$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + h\phi + \frac{c}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4$$

If $h = c = 0$ counterterms odd in ϕ are forbidden by symmetry $\phi \to -\phi$

If $h = 0$, $c \neq 0$ (resp. $h \neq 0$, $c = 0$) counterterm linear (resp. cubic) in ϕ can (almost certainly will) be generated by the renormalisation procedure

Example 2: Fermi theory

$$\mathcal{L}_I = \sum_i G_i (\bar{\psi}\Gamma_\mu \psi)(\bar{\psi}\Gamma^\mu \psi)$$

Since $d_{G_i} = 4 - 4\frac{3}{2} = -2$, the theory is non-renormalisable

Example 3: Proca field A^μ (massive vector field)

Propagator:

$$D_{\mu\nu}^{\mathrm{Proca}}(p) = \frac{-i(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2})}{p^2 - m^2 + i\epsilon} \Rightarrow d_B = 0 \Rightarrow n_B \leq d_g < 4$$

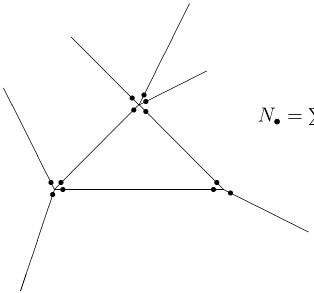
$n_B = 1$	requires one derivative for Lorentz invariance
	$g\partial_\mu A^\mu \Rightarrow d_g = 2 > 1 = n_B$
	$J_\mu A^\mu \Rightarrow n_B = 1 \leq d_J = 3$
$n_B = 2$	are the kinetic and mass terms
$n_B = 3$	requires one derivative for Lorentz invariance
	$g(\partial_\mu A^\nu)A_\nu A^\mu$, $g(\partial_\mu A^\mu)A_\nu A^\nu$
	$\Rightarrow d_g = 0 < 3 = n_B$ forbidden

No renormalisable self-interaction exists

References:

- D. Anselmi, “Renormalization”
- J. Collins, “Renormalization”
- M.E. Peskin and D.V. Schroeder, “An Introduction to Quantum Field Theory”

Addendum: topological relation for diagrams



$$N_\bullet = \sum_i n_i V_i = E + 2I$$

Dots can be counted in two ways: as a property of vertices, or as a property of internal and external lines