LECTURE 9:

GAUGE THEORY ON THE LATTICE

CONTINUUM GAUGE THY (SU(N))

$\phi^i(x) \quad i=1,\dots,N$ complex scalar field

$S = \int d^4x \left\{ \phi(x)(\Box + m^2)\phi(x) + U(\phi(x)\cdot\phi(x)) \right\}$ where $\phi\cdot\varphi = \phi^+\varphi = \sum_{i=1}^{N} \phi^{*i}\varphi^i$

$\exists$ global symmetry $\phi(x) \to \phi'(x) = \Lambda^{-1}\phi(x)$ $\quad \Lambda \in SU(N) \circ\circ\circ$ $\Lambda^+\Lambda = 1$, $\det\Lambda = 1$

we would like to make the symm. local: $\Lambda \to \Lambda(x)$

i.e. $\phi(x) \to \phi'(x) = \Lambda^{-1}(x)\phi(x)$

$\phi\cdot\phi$ invariant, but $\phi\,\Box\phi$ is not

<u>Goal:</u> a kind of derivative, s.t. $D_\mu\phi(x) \to D'_\mu\phi'(x) = \Lambda^{-1}(x)D_\mu\phi(x) \Rightarrow D_\mu\phi\, D_\mu\phi$ inv.

Similarly, if we had fermions $S_F = \int d^4x \; \bar\Psi_a(\not\partial + m)\Psi_a$ and we did

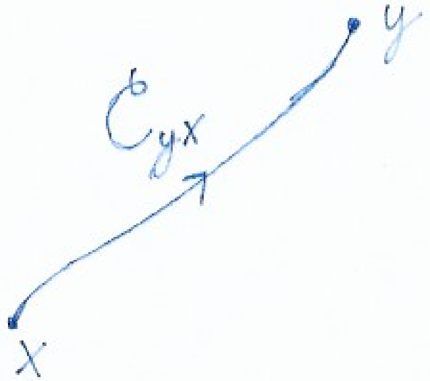
a: new SU(N) index

$\Psi(x) \to \Lambda^{-1}(x)\Psi(x)$

$\bar\Psi_a\Psi_a$ invariant

$\bar\Psi_a \not\partial \Psi_a$ not $\Rightarrow$ same goal

PARALLEL TRANSPORTER



$\phi(x) \in V_x \qquad \phi(y) \in V_y$

def. a <u>parallel transporter</u>

$\mathcal{C}_{yx}$ = a curve from x to y

$U(\mathcal{C}_{yx}) : V_x \to V_y$ linear map ($SU(N)$ matrix)

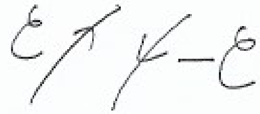
$U(\mathcal{C}_{yx})\,\phi(x) \in V_y$

<u>Properties:</u>

1) $U(\emptyset) = \mathbb{1}$ empty curve

2) $U(\mathcal{C}_2 \circ \mathcal{C}_1) = U(\mathcal{C}_2)\,U(\mathcal{C}_1)$

3) $U(-\mathcal{C}) = U(\mathcal{C})^{-1}$



Furthermore, we want $U(\mathcal{C}_{yx})\,\phi(x) \in V_y$ to transform with $\Lambda^{-1}(y)$

$$\Rightarrow \quad \boxed{U(\mathcal{C}_{yx}) \to U'(\mathcal{C}_{yx}) = \Lambda^{-1}(y)\,U(\mathcal{C}_{yx})\,\Lambda(x)}$$

INFINITESIMAL PARALLEL TRANSPORTER

$$U(\mathcal{C}_{x+dx,x}) = \mathbb{1} - A_\mu(x)\,dx^\mu \qquad (*)$$

U unitary $\Rightarrow$ $UU^\dagger = \mathbb{1} - (A_\mu + A_\mu^\dagger)dx^\mu + O(dx^2) = \mathbb{1}$ $\Rightarrow$ A_μ antihermitian

$\det U = 1$ $\Rightarrow$ $0 = \ln\det U = \mathrm{tr}\ln(1 - A_\mu dx^\mu) = \mathrm{Tr}(-A_\mu dx^\mu + O(dx^2))$ $\Rightarrow$ $\mathrm{Tr}\,A_\mu = 0$

$U \in SU(N) \leftarrow$ Lie group

$A_\mu \in su(N) \leftarrow$ Lie algebra

- In continuum field theory, one usually works with fields that are elements of the Lie algebra. In lattice field theory the fields will be elements of the Lie group.
- Most textbooks on cont. field thy use ~~anti~~ hermitian $A_\mu(x)$ (they have an extra (i) in eq. $(*)$)

COVARIANT DERIVATIVE

$$D\phi := \lim_{dx\to 0}\left(U^{-1}(\mathcal{C}_{x+dx,x})\,\phi(x+dx) - \phi(x)\right) = D_\mu \phi(x)\; dx^\mu \qquad (*)$$

$$D_\mu \phi = \lim_{dx\to 0} \frac{\phi(x+dx^\mu) - \phi(x)}{dx} + A_\mu(x)\phi(x) = \left[\partial_\mu + A_\mu(x)\right]\phi(x)$$

(*) guarantees that $D_\mu \phi(x) \to \Lambda^{-1}(x) D_\mu \phi(x)$ like we wanted

$A_\mu(x)$ is a new field, the <u>gauge field</u>

How does it transform?

$$U(\mathcal{C}_{x+dx,x}) \to \Lambda^{-1}(x+dx)\, U(\mathcal{C}_{x+dx,x})\, \Lambda(x) = \left[\Lambda^{-1}(x) + \partial_\mu \Lambda^{-1}(x)\, dx^\mu\right]\left[1 - A_\mu(x)\, dx^\mu\right]\Lambda(x)$$

$$= \mathbb{1} - \left[\Lambda^{-1}(x) A_\mu \Lambda(x) - \left(\partial_\mu \Lambda^{-1}(x)\right)\Lambda(x)\right] dx^\mu + O(dx^2)$$

$$\Rightarrow \quad A'_\mu(x) = \Lambda^{-1}(x) A_\mu(x) \Lambda(x) - \left(\left(\partial_\mu \Lambda^{-1}(x)\right)\Lambda(x)\right) = \Lambda^{-1}(x)\left[\partial_\mu + A_\mu(x)\right]\Lambda(x)$$

$$\partial_\mu(\Lambda\Lambda^{-1}) = 0$$

$$(\partial_\mu \Lambda^{-1})\Lambda = -\Lambda^{-1}\partial_\mu \Lambda$$

THE FIELD STRENGTH TENSOR

$$\boxed{F_{\mu\nu}(x) = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu(x), A_\nu(x)]}$$

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = \Lambda^{-1}(x)\, F_{\mu\nu}(x)\, \Lambda(x)$$ ← EXERCISE (use A'_μ formula from prev. slide)

$$\mathrm{Tr}(F'_{\mu\nu} F'_{\mu\nu}) = \mathrm{Tr}(\Lambda^{-1} F_{\mu\nu} \cancel{\Lambda \Lambda^{-1}} F_{\mu\nu} \Lambda) = \mathrm{Tr}(\Lambda \Lambda^{-1} F_{\mu\nu} F_{\mu\nu}) = \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu})$$

$\Longrightarrow$

YANG-MILLS ACTION

$$\boxed{S_{YM} = \pm \frac{1}{2g^2} \int d^4x \; \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu})}$$

← describes dynamics of pure gauge theory

conventions:

+ if you defined the $F_{\mu\nu}$ and A_μ to be hermitian

− if you def. it to be anti-hermitian (this is us)

with matter fields included, add:

$$S_{SCALAR} = \int d^4x \left[(D_\mu \phi)^2 + m^2 \bar{\phi}\phi + U(\bar{\phi}\cdot\phi) \right]$$

$$S_{FERMION} = \int d^4x \; \bar{\Psi} [\gamma_\mu D_\mu + m] \Psi$$

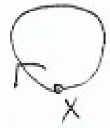
COMMENTS

$\phi(x) \leftarrow \phi(y)$

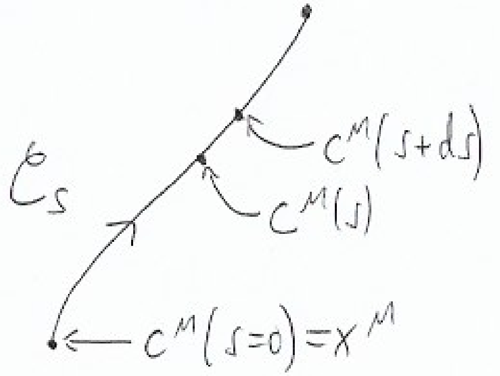
① general gauge invariant quantities constructed from matter fields and parallel transporters

i) $\phi(x) \cdot U(\mathcal{C}_{xy})\, \phi(y)$ or $\bar{\Psi}(x)\, U(\mathcal{C}_{xy})\, \psi(y)$

ii) $\mathrm{Tr}\left(U(\mathcal{C}_{xx})\right)$



② path ordered exponential form of parallel transporters



$$U(\mathcal{C}_{s+ds}) = \left[1 - A_\mu(x)\, \underbrace{dx^\mu}_{\frac{dc^\mu}{ds} ds}\right] U(\mathcal{C}_s)$$

$$\Rightarrow \boxed{\frac{d}{ds} U(\mathcal{C}_s) = - A_\mu(x) \frac{dc^\mu}{ds}\, U(\mathcal{C}_s)}$$

initial cond.s $U(\mathcal{C}_0) = \mathbb{1}$

this is the same eq. as the eq. satisfied by the time evolution operator in interaction picture (Matteo's particle phys. lecture notes L10, page 2) $\left(\begin{array}{l} s \to t \\ \frac{dc^\mu}{ds} A_\mu \to H_I \end{array}\right)$

Solution:

path order, i.e. ordered in s

$$\boxed{U(\mathcal{C}_s) = \mathcal{P} \exp\left[-\int_0^s A_\mu(x) \frac{dc^\mu}{ds}\, ds\right] = \mathcal{P} \exp\left[-\int_{\mathcal{C}_s} A_\mu dx^\mu\right]}$$

GAUGE THY ON THE LATTICE

hopping term: $-2\kappa \sum_{\langle xy \rangle} \phi(x)\cdot\phi(y)$ not invariant

scalar thy: $S = \sum_x U(\phi\cdot\phi) - 2\kappa \sum_{\langle xy\rangle} \phi(x)\cdot\phi(y)$

$\Rightarrow$ parallel transporters for the elementary lattice vectors

$$\mathcal{C}_{x+a\hat\mu, x} = \underset{x}{\bullet} \longrightarrow \underset{x+a\hat\mu}{\bullet} \qquad \Rightarrow U(\mathcal{C}_{x+a\hat\mu,x}) \equiv U_{x\mu} = U(x+a\hat\mu, x)$$

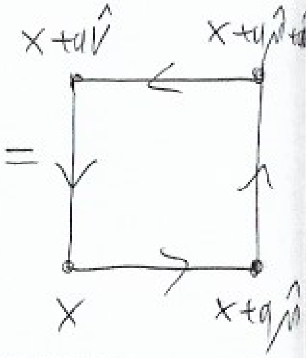
link variables

$$\sum_{\langle xy\rangle} \phi(x)\phi(y) \longrightarrow \sum_{\langle xy\rangle} \phi(x)\, U(x,y)\, \phi(y)$$ and similarly for fermions

dynamics of gauge fields

simplest gauge invariant loop: plaquette

$$U_p = U_{x;\mu\nu} = U(x, x+a\hat\nu)\, U(x+a\hat\nu, x+a\hat\mu+a\hat\nu)\, U(x+a\hat\mu+a\hat\nu, x+a\hat\mu)\, U(x+a\hat\mu, x) =$$



$\mathrm{Tr}(U_p) \leftarrow$ gauge inv.

Wilson/plaquette action: $S = \sum_p S_p(U_p)$	$\sum_p = \sum_x \sum_{1\le\mu<\nu\le 4}$	$S_p = -\beta\left(1 - \frac{1}{N}\mathrm{Re}\,\mathrm{Tr}\, U\right)$

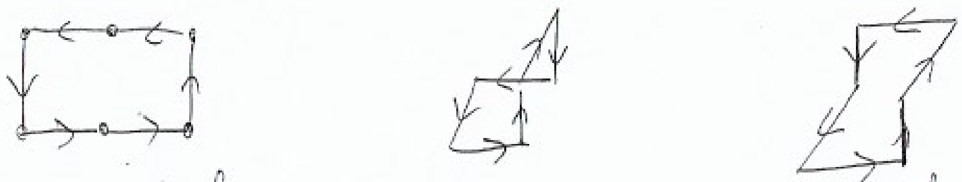
COMMENTS

ⓐ $\mathrm{Re}\,\mathrm{Tr}\,U = \cos\theta_1 + \dots + \cos\theta_N \leq N \Rightarrow S_p(U) \geq 0$

U unitary $\Rightarrow \lambda = e^{i\theta}$

This is how we fixed the const. β in S_p, but anyway it is unimportant

ⓑ The trace of an arbitrary closed loop is gauge invariant, e.g. for 6 links we have



2×1 rectangle   chair   twisted

we could have added those to the action, we just choose the simplest possibility

PLAQUETTE $\Rightarrow$ CONTINUUM YM

$$U(x+a\hat{\mu},x) = e^{-aA_\mu(x)} = 1 - aA_\mu(x) + \frac{a^2}{2}A_\mu(x)^2 + \dots$$

$$\Delta^f_\mu f(x) := \frac{1}{a}\left[f(x+a\hat{\mu}) - f(x)\right]$$

$$F^f_{\mu\nu}(x) := \Delta^f_\mu A_\nu(x) - \Delta^f_\nu A_\mu(x) + \left[A_\mu(x), A_\nu(x)\right]$$

<u>Claim:</u> $U_p = U_{x;\mu\nu} = \exp\left[-a^2 F^f_{\mu\nu}(x) + O(a^3)\right]$

suppose it is true, then

$$\mathrm{Tr}(U_p + U_p^+) = 2\,\mathrm{Tr}\left[1 + \frac{a^4}{2} F^f_{\mu\nu} \underset{\text{no sum}}{F^f_{\mu\nu}}\right] + O(a^5)$$

$$\sum_p = \sum_x \sum_{\mu<\nu} = \frac{1}{2}\sum_x\sum_{\mu\nu} \qquad (\mu=\nu \text{ doesn't count, as } F_{\mu\nu} = -F_{\nu\mu})$$

$$\Rightarrow S_p = \frac{1}{2}\sum_x\sum_{\mu\nu}\beta\left(1 - \frac{1}{N}\mathrm{Re\,tr}\,U_p\right) = \frac{1}{2}\sum_x\sum_{\mu\nu}\left(\frac{1}{2N}\mathrm{tr}(U_p+U_p^+) - 1\right)(-\beta) = \frac{-\beta}{4N}\sum_x\sum_{\mu\nu} a^4\,\mathrm{Tr}\left[F_{\mu\nu}F_{\mu\nu}\right] + O(a^5)$$

$\Rightarrow$ the prefactor $\dfrac{\beta}{4N} = \dfrac{1}{2g^2}$ $\Rightarrow$ $\boxed{\beta = \dfrac{2N}{g^2}}$ inverse bare coupling

PROOF OF THE CLAIM

Baker – Campbell – Hausdorff: $e^{A} e^{B} = e^{A+B+\frac{1}{2}[A,B]+\dots} \overset{\text{exercise}}{\Longrightarrow} e^{B_1} e^{B_2} e^{B_3} e^{B_4} = e^{\sum_i B_i + \frac{1}{2}\sum_{i<j}[B_i,B_j]+\dots}$

$$U_{x,\mu\nu} = \underbrace{U(x,x+a\hat{\nu})}_{U^+(x+a\hat{\nu},x)} \underbrace{U(x+a\hat{\nu},x+a\hat{\mu}+a\hat{\nu})}_{U^+(x+a\hat{\mu}+a\hat{\nu},x+a\hat{\nu})} U(x+a\hat{\mu}+a\hat{\nu},x+a\hat{\mu})\, U(x+a\hat{\mu},x) = e^{\overset{B_1}{a A_\nu(x)}} e^{\overset{B_2}{a A_\mu(x+a\hat{\nu})}} e^{\overset{B_3}{-a A_\nu(x+a\hat{\mu})}} e^{\overset{B_4}{-a A_\mu(x)}}$$

$$\equiv \exp(\phi_{\mu\nu})$$

$$A_\nu(x+a\hat{\mu}) = A_\nu(x) + a\,\frac{A_\nu(x+a\hat{\mu})-A_\nu(x)}{a} = A_\nu(x) + a\,\Delta^f_\mu A_\nu(x)$$

$$\phi_{\mu\nu} = a A_\nu(x) + a A_\mu(x+a\hat{\nu}) - a A_\nu(x+a\hat{\mu}) - a A_\mu(x) + \frac{1}{2}\sum_{i<j}[B_i,B_j]$$

$$= \cancel{a A_\nu(x)} + \cancel{a A_\mu(x)} + a^2 \Delta^f_\nu A_\mu(x) - \cancel{a A_\nu(x)} - a^2 \Delta^f_\mu A_\nu(x) - \cancel{a A_\mu(x)} + \frac{1}{2}\sum_{i<j}[B_i,B_j]$$

$$= a^2\left[\Delta^f_\nu A_\mu(x) - \Delta^f_\mu A_\nu(x)\right] + \frac{a^2}{2}\left(\cancel{[A_\nu A_\mu]} - \cancel{[A_\nu,A_\nu]} - \cancel{[A_\nu,A_\mu]} - [A_\mu A_\nu] - \cancel{[A_\mu A_\mu]} + [A_\nu,A_\mu]\right)$$

$$+ O(a^3)$$

$$= a^2\left(\Delta^f_\nu A_\mu(x) - \Delta^f_\mu A_\nu(x) - [A_\mu,A_\nu]\right) + O(a^3) = -a^2 F^f_{\mu\nu}(x) + O(a^3)$$

$\square$

PATH INTEGRAL MEASURE?

$$Z = \int \mathcal{D}U \, e^{-S[U]}$$

$$\prod_{x,\mu} dU_{x,\mu}$$

? integral over links, but how to integrate over a single link?

real parameters

generators

$U \in SU(N)$ can e.g. be parametrized like this $U = \exp\left[i \sum_{a=1}^{N^2-1} w^a T^a\right]$

1st guess: $\int dU \sim \int \prod_{a=1}^{N^2-1} dw^a$ <u>WRONG</u>

we want the measure to be

(I) PARAMETRIZATION INDEPENDENT ⇝ we need a metric on the group manifold

(II) GROUP INVARIANT ⇝ tells us what the metric can be

PARAMETRIZATION INDEPENDENCE

usual construction in differential geometry (like in GR)

$\Rightarrow$ def. a <u>metric tensor</u>

$$g'_{a'b'} = g_{ab} \frac{\partial w^a}{\partial w'^{a'}} \frac{\partial w^b}{\partial w'^{b'}} \qquad g \text{ pos. definite}$$

$\Rightarrow$ $\sqrt{\det g} \prod_a dw^a$ is invariant, as

$$\sqrt{\det g'} = \sqrt{\det g \left(\det \frac{\partial w}{\partial w'}\right)^2} = \sqrt{\det g} \left|\det \frac{\partial w}{\partial w'}\right|$$

$$\prod dw'^a = \left|\det \frac{\partial w'}{\partial w}\right| \prod_a dw^a = \left|\det \frac{\partial w}{\partial w'}\right|^{-1} \prod_a dw^a$$

$$\Rightarrow \quad \sqrt{\det g'} \prod_a w'^a = \sqrt{\det g} \prod_a dw^a$$

BUT: $\boxed{g_{ab} = ?}$ $\Leftarrow$ fixed by the group invariant property

GROUP INVARIANCE

WHY? gauge transform $U_{x\mu} \rightarrow \Lambda^{-1}(x+a\hat{\mu})\, U_{x\mu}\, \Lambda(x)$

leaves action invariant $S[U] = S[U']$

we want: $\int DU\, e^{-S[U']} = \int DU'\, e^{-S[U']}$

$$\Rightarrow dU_{x\mu} = d[\Lambda^{-1}(x+a\hat{\mu})\, U_{x\mu}\, \Lambda(x)]$$

$\Lambda^{-1}(x+a\hat{\mu})$ and $\Lambda(x)$: can be chosen independently

$$\Rightarrow \boxed{\underline{\text{GOAL}} \quad dU = d(UV) = d(VU)} \quad V \in SU(N)$$

GROUP INV. DISTANCE

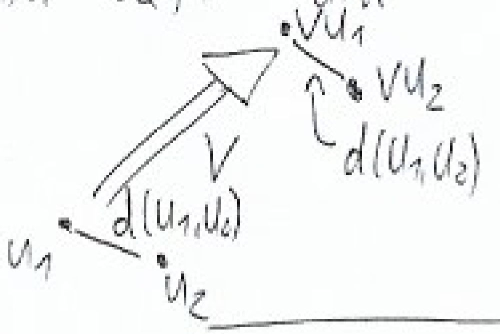
$$d(U_1, U_2) := \mathrm{Tr}[(U_1^{+} - U_2^{+})(U_1 - U_2)]$$

$$d(U_1, U_2) = d(U_1 V, U_2 V) = d(V U_1, V U_2)$$

invariant under both left- and righ-mult.

$$d(U_1 V, U_2 V) = \mathrm{Tr}[V^{+}(U_1^{+} - U_2^{+})(U_1 - U_2)V]$$

$$d(V U_1, V U_2) = \mathrm{Tr}[(U_1^{+} - U_2^{+})V^{+}V(U_1 - U_2)]$$



INFINITESIMAL VERSION: $ds^2 = \mathrm{Tr}[dU^{+} dU] = \sum_{ab} \mathrm{Tr}\left[\frac{\partial U^{+}}{\partial w^a} \frac{\partial U}{\partial w^b}\right] dw^a\, dw^b \Rightarrow \boxed{g_{ab}(w) = \mathrm{Tr}\left[\frac{\partial U^{+}}{\partial w^a} \frac{\partial U}{\partial w^b}\right]}$

$$dU = \mathcal{N} \sqrt{\det g(w)} \prod_a dw^a$$

$$g_{ab}(w) = \mathrm{Tr}\left[\frac{\partial U^+}{\partial w^a} \frac{\partial U}{\partial w^b}\right]$$

normalization: $\int dU = 1 \Rightarrow \mathcal{N}$

$\Rightarrow$ parametrization invariant

$\Rightarrow$ group invariant $\int_{SU(N)} dU\, f(U) = \int_{SU(N)} dU\, f(UV) = \int_{SU(N)} dU\, f(VU)$

COMMENTS

we only looked at SU(N)

we didn't prove uniqueness

① For any compact group there exists a unique measure s.t.

1) $\int_G f(U)\,dU = \int_G f(UV)\,dU = \int_G f(VU)\,dU$ and 2) $\int_G dU = 1$ **HAAR MEASURE**

② Also works for discrete groups

e.g. $\mathbb{Z}_2$ gauge thy $\sigma(b) = \pm 1$

$S = -\beta \sum_p (\sigma_p - 1)$ gauge tr. $\sigma(x+\hat{\mu}, x) \to \Lambda^{-1}(x+\hat{\mu})\, \sigma(x+\hat{\mu}, x)\, \Lambda(x)$, ± 1

$\sigma_p = \sigma(b_1)\sigma(b_2)\sigma(b_3)\sigma(b_4) = \sigma(x, x+\hat{\nu})\, \sigma(x+\hat{\nu}, x+\hat{\mu}+\hat{\nu})\, \sigma(x+\hat{\mu}+\hat{\nu}, x+\hat{\mu})\, \sigma(x+\hat{\mu}, \sigma)$

$$\int_{\mathbb{Z}_2} f(\sigma)\, d\sigma = \frac{1}{2}\left[f(+1) + f(-1)\right]$$

③ For $U(1)$ $S = -\beta \sum (\cos\varphi - 1)$ Haar-measure $\frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi$

④ To define the PI we didn't need gauge fixing

EXAMPLE: SU(2)

$$U = x^0 \mathbb{1} + i\vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x^0 + ix^3 & x^2 + ix^1 \\ -x^2 + ix^1 & x^0 - ix^3 \end{pmatrix}$$

$$\det U = 1 \Rightarrow (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 1$$

4D spherical coord.s

$w^1 = \phi \in [0, 2\pi)$
$w^2 = \theta \in [0, \pi)$
$w^3 = \psi \in [0, \pi)$

$$\begin{aligned} x^0 &= r\cos\theta \\ x^1 &= r\sin\theta\cos\psi \\ x^2 &= r\sin\theta\sin\psi\cos\phi \\ x^3 &= r\sin\theta\sin\psi\sin\phi \end{aligned} \xrightarrow{r=1} \begin{aligned} x^0 &= \cos\theta \\ x^1 &= \sin\theta\cos\psi \\ x^2 &= \sin\theta\sin\psi\cos\phi \\ x^3 &= \sin\theta\sin\psi\sin\phi \end{aligned}$$

EXERCISE:

$$g_{ab} = \mathrm{Tr}\left[\frac{\partial U^+}{\partial w^a}\frac{\partial U}{\partial w^b}\right] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2\sin^2\theta & 0 \\ 0 & 0 & 2\sin^2\theta\sin^2\psi \end{bmatrix} \Rightarrow dU = \mathcal{N}\sin^2\theta\sin\psi\, d\theta\, d\psi\, d\phi$$

$$\int_0^\pi \sin^2\theta\, d\theta \int_0^\pi \sin\psi\, d\psi \int_0^{2\pi} d\phi = \frac{\pi}{2}2(2\pi) = 2\pi^2 \Rightarrow \boxed{dU = \frac{1}{2\pi^2}\sin^2\theta\sin\psi\, d\theta\, d\psi\, d\phi}$$

EXERCISE:

$$\left|\det\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(r, \theta, \psi, \phi)}\right| = r^3\sin^2\theta\sin\psi \Rightarrow \frac{1}{\pi^2}\delta(r^2-1)\overbrace{r^3 dr\sin^2\theta\, d\theta\sin\psi\, d\psi\, d\phi}^{d^4x}$$

$$\xrightarrow{\int dr} \frac{1}{2\pi^2}\sin^2\theta\, d\theta\sin\psi\, d\psi\, d\phi$$

$$\Rightarrow \boxed{dU = \frac{1}{\pi^2}\delta(x^2-1)\, d^4x}$$

COMMENT: Explicit form of the Haar measure is often not needed

- certain algorithms automatically generate matrices w. the correct measure ("uniform" measure on the matrix group)
- analytical integrals can often be calculated via symmetry considerations

PATH INTEGRAL FOR LATTICE QCD

1 FLAVOR

$$Z = \int \underbrace{DU}_{\text{Haar}} \overbrace{D\bar{\Psi} D\Psi}^{\text{Berezin/Grassmann}} e^{-S_{GAUGE} - \bar{\Psi} \overbrace{M}^{\text{e.g. Wilson}} \Psi} = \int DU \det M[U] \, e^{-S_{GAUGE}}$$

2+1 FLAVOR

$$\det M \rightarrow \underbrace{\det M_u \det M_d}_{\text{assume } m_u = m_d} \det M_s = (\det M_{ud})^2 \det M_s$$

STAGGERED?

staggered $\det M$ describes 4 fermions

rooting $\det M_{staggered} \rightarrow (\det M_{staggered})^{N_f/4}$

Has conceptual issues.

90% consensus: OK

PURE GAUGE

No $(\det M)$. Also called quenched QCD.