LECTURE 10:

MORE ON GAUGE THEORY

- GROUP INTEGRATION
- GAUGE INVARIANCE, GAUGE FIXING
- ELITZUR'S THM
- TRANSFER MATRIX

CALCULATING SU(N) GROUP INTEGRALS - 1

defining properties of the Haar measure: $\int dU = 1$

$$\int dU\, f(U) = \int dU\, f(UV) = \int dU\, f(VU)$$

EXAMPLE #1

$$\boxed{\int dU\, U_{ab} = 0}$$

similarly $\boxed{\int dU\, U^{+}_{ab} = 0}$

Proof

Let $M_{ab} := \int dU\, U_{ab} = \int dU (VU)_{ab} = V_{am} M_{mb}$

$\Rightarrow (1 - V)_{am} M_{mb} = 0$

choose a V s.t. $(1 - V)^{-1}$ exists

$\Rightarrow M_{ab} = 0$ $\square$

EXAMPLE #2

$$\boxed{\int_{SU(N)} dU\, U_{ab} U_{cd} = \begin{cases} 0 & N \geq 3 \\ \frac{1}{2}\epsilon_{ac}\epsilon_{bd} & N = 2 \end{cases}}$$

Proof $\boxed{N \geq 3}$

$$\int dU\, U_{ab} U_{cd} = \int dU (UV)_{ab}(UV)_{cd} = M_{abcd}$$

Let $V = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N})$

$\Rightarrow (UV)_{ab} = \sum_c U_{ac} V_{cb} = U_{ac}\delta_{cb} e^{i\theta_c} = e^{i\theta_b} U_{ab}$

$\Rightarrow M_{abcd} = e^{i(\theta_b + \theta_d)} M_{abcd} \Rightarrow M_{abcd} = 0$

$\boxed{N=2}$ we cannot choose 2 independent phases, since $\det V = 1$

cases: $\boxed{b=d}$ $M_{abcb} = \int dU\, U_{ab} U_{cb} = \int dU (UV)_{ab} (UV)_{cb} = e^{i2\theta_b} M_{abcb}$

$\uparrow$ $V = \mathrm{diag}(e^{i\theta}, e^{-i\theta}) = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2})$

$\Rightarrow M_{abcb} = 0$

$\boxed{a=c}$ $M_{abad} = \int dU (VU)_{ab} (VU)_{ad} = e^{2i\theta_a} M_{abad} \Rightarrow M_{abad} = 0$

$V = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2})$

$$VU = \delta_{ac} e^{i\theta_a} U_{cb} = e^{i\theta_a} U_{ab}$$

we are left w. $\int dU\, U_{11} U_{22}$ and $\int dU\, U_{12} U_{21}$

$$\int dU\, U_{12} U_{21} = \int dU (UV)_{12} (UV)_{21} = \int dU\, U_{11}(-U_{22})$$

$V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -U_{12} & U_{11} \\ -U_{22} & U_{21} \end{pmatrix}$$

$$\int dU \det U = 1 = \int dU\, U_{11} U_{22} - \int dU\, U_{12} U_{21}$$

$\Rightarrow \int dU\, U_{11} U_{22} = \frac{1}{2}$ and $\int dU\, U_{12} U_{21} = -\frac{1}{2}$ $\square$

SU(N) GROUP INTEGRALS -2

EXAMPLE #3

$$\boxed{\int dU\, U_{a_1b_1} U_{a_2b_2} U_{a_3b_3} = \begin{cases} 0 & N>3 \\ \frac{1}{3!}\epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3} & N=3 \end{cases}}$$

Proof: similar to example 2, but slightly longer

EXAMPLE #4

$$\boxed{\int_{SU(N)} dU\, U_{a_1b_1} U_{a_2b_2} \dots U_{a_Nb_N} = \frac{1}{N!}\epsilon_{a_1a_2\dots a_N}\epsilon_{b_1b_2\dots b_N}}$$

also true

EXAMPLE #5

$$\boxed{\int_{SU(N)} dU\, U_{ab} U^+_{cd} = \frac{1}{N}\delta_{ad}\delta_{bc}}$$

Proof first set $b=c$ and sum over b $\Rightarrow \int dU \sum_b U_{ab}U^+_{bd} = \int dU (UU^+)_{ad} = \int dU \delta_{ad} = \delta_{ad}$

But $\int dU\, U_{a1} U^+_{1d} = \int dU\, U_{a2} U^+_{2d} = \dots = \int dU\, U_{aN} U^+_{Nd} \Rightarrow \int dU\, U_{ab} U^+_{bd} = \frac{1}{N}\delta_{ad}$ no sum over b

e.g. to establish this we $V_{ab} = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & 0 \\ & 0 & \mathbb{1} \end{pmatrix}_{ab} = \delta_{1a}\delta_{2b} - \delta_{2a}\delta_{1b} \Rightarrow (V^+)_{ab} = \delta_{2a}\delta_{1b} - \delta_{1a}\delta_{2b}$

$$\left.\begin{aligned} \Rightarrow (UV)_{a1} &= \sum_c U_{ac}V_{c1} = \sum_c U_{ac}(\overset{0}{\cancel{\delta_{1c}\delta_{21}}} - \delta_{2c}\delta_{11}) = -U_{a2} \\ (UV)^+_{1d} &= \sum_c V^+_{1c}U^+_{cd} = \sum_c(\delta_{21}\delta_{1c} - \delta_{11}\delta_{2c})U^+_{cd} = -U^+_{2d} \end{aligned}\right\} \Rightarrow \int dU\, U_{a1}U^+_{1d} = \int dU\, U_{a2}U^+_{2d}\ \checkmark$$

example 5 cont.d

$b \neq c$ case: $$\int dU\, U_{ab}\, U^{+}_{cd} = e^{i(\theta_b-\theta_c)} \int dU\, U_{ab}\, U^{+}_{cd} = 0$$

$$V = \mathrm{diag}(e^{i\theta_1} \dots e^{i\theta_N})$$

$$(UV)_{ab} = U_{ac}\,\delta_{cb}\, e^{i\theta_c} = U_{ab}\, e^{i\theta_b}$$

$$(UV)^{+}_{cd} = V^{+}_{ca}\, U^{+}_{ad} = e^{-i\theta_c}\,\delta_{ca}\, U^{+}_{ad} = e^{-i\theta_c}\, U^{+}_{cd}$$

either we can choose θ_b and θ_c independently $(N \geq 3)$

or $\theta_b - \theta_c = 2\theta_b$ $(N=2)$ $\square$

EXAMPLE #6

$$\boxed{\int dU\, \mathrm{Tr}[VU]\, \mathrm{Tr}[U^{+}W] = \frac{1}{N}\mathrm{Tr}[VW]}$$ a consequence of example 5

Proof: $$\sum_{abcd}\int dU\, V_{ba}\, U_{ab}\, (U^{+})_{cd}\, W_{dc} = V_{ba} W_{dc} \int dU\, U_{ab}\, U^{+}_{cd} = V_{ba} W_{dc} \frac{1}{N}\delta_{ad}\delta_{bc}$$

$$= \frac{1}{N} V_{ba} W_{ab} = \frac{1}{N}\mathrm{Tr}[VW]$$

In particular, if we have two plaquettes, we can integrate out the common link

GAUGE INVARIANCE OF EXPECTATION VALUES

Notation: $U \to {}^{\Lambda}U$ means $U_{x\mu} \to \Lambda^{-1}(x+a\hat{\mu})\, U_{x\mu}\, \Lambda(x)$

$$\langle O[U] \rangle = \frac{1}{Z}\int DU\, O[U] e^{-S[U]} = \frac{1}{Z}\int D({}^{\Lambda}U)\, e^{-S[{}^{\Lambda}U]}\, O[{}^{\Lambda}U] = \frac{1}{Z}\int DU\, e^{-S[U]}\, O[{}^{\Lambda}U]$$

$$\Rightarrow \boxed{\langle O[U] \rangle = \langle O[{}^{\Lambda}U] \rangle}$$

even if the observable is constructed in a non gauge invariant manner, the expectation value will be gauge inv.

also

$$\boxed{\langle O[U] \rangle = \int D\Lambda\, \langle O[{}^{\Lambda}U] \rangle}$$

since $\Lambda(x)$ is arbitrary, we might as well integrate $D\Lambda = \prod_x \underbrace{d\Lambda(x)}_{\text{Haar}}$

Example $\boxed{\langle U_{x\mu} \rangle = 0}$ since

$$\int d\Lambda(x)\, U_{x\mu}\, \Lambda(x) = 0$$

GAUGE FIXING

Not necessary, but possible, if one only considers gauge invariant observables

GAUGE FIXING ON A GRAPH

suppose $O[U] = O[{}^{\Lambda}U]$ and $S[U] = S[{}^{\Lambda}U]$ $\forall \Lambda$

try to fix some link variables to $\mathbb{1}$ without changing $\langle O \rangle$

step #1 $U_{x\mu} \longrightarrow U'_{x\mu} = \underbrace{\Lambda^{-1}(x+a\hat{\mu})}_{\text{let} = \mathbb{1}} U_{x\mu} \underbrace{\Lambda(x)}_{\text{let} = U^{\dagger}_{x\mu}} = \mathbb{1}$

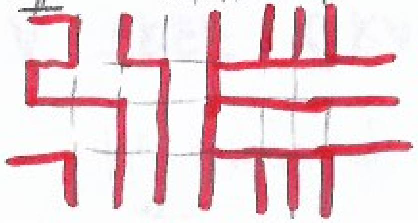
step #2 our choice of $\Lambda(x) = U^{\dagger}_{x\mu}$ now transforms all links starting from x

$$U_{x\nu} \rightarrow U'_{x\nu} = \underbrace{\Lambda^{-1}(x+a\hat{\nu})}_{\text{let} = U_{x\mu} U^{\dagger}_{x\nu}} U_{x\nu} \underbrace{\Lambda(x)}_{U^{\dagger}_{x\mu}} = \mathbb{1}$$

step #3 we can repeat until we come to a point y where we already fixed $\Lambda(y)$

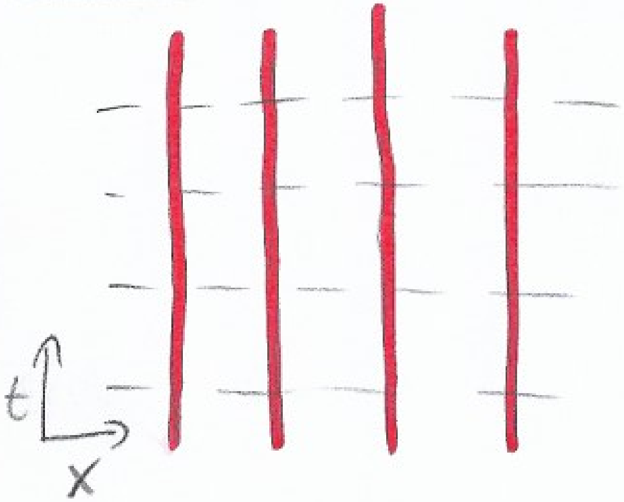
$\Rightarrow$ the links we make $= \mathbb{1}$ cannot form a loop

$\Rightarrow$ the links we make $= \mathbb{1}$ either form a maximal tree graph or are subsets of one

e.g. we can make links $= \mathbb{1}$ like this:  (since $\int dU = 1$ we can leave out the links we set $= \mathbb{1}$ from the path int.)

EXAMPE: TEMPORAL GAUGE

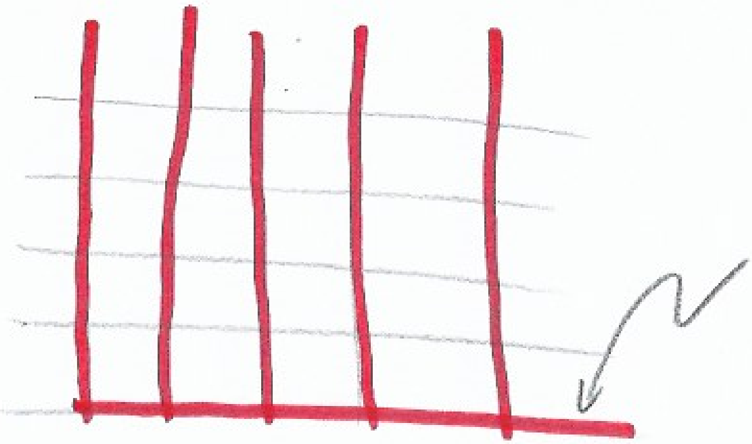
(This is a really awful gauge to discuss e.g. the Hamiltonian formalism)



$\boxed{U_4(x) = \mathbb{1}}$ $\forall x$ continuum version? $U_4 = e^{-aA_4} \Rightarrow \boxed{A_4 = 0}$

Note: I cannot do this on the last link, as I would have a closed loop that wraps around the lattice

Note: this is not a maximal tree yet



I can make this $= \mathbb{1}$ as well

LANDAU GAUGE

(e.g. to compare lattice results w. continuum pert. thy)

$$\boxed{\partial_\mu A_\mu(x) = 0}$$ continuum version

Claim: Let $W[A] = \sum_{\mu=1}^{4} \int d^4x \; tr[A_\mu^2(x)]$

then $\partial_\mu A_\mu(x) = 0 \iff W[A]$ extremal

Proof infinitesimal gauge transform $\Lambda(x) = \exp[\varepsilon H(x)]$ where $Tr\, H = 0$ and $H^\dagger = -H$

$$A_\mu \to \Lambda^{-1}(\partial_\mu + A_\mu)\Lambda = (1 - \varepsilon H)(\partial_\mu + A_\mu)(1 + \varepsilon H) = A_\mu + \varepsilon(\partial_\mu H + [A_\mu, H]) + O(\varepsilon^2)$$

$$W[A] \to W[A] - 2\varepsilon \sum_{\mu=1}^{4} \int d^4x \; tr[A_\mu(x)\, \partial_\mu H(x)] + O(\varepsilon^2) = W[A] + 2\varepsilon \int d^4x \sum_\mu tr[(\partial_\mu A_\mu) H] + O(\varepsilon^2)$$

$$tr[A_\mu [A_\mu H]] = tr(A_\mu A_\mu H - A_\mu H A_\mu) = tr(A_\mu A_\mu H - A_\mu A_\mu H) = 0$$

integration by parts

assume $H(x) \to 0$ as $|x| \to \infty$ (no surface term)

$\square$

$$W_{LATT} = a^2 \sum_{\mu=1}^{4} \sum_{x} \mathrm{tr}\left(U_{x\mu} + U^{\dagger}_{x\mu}\right) \underset{\uparrow}{=} \text{const.} + \sum_{x} a^4 \sum_{\mu} \mathrm{tr}\left[A_\mu^2(x)\right] + O(a^5)$$

$$U_{x\mu} = e^{-aA_\mu} = 1 - aA_\mu + \frac{a^2}{2} A_\mu^2 + \ldots$$

Goal: find a gauge transform s.t. $W_{LATT}[\Lambda]$ is extremal, i.e. we look for the minimum of

$$F[\Lambda] = -a^2 \sum_{x} \sum_{\mu} \mathrm{tr}\left[\Lambda^{\dagger}(x+a\hat{\mu})\, U_{x\mu}\, \Lambda(x) + \Lambda^{\dagger}(x)\, U^{\dagger}_{x\mu}\, \Lambda(x+a\hat{\mu})\right]$$

in the variables $\Lambda(x)$

Note: Coulomb gauge $\sum_{j=1}^{3} \partial_j A_j = 0$ can be fixed in a similar manner

Note: in courses on continuum gauge thy / perturbative gauge thy we often just assume the gauge can be fixed, but don't show how to actually fix it for a given gauge config., but as can be seen here, it is a non-trivial procedure

SUMMARY OF GAUGE INVARIANCE OF $\langle \ldots \rangle$'s

TWO CASES

① NO GAUGE FIXING

$$\langle O[U] \rangle = \langle O[{}^{\Lambda}U] \rangle = \int D\Lambda \, \langle O[{}^{\Lambda}U] \rangle$$

path integral projects out the gauge invariant part of any observable

② If we restrict ourselves to observables w. $O[U] = O[{}^{\Lambda}U]$ then

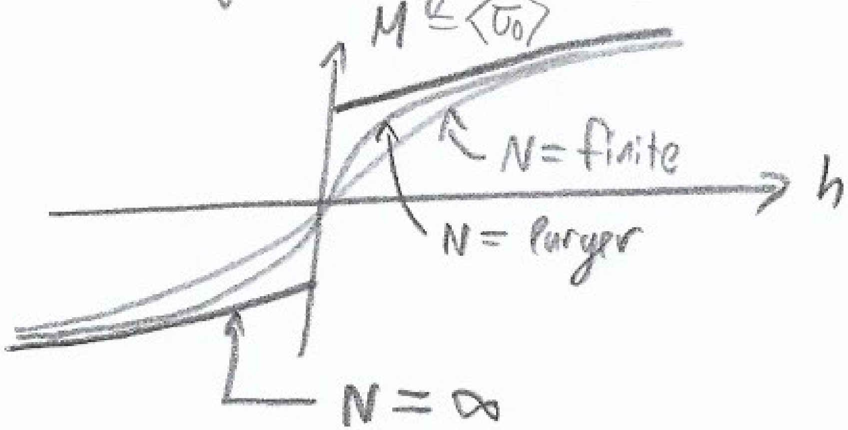
$$\langle O[U] \rangle_{\substack{\text{gauge} \\ \text{fixed}}} = \langle O[U] \rangle$$

any we can fix the gauge if we want

ORDER PARAMETERS AND ELITZUR'S THEOREM

REMINDER: SPONTANEOUS BREAKING OF GLOBAL SYMM.

involves a double limit e.g. Ising model



local order parameter

$$\underbrace{\lim_{h \to 0} \lim_{N \to \infty}}_{} \langle \sigma_0 \rangle \neq 0$$

order of limits important, since

$$\lim_{N \to \infty} \lim_{h \to 0} \langle \sigma_0 \rangle = 0$$

always

Elitzur's thm states that this behavior is impossible in a gauge thy, and e.g.

$$\lim_{h \to 0} \lim_{N \to \infty} \langle U_{x_\mu} \rangle = 0 \quad \text{always}$$

there are no local order parameters in a gauge thy, no spontaneous breaking of gauge symmetries

A VERSION OF THE THM

ASSUME:

1) $\int f[{}^{\Lambda}U] \prod_x d\Lambda(x) = 0$ non gauge invariant observable
can observable w. no gauge inv. part

2) $f[U]$ depends on a finite num. of links

3) compact gauge group

4) symmetry breaking term is real and of the form $J \cdot U$

$$\Longrightarrow \boxed{\lim_{J \to 0} \lim_{N \to \infty} \langle f[U] \rangle = 0}$$

PROOF

$$\langle f[U] \rangle_{N,J} = \frac{1}{Z_{N,J}} \int DU \, f[U] \, e^{-S[U] + J \cdot U}$$

e.g.

$$\varepsilon \Big(\sum_{x\mu} \sum_{ab} J_{x\mu ab} (U_{x\mu})_{\dots} + \sum_{x\mu} \sum_{ab} K_{x\mu ab} (U^{+}_{x\mu})_{\dots}$$

$$\langle f[U] \rangle_{N,J} = \frac{1}{Z_{N,J}} \int D[U] \, f[{}^{\Lambda}U] \, e^{-S[{}^{\Lambda}U] + J \cdot {}^{\Lambda}U} \quad D\Lambda = *$$

Now choose $\Lambda(x)$ such that

$U' \to {}^{\Lambda}U' \neq U'$ for links U' where $f[U]$ has suppor[t]

$U'' \to {}^{\Lambda}U'' = U''$ for links U'' where $f[U]$ has no supp[ort]

$$* = \frac{1}{Z_{N,\mathcal{F}}}\int DU\, e^{-S[U]}\, e^{\mathcal{F}''U''}\left[(e^{\mathcal{F}'{}^\Lambda U'}-1)+\cancel{1}\right] f[{}^\Lambda U]\, D\Lambda$$

disappears due to the condition 1) $\int f[{}^\Lambda U]\, D\Lambda = 0$

$|e^{\mathcal{F}'{}^\Lambda U'}-1| \leq \eta(\varepsilon)$ uniform upper bound in N due to conditions 2) $f[U]$ has finite support 3) gauge group is compact

where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$

$$\Rightarrow \left|\langle f[U]\rangle_{N,\mathcal{F}}\right| \leq \eta(\varepsilon)\,(\sup f)\, \frac{\int DU\, e^{-S} e^{\mathcal{F}''U''}}{\int DU\, e^{-S} e^{\mathcal{F}''U''}\left[(e^{\mathcal{F}'U'}-1)+1\right]} = \frac{\eta(\varepsilon)\sup f}{1+\langle e^{\mathcal{F}'U'}-1\rangle_{\mathcal{F}'=0}}$$

$$\leq \frac{\eta(\varepsilon)\sup f}{1-\eta(\varepsilon)} = \eta'(\varepsilon)\sup f$$ independent of N

$\square$

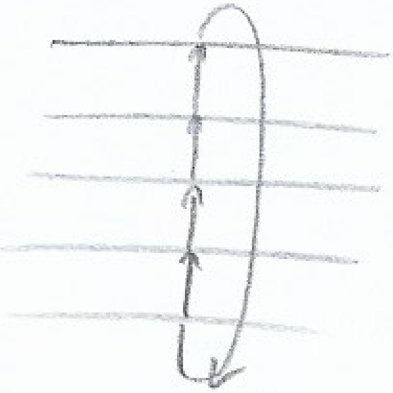
COMMENTS

① the proof doesn't go through for a global symmetry, as we have to transform all var.s at once. $\{U\} = \{U'\}$

② the thm does <u>not</u> imply that there are no PhTr's in gauge thy only that you cannot have local order parameters

③ E.g. an important observable for which the thm doesn't apply is the <u>Polyakov loop</u>

$$\boxed{P(\vec{x}) = \frac{1}{N} \mathrm{Tr} \prod_{x_4=0}^{N_t-1} U_4(\vec{x}, x_0)}$$



It will turn out that this can be an order parameter.
In general interesting observables: <u>traces of closed loops</u>

④ The same proof works if we also have compact matter fields (scalars)

⑤ For non-compact scalar fields, the thm is still true, but the proof is much harder

⑥ Implication for the Hamiltonian formulation: no separate vacua connected by symm. transformations, just the single vacuum, which is gauge invariant

TRANSFER MATRIX

Notation: B_t links in $x_4 = t$ hyperplane $\qquad U_t = \{U(b) : b \in B_t\}$

$B_{t+1,t}$ links between the $x_4 = t+1$ and t hyperplanes $\qquad U_{t+1,t} = \{U(b) : b \in B_{t+1,t}\}$

P_t = plaquettes with all links from U_t

$P_{t+1,t}$ = plaquettes with links from U_t, $U_{t+1,t}$ and U_{t+1}

$$S = \sum_t \underbrace{L[U_{t+1}; U_{t+1,t}; U_t]}_{\frac{1}{2} L_1[U_{t+1}] + L_2[U_{t+1}; U_{t+1,t}; U_t] + \frac{1}{2} L_1[U_t]} + \cancel{\text{const.}}$$

where $L_1[U_t] := -\frac{\beta}{N} \sum_{p \in P_t} \mathrm{Re}\,\mathrm{Tr}\, U_p \qquad L_2[U_{t+1}; U_{t+1,t}; U_t] = -\frac{\beta}{N} \sum_{p \in P_{t+1,t}} \mathrm{Re}\,\mathrm{Tr}\, U_p$

wave functionals $\Psi[U_0]$

- functions of links living on a single timeslice
- T acts on these, with matrix elements

$$T[U_{t+1}, U_t] = \int \prod_{b \in B_{t+1,t}} dU(b)\, e^{-L[U_{t+1}; U_{t+1,t}; U_t]}$$

partition fn.: $Z = \int DU e^{-S} = \int DU_1\, DU_{21}\, DU_2 \dots DU_{N_t, N_t - 1}\, DU_{N_t}\, e^{-\sum_t L}$

$$= \int DU_1\, DU_2 \dots DU_{N_t}\, T[U_{N_t}, U_{N_t-1}] \dots T[U_2\, U_1]\, T[U_1\, U_{N_t - 1}] = \mathrm{Tr}\left[T^{N_t}\right]$$

To def. the Hamiltonian

$$Z = \mathrm{Tr}(e^{-H/T}) = \mathrm{Tr}(e^{-aN_t H}) \stackrel{!}{=} \mathrm{Tr}(T^{N_t}) \Rightarrow T = e^{-aH} \Rightarrow H = -\frac{1}{a}\ln T$$

T has to be positive definite, for this formula to make sense

<u>reflection positivity</u>
(reminder from lecture 2)

point reflection $x_4 \to \theta x_4 = -x_4$
link reflection $x_4 \to \theta x_4 = 1 - x_4$

with $\Theta f(x) = \overline{f(\theta x)}$
on the links $\Theta U(x,y) = \overline{U(\theta x, \theta y)}$

$$\boxed{\langle (\Theta F) F \rangle \geq 0}$$

where F depends on links only at $x_4 \geq 0$ (point refl.) $x_4 \geq 1$ (link refl.)

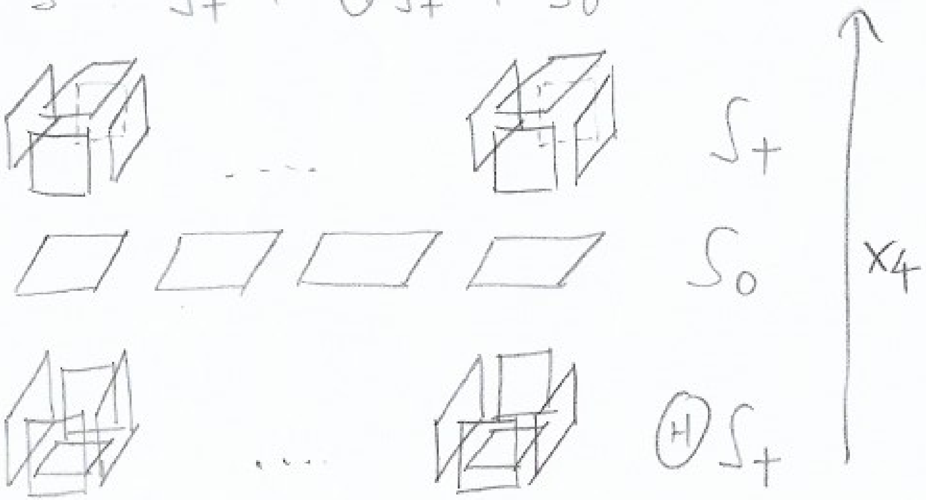
$|\psi\rangle = F[U_0]|\Omega\rangle$ $\quad T|\psi\rangle = \int DU_0\, T[U_0, U_0'] F[U_0]|\Omega\rangle$ etc.

point $\Rightarrow$ $\langle\psi|\psi\rangle \geq 0$, $\langle\psi|T^2|\psi\rangle \geq 0, \ldots$ $\quad\Rightarrow T^2$ positive

link $\Rightarrow$ $\langle\psi|T|\psi\rangle \geq 0$, $\langle\psi|T^3|\psi\rangle \geq 0, \ldots$ $\quad\Rightarrow T$ itself positive

<u>Point reflection positivity for the plaquette action</u>

split:
$$S = S_+ + \Theta S_+ + S_0$$



$$\int \prod_b dU(b)\, F(\Theta F) e^{-S} = \int \prod_b dU(b)\, e^{-S_0}\, \Theta(F e^{-S_+})\, (F e^{-S_+})$$

Let $\mathcal{F}[U_0] = \int \prod_{b \in B_+} dU(b)\, F e^{-S_+} \Rightarrow \Theta \mathcal{F}[U_0] = \overline{\mathcal{F}[U_0]}$
$$= \int \prod_{b \in B_-} dU(b)\, \Theta(F e^{-S_+})$$

$$\Rightarrow Z \langle (\Theta F) F \rangle = \int \prod_{b \in B_0} dU(b)\, e^{-S_0[U]} |\mathcal{F}[U_0]|^2 \geq 0$$

<u>Link reflection positivity:</u> Also true, proof harder (see e.g. Montvay book) (temporal gauge)