LECTURE 11:

THE WILSON LOOP & THE STATIC QUARK POTENTIAL

QUARK CONFINEMENT

STATIC QUARK POTENTIAL - 1

Classical field thy, Minkowski

$$S = \frac{1}{2g^2}\int d^4x \, \mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) = -\frac{1}{4}\int d^4x \, F^a_{\mu\nu} F^{\mu\nu a}$$

$$F_{\mu\nu} = -ig F^a_{\mu\nu} T^a \qquad A_\mu = -ig A^a_\mu T^a$$

$$\mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu} = -g^2 F^a_{\mu\nu} F^b_{\mu\nu} \underbrace{\mathrm{Tr}(T^a T^b)}_{\frac{1}{2}\delta^{ab}} = -\frac{g^2}{2} F^a_{\mu\nu} F^{\mu\nu a}$$

canonical momenta: $\displaystyle \Pi^a_\mu(x) = \frac{\delta S}{\delta(\partial_0 A^{\mu a})} = -F^a_{0\mu}(x)$

There is no canonical momentum for A_0. Constraint $\Pi_0 = 0$

$\rightarrow$ $\boxed{A_0(x) = 0}$ temporal gauge ($U_{x4} = \mathbb{1}$ on the lattice)

$\rightarrow$ there is still a freedom to do time independent gauge transformations

Momenta: $\Pi^a_i = -F^a_{0i} = F^{0ia} \equiv -E^a_i$ = chromoelectric field

$B^a_i = -\frac{1}{2}\varepsilon_{ijk} F^{ajk}$ = chromomagnetic field

$$\mathcal{H} = \dot{A}^a_\mu \Pi^{\mu a} - \mathcal{L} = \frac{1}{2}(E^a_i E^a_i + B^a_i B^a_i) \implies \boxed{H = \frac{1}{2}\int d^3x \,(E^a_i E^a_i + B^a_i B^a_i)}$$

temporal gauge (exercise)

$\hookrightarrow$ equations of motion

INFINITESIMAL GAUGE TRANSFORMS

$\boxed{\Lambda = \mathbb{1} + iW} \Rightarrow \Lambda^{-1} A_\mu \Lambda + \Lambda^{-1}\partial_\mu \Lambda = (\mathbb{1} - iW) A_\mu (\mathbb{1} + iW) + (\mathbb{1} - iW)\partial_\mu(\mathbb{1} + iW) = A_\mu + i\partial_\mu W + i[A_\mu, W]$

to first order

Λ: $\in SU(N)$ group

W: $\in su(N)$ algebra

$$\boxed{\delta A_\mu = i\partial_\mu W + i[A_\mu, W]}$$

In components $\boxed{W = W^a T^a}$

$$-ig\,\delta A_\mu^a T^a = i(\partial_\mu W^a) T^a + i(-ig)\underbrace{[A_\mu^b T^b, W^c T^c]}$$

$$A_\mu^b W^c [T^b T^c] = i A_\mu^b W^c f^{bcd} T^d$$

$$= i A_\mu^b W^c f^{abc} T^a$$

$$-ig\,\delta A_\mu^a T^a = [i\partial_\mu W^a + ig f^{abc} A_\mu^b W^c] T^a$$

$$\Rightarrow (-ig)\left(\frac{i}{g}\right)\delta A_\mu^a T^a = \left[\frac{-1}{g}\partial_\mu W^a - f^{abc} A_\mu^b W^c\right] T^a \Rightarrow \boxed{\delta A_\mu^a = \frac{-1}{g}\partial_\mu W^a - f^{abc} A_\mu^b W^c}$$

$$\Lambda^{-1} F_{\mu\nu} \Lambda = (1 - iW) F_{\mu\nu} (1 + iW) = F_{\mu\nu} - i[W, F_{\mu\nu}] \Rightarrow \boxed{\delta F_{\mu\nu} = i[F_{\mu\nu}, W]}$$

$$E_i = F_{0i} \rightarrow \Lambda^{-1} E_i \Lambda \rightarrow \boxed{\delta E_i = i[E_i, W]}$$

due to our fixing temporal gauge <u>Gauss' law</u> $D^i E_i = 0$ is not part of the equations of motion anymore, but rather has to be imposed as an inital condition

then $\frac{\partial}{\partial t}(D^i E_i) = 0$ follow from the eq. of motion (<u>exercise</u>)

QUANTUM THY

$\Psi[A] \leftarrow$ functional of fields $A_i^a(\vec{x}, t=0)$ $\quad i = 1,2,3$

(E operator) = (-1)(momentum operator) i.e. $E_i^a = i \frac{\delta}{\delta A^{ia}(\vec{x})}$

time independent gauge transform $\Lambda(\vec{x})$ repr. on Hilbert space

$$R(\Lambda)\Psi[A_\mu] = \Psi[\Lambda^{-1} A_\mu \Lambda + \Lambda^{-1}\partial_\mu \Lambda]$$

<u>Claim:</u> For an infinitesimal gauge transform $\Lambda = \mathbb{1} + i w^a T^a$

$$\Rightarrow \boxed{\delta \Psi = \int d^3x \left(-i w^a \frac{1}{g} D^i E_i^a\right) \Psi} = (R - id)\Psi \quad \text{to first order in } w$$

$$\Psi[A_i^a + \delta A_i^a] - \Psi[A_i^a] = \int d^3x \, \frac{\delta\Psi}{\delta A_i^a(\vec{x})} \delta A_i^a(\vec{x}) = \int d^3x \frac{\delta\Psi}{\delta A_i^a}\left(-\frac{1}{g}\partial_i w^a - f^{abc} A_i^b w^c\right)$$

$$= \frac{i}{g}\int d^3x \; i \frac{\delta\Psi}{\delta A_i^a}\left(\partial_i w^a + g f^{abc} A_i^b w^c\right) = \frac{i}{g}\int d^3x \left(D_i w^a\right) E^{ia}\Psi$$

$$= -\frac{i}{g}\int d^3x \, W^a (D_i E^{ia})\Psi = 0 \quad \forall w^a \iff \boxed{D_i E^{ia} = 0}$$

IN TEMPORAL GAUGE:

GAUSS' LAW	$\Longleftrightarrow$	GAUGE INV. OF WAVE FUNCTIONAL UNDER TIME INDEPENDENT SMALL GAUGE TRANSFORMATIONS

$\boxed{D_i w^a = ?}$

$$(\partial_\mu w)^a - i A_\mu^c \underbrace{(T^c_{(adj)})^{ab}}_{-if^{cab}} w^b = \partial_\mu w^a - g f^{cab} A_\mu^c w^b$$

$$= \partial_\mu w^a - g f^{abc} A_\mu^c w^b$$

$$= \partial_\mu w^a + g f^{abc} A_\mu^b w^c$$

STATIC QUARKS

Involve non-trivial gauge transformations.

Suppose $R(\Lambda)\,\Psi_\alpha = \Lambda_{\alpha\beta}(\vec{x}_0)\,\Psi_\beta$

color indices

$\Leftrightarrow D^i E_i^a = -g T^a \delta^3(\vec{x}-\vec{x}_0)$ a static charge in the fundamental representation of the gauge group

since:

$$\delta\Psi_\alpha = i\,w_{\alpha\beta}(\vec{x}_0)\,\Psi_\beta = \int d^3x \left(-\frac{i}{g}\left[w^a(\vec{x})\right] D^i E_i^a\right)_{\alpha\beta}\Psi_\beta$$

$$= i\,w^a(\vec{x}_0)\,T^a_{\alpha\beta}\,\Psi_\beta = \frac{-i}{g}\int d^3x \;\; w^a(\vec{x})\,\underbrace{\left(D^i E_i^a\right)_{\alpha\beta}}_{-g\,T^a{}_{\alpha\beta}\,\delta^{(3)}(\vec{x}-\vec{x}_0)}\Psi_\beta \qquad \forall\, w^a(\vec{x})$$

more generally: a configuration of static charges $\Longleftrightarrow$ a transformation law for the wave functional

e.g. a static quark at $\vec{x}$ and a static anti-quark at $\vec{y}$

$$\Psi_{\alpha\beta}[A] \longrightarrow \Lambda_{\alpha\gamma}(\vec{x})\,\Lambda^{-1}{}_{\delta\beta}(\vec{y})\,\Psi_{\gamma\delta}[A]$$

STATIC QUARKS ON THE LATTICE

$\Psi[\overset{\curvearrowleft}{U}]$ links on $x_4=0$ timeslice   temporal gauge $U_4(x)=1$

$U(x,y) \longrightarrow \Lambda^{-1}(\vec{y})\, U(y,x)\, \Lambda(\vec{x})$ time indep. gauge tr.

$\Psi_{\alpha\beta}[U] \longrightarrow \Lambda_{\alpha\gamma}(\vec{x})\, \Psi_{\gamma\delta}[U]\, \Lambda^{-1}_{\delta\beta}(\vec{y})$

H gauge inv. (under time indep. gauge tr.) $\Rightarrow$ different static quark configs don't mix, i.e. $\exists\, \mathcal{H}_{\vec{x},\vec{y}}$ Hilbert subspaces

energy eigenstates on $\mathcal{H}_{\vec{x},\vec{y}}$: $H\Psi^{(n)} = E_n \Psi^{(n)}$

$\Psi^{(0)}$ = ground state of $\mathcal{H}_{\vec{x},\vec{y}}$ $E_0(|\vec{x}-\vec{y}|)$

$$\boxed{\underline{\text{Def.}}\quad V(R) = E_0(R=|\vec{x}-\vec{y}|) = \min_{|\Psi\rangle\in\mathcal{H}_{\vec{x},\vec{y}}} \langle\Psi|E|\Psi\rangle \qquad \underline{\text{static quark potential}}}$$

$$\langle\Psi^{(0)}|\Psi\rangle \neq 0 \Rightarrow \langle\Psi|e^{-\tau H}|\Psi\rangle = \sum_n |\langle\Psi^{(n)}|\Psi\rangle|^2 e^{-\tau E_n} \xrightarrow[\tau\to\infty]{} |\langle\Psi^{(0)}|\Psi\rangle|^2 e^{-\tau V(R)}$$

Let us make an ansatz/trial fn.

$$\Psi_{\alpha\beta}[U] \equiv U_{\alpha\beta}(\vec{x},\vec{y})\,\Omega[U]$$

$\Omega[U]$ ← gauge invariant vacuum on Hilbert space without static quarks

$U_{\alpha\beta}(\vec{x},\vec{y})$: parallel transporter $\vec{x} \to \vec{y}$

due to the tr. law of the parallel transporters $\Psi_{\alpha\beta}[U] \in \mathcal{H}_{\vec{x},\vec{y}}$

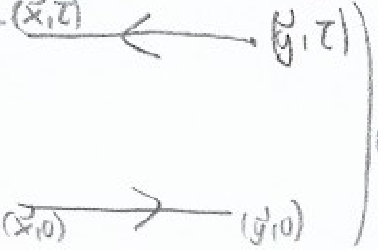
$$e^{-\tau H}\Psi_{\alpha\beta}[U] = e^{-\tau H} U_{\alpha\beta}(\vec{x},\vec{y}) e^{\tau H} e^{-\tau H}\Omega[U] = U(\vec{x}+\tau\hat{4},\vec{y}+\tau\hat{4})\, e^{-\tau H}\Omega[U]$$

$$= U(\vec{x}+\tau\hat{4},\vec{y}+\tau\hat{4})\,\Omega[U]$$

from this point on, we measure energy from ground state energy in the static quarkless $\mathcal{H}$ i.e. let this energy $=0$

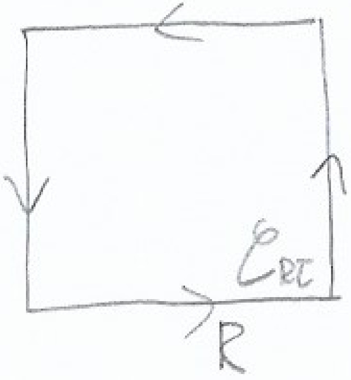
$$\rightarrow \langle \Psi_{\alpha\beta} | e^{-\tau H} | \Psi_{\alpha\beta} \rangle = \langle \Omega | \overline{U_{\alpha\beta}(\vec{x}+\tau 4,\vec{y}+\tau 4)}\; U_{\alpha\beta}(\vec{x},\vec{y}) | \Omega \rangle$$

$$= \frac{1}{Z}\int \mathcal{D}U\; \overline{U_{\alpha\beta}(\vec{x}+\tau\hat{4},\vec{y}+\tau\hat{4})}\; U_{\alpha\beta}(\vec{x},\vec{y})\, e^{-S[U]}$$

$= \frac{1}{Z}\int \mathcal{D}U\, \mathrm{Tr}\Big[$  $\Big] e^{-S}$ $\underset{\text{temporal gauge}}{=}$ $\frac{1}{Z}\int \mathcal{D}U\, \mathrm{Tr}\left(\boxed{C_{R\tau}}\right) e^{-S[U]}$

WILSON LOOP

$$W_{R\tau} \equiv \left\langle \mathrm{Tr} \right.$$



$$\left. \right\rangle \sim e^{-\tau V(R)} \quad \text{as } \tau \text{ large}$$

$$V(R) = -\lim_{\tau \to \infty} \frac{1}{\tau} \log W_{R\tau} \qquad \text{where} \quad W_{R\tau} = \frac{1}{Z} \int DU\, e^{-S[U]}\, \mathrm{Tr}\left[U(\mathcal{C}_{R\tau})\right]$$

Two types of behavior

<u>area law</u> $\quad W_{R\tau} \sim e^{-\sigma(R\tau)} \quad \rightarrow \quad V(R) \sim \sigma R$ <u>confinement phase</u>

(σ: string tension)

<u>perimeter law</u> $\quad W_{R\tau} \sim e^{-A(R+\tau)}$

$m_A = 0 \qquad$ <u>Coulomb phase</u>

$m_A \neq 0 \qquad$ <u>Higgs phase</u> (in models w. matter fields)

STRONG COUPLING EXPANSION OF THE WILSON LOOP

$(SU(3))$ strong coupling = small β

$$W_{RT} = \frac{1}{Z}\int DU \exp\left(\frac{\beta}{3}\sum_p \mathrm{Re}\,\mathrm{Tr}\,U_p\right) \mathrm{tr}\prod_{e\in C_{RT}} U_e = \frac{1}{Z}\int DU \exp\left(\frac{\beta}{6}\sum_p (\mathrm{Tr}\,U_p + \mathrm{Tr}\,U_p^\dagger)\right) \mathrm{tr}\prod_{e\in C_{RT}} U_e$$

$$Z = \int DU \exp\left(\frac{\beta}{3}\sum_p \mathrm{Re}\,\mathrm{Tr}\,U_p\right) = \int DU \exp\left(\frac{\beta}{6}\sum_p (\mathrm{Tr}\,U_p + \mathrm{Tr}\,U_p^\dagger)\right)$$

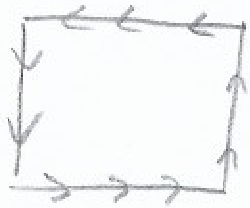
$$\exp\left(\frac{\beta}{6}\sum_p (\mathrm{tr}\,U_p + \mathrm{tr}\,U_p^\dagger)\right) = \sum_{i,j=0}^{\infty} \frac{1}{i!}\frac{1}{j!}\left(\frac{\beta}{6}\right)^{i+j}\left(\sum_p \mathrm{tr}\,U_p\right)^i \left(\sum_{p'} \mathrm{tr}\,U_{p'}^\dagger\right)^j$$

$$Z = \int DU \exp\left(\frac{\beta}{6}\sum_p (\mathrm{Tr}\,U_p + \mathrm{Tr}\,U_p^\dagger)\right) = \int DU\,[1 + O(\beta)] = 1 + O(\beta)$$

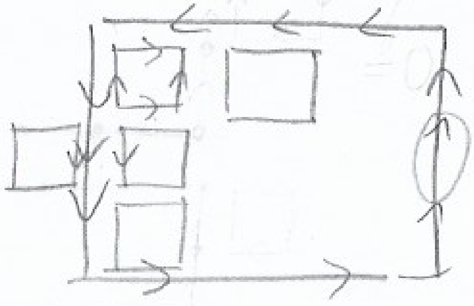
numerator is harder.

$\int dU\, U_{ab} = 0$

1st term
no plaquettes $\quad \int DU\ \mathrm{tr}\prod_e U_e = 0$

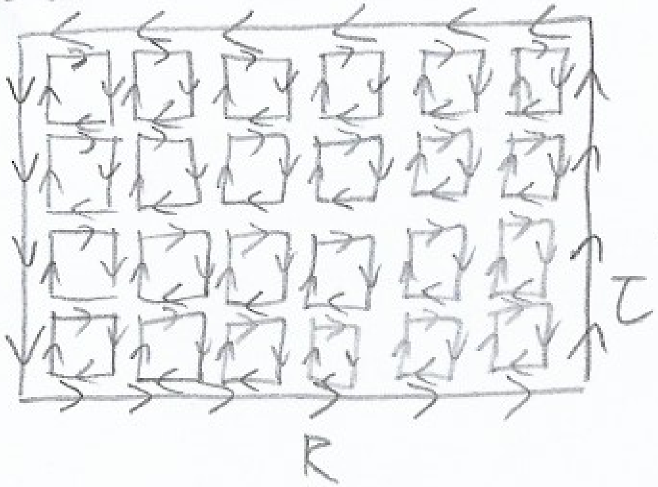
$\int DU$  $= 0$

in general to every link we have to put a plaquette to get a nonzero integral e.g.



$\longrightarrow$ when integrating over this we get $= 0$

lowest term:



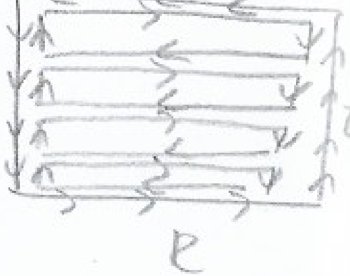
$$\Rightarrow \int DU \frac{1}{n_A!}\left(\frac{\beta}{6}\right)^{n_A}\left(\sum_p \mathrm{tr}\, U_p^{+}\right)^{n_A} \mathrm{tr}\prod_{e\in C_{R\tau}} U_e + \dots$$

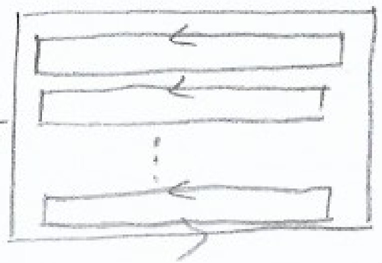
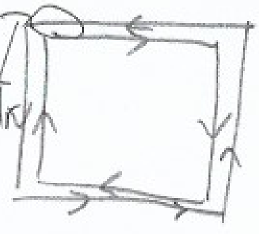
$$n_A = n_R n_\tau$$

(lowest order term has only $\rightleftarrows$ since $\rightrightarrows \Rightarrow 0$ and
only $\substack{\to\\ \to\\ \to} \Rightarrow$ non-zero)

$$\Rightarrow \left(\frac{\beta}{6}\right)^{n_A}\int DU \prod_{p\in A}\mathrm{tr}\, U_p^{+}\ \mathrm{tr}\prod_{e\in C_{R\tau}} U_e$$

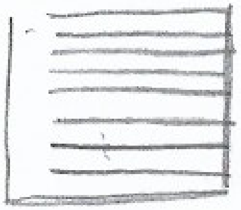
($\frac{1}{n_A!}$ canceled by which term we take in $\left(\sum_p \mathrm{tr}\, U_p^{+}\right)^{n_A}$)

now  $\longrightarrow \frac{1}{3}$  $\Rightarrow \left(\frac{\beta}{6}\right)^{n_A}\left(\frac{1}{3}\right)^{(n_R-1)n_\tau}\int DU$ 

$$\left(\frac{\beta}{6}\right)^{n_A}\left(\frac{1}{3}\right)^{(n_R-1)n_\tau}\int DU \operatorname{Tr}$$  $$= \left(\frac{\beta}{6}\right)^{n_A}\left(\frac{1}{3}\right)^{(n_R-1)n_\tau}\left(\frac{1}{3}\right)^{n_\tau-1}\int DU \operatorname{Tr}$$ 

gauge fix $\Rightarrow$ now we can join the inner loops

only 1 integral left



$$= \left(\frac{\beta}{6}\right)^{n_A}\left(\frac{1}{3}\right)^{n_R n_\tau-1}\frac{1}{3}\operatorname{Tr}\mathbb{1} = \operatorname{Tr}\mathbb{1}\left(\frac{\beta}{6\cdot 3}\right)^{n_A} = 3\exp\left(n_A \ln\frac{\beta}{18}\right)$$

<u>area law</u>

$n_R n_\tau - n_\tau + n_\tau - 1$

$$\Rightarrow \boxed{\sigma = -\frac{1}{a^2}\ln\frac{\beta}{18}\left[1+O(\beta)\right]}$$

string tension to leading order in strong coupling expansion

COMMENTS:

- systematic higher order expansion: instead of Taylor expansion of exponential, a character expansion is more useful (Montvay or Itzykson-Drouffe book)

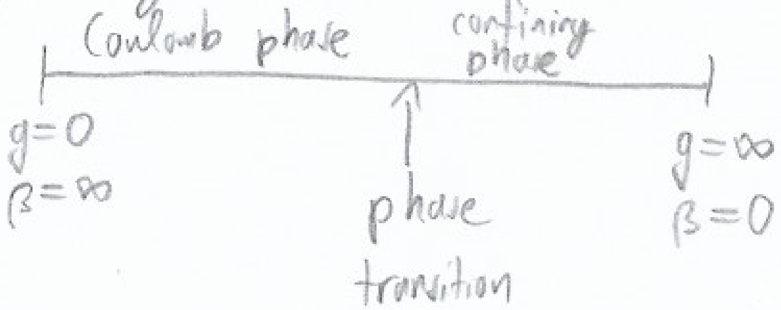
  graphically: higher order diagrams for σ involve surfaces, e.g. 

- strong coupling expansions do not play a central role in modern lattice QCD

- continuum limit: is at $\beta \to \infty$ $(g_0 \to 0)$ asymptotic freedom (next semester)

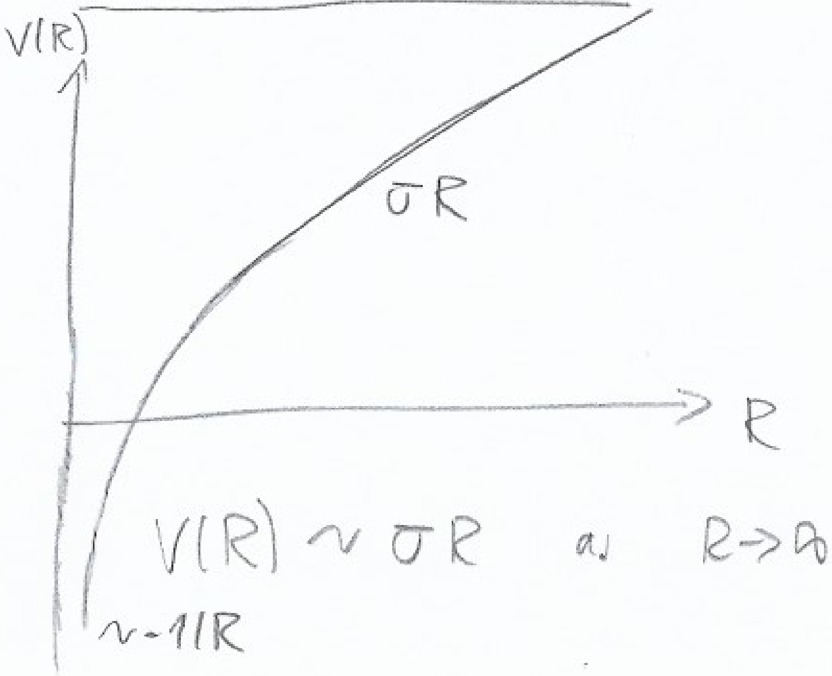
- we didn't prove confinement for continuum gauge theory

  counterexample: $U(1)$ gauge thy on the lattice



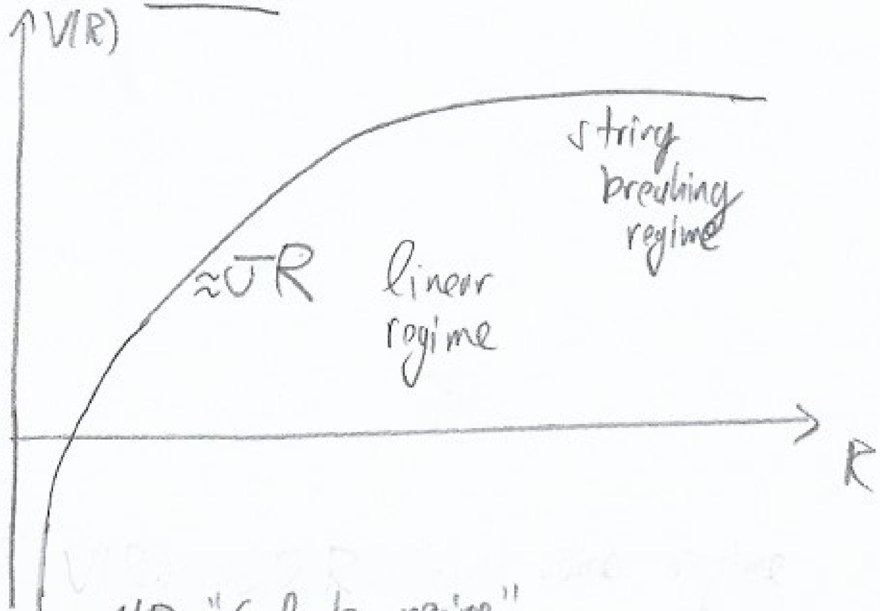
BEHAVIOR OF STATIC POTENTIAL IN

PURE GAUGE THY



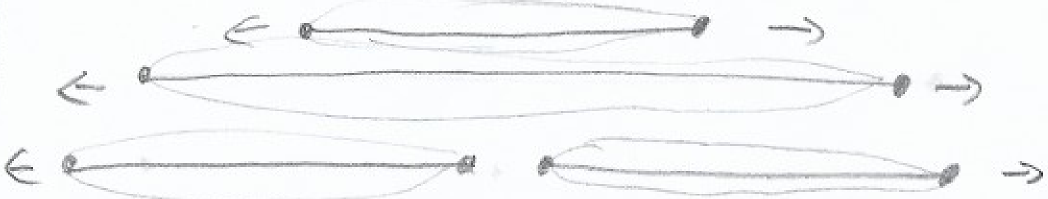
$V(R) \sim \sigma R$ as $R \to \infty$

QCD



eventually pair creation becomes energetically favorable

string breaking



confinement cannot be defined with $V(R)$

$$V(r) \approx A + \frac{B}{r} + \sigma r$$

Why do we expect a Coulomb part?

$$S_G = \frac{1}{4}\int d^4x \; F^a_{\mu\nu} F^{\mu\nu a}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

when $g \to 0$ $\qquad F^a_{\mu\nu} \to \left(F^a_{\mu\nu}\right)_{Abelian}$

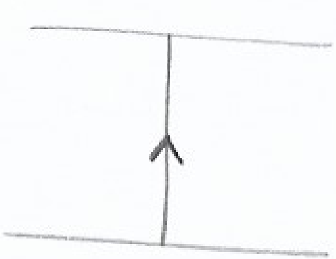
and $\qquad S_G \to \left(N_c^2-1\right) S_G^{Abelian}$

QED $\Rightarrow$ Coulomb potential

thus we also expect the Coulomb part for short distances in the non-Abelian thy

FINITE TEMPERATURE PHASE TRANSITION IN PURE GAUGE THY

Polyakov loop $\quad P(\vec{x}) = \frac{1}{N_c}\,\mathrm{tr}\left[\prod_{j=0}^{N_\tau-1} U_4(\vec{x}, j\hat{4})\right]$



trace of a closed loop in the temporal direction

A similar argument leads to the interpretation

$$|\langle P\rangle| \sim e^{-F_Q/T}$$

free energy of a static color charge

$\langle P\rangle = 0 \iff$ confinement $\quad (F_Q = \infty)$

$\langle P\rangle \neq 0 \iff$ no confinement $\quad (F_Q = \text{finite})$

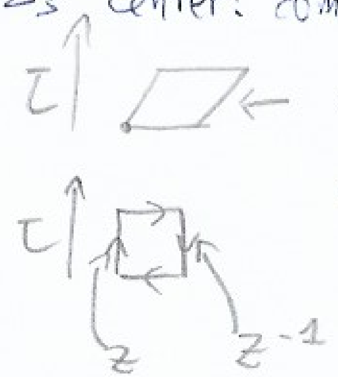
CENTER SYMMETRY (only a symm. in pure gauge: no quarks)

On some fixed timeslice $x_4 = t_0$ we transform links in time direction as:

$$U_4(\vec{x}, t_0) \rightarrow z\, U_4(\vec{x}, t_0) \qquad z \in \left(\mathbb{1}, \mathbb{1}e^{2\pi i/3}, \mathbb{1}e^{-2\pi i/3}\right)$$

$z \in \mathbb{Z}_3$ center: commutes w. all $SU(3)$ elements

GAUGE ACTION INVARIANT

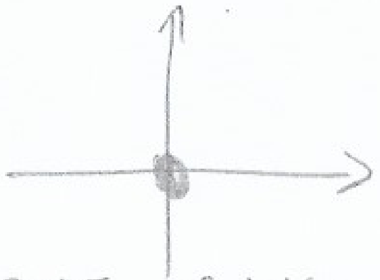


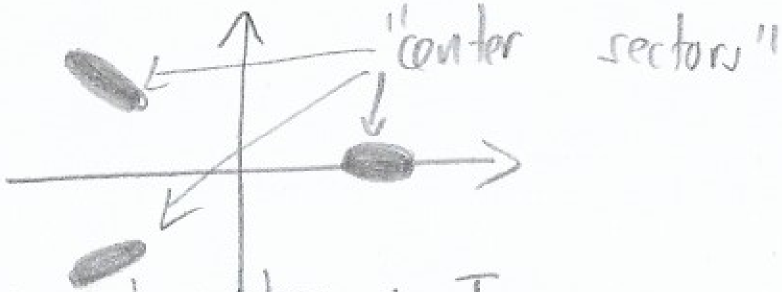
nothing transformed or can commute w. the links and $zz^{-1} = \mathbb{1}$ invariant again

POLYAKOV LOOP NOT INVARIANT $\quad P \rightarrow zP$

POLYAKOV LOOP IS AN ORDER PARAMETER OF THE SPONTANEOUS BREAKING OF CENTER SYMMETRY

HISTOGRAM OF P: low $T < T_c$ $\qquad$ high $T > T_c$





IN PURE GAUGE THY: 1st order transition in T