LECTURE 8:

FERMION DOUBLING AND A SMALL TASTE OF LATTICE FERMIONS

THE DIRAC FIELD

Minkowski $S = \int d^4x \, \overline{\Psi} \, (i \gamma^\mu_{(M)} \partial_\mu - m) \Psi$ where $\{\gamma^\mu_{(M)}, \gamma^\nu_{(M)}\} = 2 \eta^{\mu\nu} \mathbb{1}_{4\times4}$

Minkowski

canonical momenta $\pi = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} = i \Psi^\dagger$

Hamiltonian $H = \int d^3x \left(\pi \partial_0 \Psi - \mathcal{L} \right) = \int d^3x \, \overline{\Psi} (-i \vec{\gamma}_{(M)} \cdot \vec{\partial} + m) \Psi$

Let the Euclidean γ matrices be

$$\left.\begin{aligned} \gamma_1 &= \gamma_1^{(E)} = -i \gamma_1^{(M)} \\ \gamma_2 &= \gamma_2^{(E)} = -i \gamma_2^{(M)} \\ \gamma_3 &= \gamma_3^{(E)} = -i \gamma_3^{(M)} \\ \text{and } \gamma_4 &= \gamma_4^{(M)} \end{aligned}\right\} \quad \{\gamma^\mu, \gamma^\nu\} = 2 \delta^{\mu\nu} \mathbb{1}$$

$$\Rightarrow \boxed{H = \int d^3x \, \overline{\Psi} (\vec{\gamma} \cdot \vec{\partial} + m) \Psi}$$

REMINDER OF LAST TIME

we had creation/annihilation operators s.t.

$$a_i|0\rangle = 0 \quad \text{Fock vacuum}$$

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0$$

$$\{a_i, a_j^\dagger\} = \delta_{ij}$$

and a Hamiltonian in normal form $H(a_i^\dagger, a_j)$

we introduced coherent states $a_i|\eta\rangle = \eta_i|\eta\rangle$ eigenstates of annihilation op-s

Grassmann numbers

$$|\eta\rangle = \exp\left(-\sum_i \eta_i a_i^\dagger\right)|0\rangle$$

we used these as intermediate states to derive the path integral

$$Z = \int_{\eta(0)=-\eta(\beta)} \mathcal{D}(\bar{\eta}_i, \eta_i)\, e^{-S_E}$$

$$S_E = \int_0^\beta \left[\bar{\eta}_i \partial_\tau \eta_i + H(\bar{\eta}_i, \eta_j)\right] d\tau$$

PATH INTEGRAL FOR THE DIRAC FIELD - 1

a few little adjustments and we can use the formalism from last time

* instead of creation/annih op-s we want to construct the coherent states for the field op. ψ

$\{\psi_\alpha(\vec{x})\,\psi_\beta(\vec{y})\}=0=\{\psi^+_\alpha(\vec{x})\,\psi^+_\beta(\vec{y})\}$ and $\{\psi_\alpha(\vec{x}),\ \psi^+_\beta(\vec{y})\} = \delta_{\alpha\beta}\,\delta_{\vec{x}\vec{y}}$ commutation relation OK

but $\psi_\alpha(\vec{x}) \sim \int_p \sum_s \Big(a(\vec{p},s)\, u_\alpha(\vec{p},s)\, e^{i\vec{p}\vec{x}} + b^+(\vec{p},s)\, v_\alpha(\vec{p},s)\, e^{-i\vec{p}\vec{x}} \Big)$

eliminates the Fock vacuum

doesn't eliminate

so we construct a different "vacuum"

$$|0,\psi\rangle \equiv \prod_{\alpha,\vec{x}} \psi_\alpha(\vec{x})\,|f\rangle$$

where $|f\rangle$ is any state s.t. $|0,\psi\rangle \neq 0$ (e.g. $|f\rangle = |0,\text{Fock}\rangle$ works)

now by construction the CARs imply

$$\psi_\alpha(\vec{x})\,|0,\psi\rangle = 0$$

and we can define the coherent states as

$$|\eta\rangle = \exp\Big(-\sum_{\vec{x},\alpha} \eta_{\alpha\vec{x}}\, \psi^+_{\alpha\vec{x}}\Big)\,|0,\psi\rangle$$

and now the derivation goes through as before

PATH INTEGRAL FOR THE DIRAC FIELD -2

Now we change $\eta \to \Psi$ as there is now chance of confusing the Grassmann vars with the field operators

$$\Rightarrow Z = \int \prod_{x\alpha} d\bar{\Psi}_{x\alpha}\, d\Psi_{x\alpha}\; e^{-S_E}$$

$$S_E = \int_0^\beta d\tau \left[\sum_{\vec{x}\alpha} \Psi^+_\alpha(\vec{x})\, \partial_0 \Psi_\alpha + H \right]$$

$$\boxed{\rightarrow \int_0^\beta d\tau \int d^3x \left[\bar{\Psi} \, (\underbrace{\gamma_\mu \partial_\mu + m}) \, \Psi \right] = S_E \quad \Psi(\vec{x},0) = -\Psi(\vec{x},\beta) \;\; \text{anti periodic}}$$

Terminology: Dirac operator $M = D + m$

Substitution:

$$S_E = -iS_M \quad \text{as} \quad t_M = -i\tau$$

$$-i\mathcal{L}_M = -i\,\bar{\Psi}(i\gamma^\mu_{(M)}\partial_\mu - m)\Psi = \bar{\Psi}(\gamma^\mu_{(M)}\partial_\mu + im)\Psi = \bar{\Psi}(\gamma^0_{(E)}\, i\,\partial^0_{(E)} + \overbrace{\gamma^i_{(M)}}^{i\gamma^i_{(E)}}\partial^{(E)}_i + im)\Psi$$

$$S_E = -iS_M = \int d^4x^{(M)}(-i\mathcal{L}_M) = \underbrace{(-i)i}_{1}\int d\tau \int d^3x \;\bar{\Psi}(\gamma^\mu_{(E)}\partial^{(E)}_\mu + m)\Psi$$

CHIRAL SYMMETRY (1 FLAVOR)

$\gamma_5 \equiv \gamma_1 \gamma_2 \gamma_3 \gamma_4 \qquad \{\gamma_\mu, \gamma_5\} = 0$

$$\psi \to \psi' = e^{i\alpha\gamma_5}\psi \qquad \bar{\psi} \to \bar{\psi}' = \bar{\psi} e^{i\alpha\gamma_5}$$

kinetic term $\quad \bar{\psi}\, i\gamma_\mu \partial_\mu \psi \to \bar{\psi} e^{i\alpha\gamma_5} i\gamma_\mu \partial_\mu e^{i\alpha\gamma_5}\psi$

$$= \bar{\psi} \underbrace{e^{i\alpha\gamma_5} e^{-i\alpha\gamma_5}}_{1} i\gamma_\mu \partial_\mu \psi = \bar{\psi}\, i\gamma_\mu \partial_\mu \psi$$

invariant

mass term $\quad m\bar{\psi}\psi \to m\bar{\psi} e^{i2\alpha\gamma_5}\psi \quad$ not invariant

denoting $\quad i\gamma_\mu \partial_\mu \equiv D$

$$\delta(\bar{\psi} D \psi) = i\alpha\left(\bar{\psi}\gamma_5 D\psi + \bar{\psi} D\gamma_5 \psi\right) = i\alpha\, \bar{\psi}\{D, \gamma_5\}\psi$$

can be stated as $\boxed{\{D, \gamma_5\} = 0}$

Chiral symmetry is of central importance in QCD, and we will have many things to say about it next semester. Today, we just look at some consequences on the properties of the the Dirac operator D (mostly just linear algebra)

SOME PROP.S OF D (and M) IN THE CONTINUUM THY

$\gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4$ $\qquad \gamma_5^2 = 1$ $\qquad \{\gamma_5,\gamma_\mu\}=0$ $\qquad M = D + m$

① γ_5 hermiticity $\qquad D^\dagger = \gamma_5 D \gamma_5$ $\qquad M^\dagger = \gamma_5 M \gamma_5$

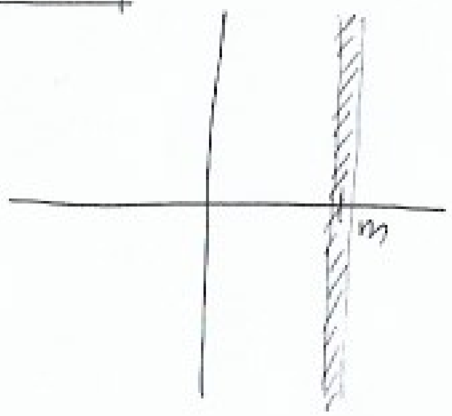
② chiral symmetry $\qquad \{D,\gamma_5\}=0$ $\qquad$ (not for M)

Consequences: **A** ① $\Rightarrow$ $(\det M)^* = \det M^\dagger = \det \gamma_5 M \gamma_5 = \det \gamma_5\gamma_5 M = \det M \in \mathbb{R}$

B ① and ② $\Rightarrow 0 = \gamma_5 D + D\gamma_5 = D^\dagger\gamma_5 + D\gamma_5 \Rightarrow \boxed{D^\dagger + D = 0 \text{ antihermitian}}$

$\Rightarrow$ eigenvalues of D are purely imaginary

$\Rightarrow$ eigenvalues of M are of the form $m + i\lambda$



C ① $\Rightarrow P(\lambda) = \det(M - \lambda \mathbb{1}) = \det[\gamma_5^2(M-\lambda\mathbb{1})] = \det[\gamma_5(M-\lambda\mathbb{1})\gamma_5]$

$$= \det[M^\dagger - \lambda\mathbb{1}] = \det[M - \lambda^*\mathbb{1}]^* = P(\lambda^*)^*$$

thus $P(\lambda)=0 \Rightarrow P(\lambda^*)=0$ eigenvalues of a γ_5-hermitian matrix are either real or come in complex conjugate pairs

D ① and ② $\Rightarrow$ B and C $\Rightarrow \det M > 0$ (provided $m>0$)

E ① $\Rightarrow \lambda(v_\lambda, \gamma_5 v_\lambda) = (v_\lambda, \gamma_5 D v_\lambda) = (v_\lambda, D^\dagger\gamma_5 v_\lambda) = (Dv_\lambda, \gamma_5 \lambda) = \lambda^*(v_\lambda, \gamma_5 v) \Rightarrow \mathrm{Im}\lambda\, (v_\lambda, \gamma_5 v_\lambda) = 0$

only e.vec.s with real e.val.s can have nonvanishing chirality, i.e. $(v_\lambda, \gamma_5 v_\lambda) \neq 0$

NAIVE FERMIONS

$$S_E = \int d^4x \; \overline{\Psi} \left(\gamma_\mu \partial_\mu + m \right) \Psi$$

(lec 2: int by parts)

∂_μ antihermitian, to keep this property we choose $\partial_\mu \to \Delta_\mu^{symm} = \frac{1}{2}\left(\Delta_\mu^f + \Delta_\mu^b \right)$ (*)

$$S_E = \sum_{xy} a^4 \overline{\Psi}_x \left(D_{xy}^{naive} + m \delta_{xy} \right) \Psi_y \quad \text{where} \quad D_{xy}^{naive} = \sum_{\mu=1}^{4} \frac{1}{2a} \gamma_\mu \left(\delta_{y,x+a\hat{\mu}} - \delta_{y,x-a\hat{\mu}} \right)$$

satisfies both γ_5 hermiticity $\left(D^{naive} \right)^+ = \gamma_5 D^{naive} \gamma_5$

and chiral symmetry $\left\{ D^{naive}, \gamma_5 \right\} = 0$

so everything from the previous page will be also true for naive lattice fermions.

BUT: the cont. lim. of naive fermions does not describe one but rather 16 Dirac fermions ↯

[(*) if we took $\partial_\mu \to \Delta_\mu^f$ we would have serious problems later (non-unitarity etc.) (Rothe book)]

[for bosons the symm. deriv. was not necessary because of the two derivatives]

FREE PROPAGATOR

$$\langle \psi(x)\bar\psi(y)\rangle = M^{-1}_{xy} = G_{xy} \qquad \sum_y M_{xy}G_{yz} = \delta_{xz}$$

momentum space

$$M(p) = \sum_x e^{-ipx}\left[m\delta_{x0} + \sum_{\mu=1}^{4}\frac{\gamma_\mu}{2a}\left(\delta_{-a\hat\mu,x} - \delta_{a\hat\mu,x}\right)\right] = m + \frac{1}{a}i\sum_{\mu=1}^{4}\gamma_\mu \underbrace{\sin(ap_\mu)}_{\equiv \hat p_\mu a}$$

$$G(p) = \frac{1}{m + i\sum_{\mu=1}^{4}\gamma_\mu \hat p_\mu} = \frac{m - i\sum_\mu \gamma_\mu \hat p_\mu}{m^2 + \sum_{\mu=1}^{4}\hat p_\mu^2}$$

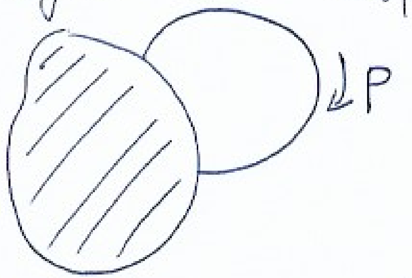
near zero momenta $\hat p_\mu = p_\mu + O(a^2) \Rightarrow$ continuum propagator

BUT ...

$$(m + i\gamma_\mu \hat p_\mu)(m - i\gamma_\nu \hat p_\nu) = m^2 + \gamma_\mu\gamma_\nu \hat p_\mu \hat p_\nu = m^2 + \underbrace{\tfrac{1}{2}\{\gamma_\mu,\gamma_\nu\}}_{\delta_{\mu\nu}}\hat p_\mu \hat p_\nu = m^2 + \hat p^2$$

DOUBLING PROBLEM

Imagine a loop in some Feynman diagram.

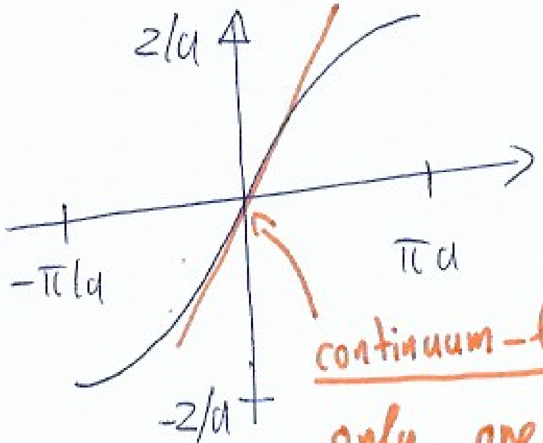


integration range: $p_\mu \in [-\pi/a, \pi/a]$ (Brillouin zone)

BOSONIC

2nd deriv $\Longrightarrow$ lattice momenta

$$\tilde{p}_\mu = \frac{2}{a} \sin \frac{a p_\mu}{2}$$



continuum-like

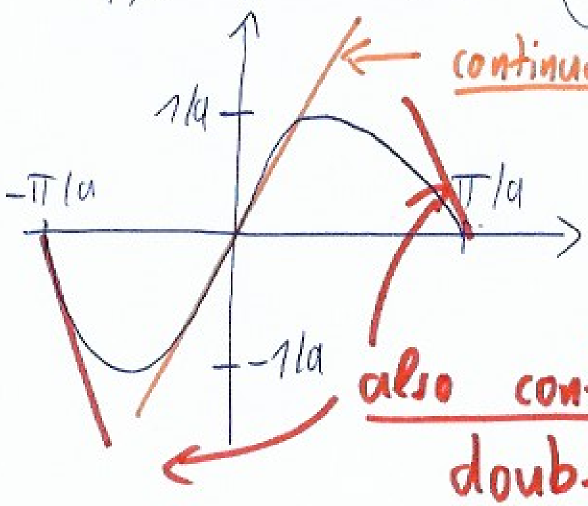
only one such region, near $p_\mu \approx 0$

FERMIONIC

these continuum-like parts $\Longrightarrow$ poles of the massless propagator

1st deriv $\Longrightarrow$ lattice momenta

$$\tilde{p}_\mu = \frac{1}{a} \sin(a p_\mu)$$



continuum-like

also continuum-like

doubler

$\exists$ two regions with continuum-like behavior

The naive discretization represents $2^4 = 16$ fermions in the continuum (p_μ near 0 or π for all components)

WILSON FERMIONS – 1

Origin of doubling was 1st deriv. in action, so add a 2nd derivative by hand

$$\frac{1}{2} r a \bar{\Psi} \Box \Psi \longleftarrow \text{Wilson term}$$

$$\bar{\Psi}(\gamma_\mu \partial_\mu + m)\Psi \longrightarrow \bar{\Psi}(\gamma_\mu \partial_\mu + \frac{1}{2} a r \Box + m)\Psi$$

r ← Wilson parameter (typically chosen to be $r=1$)

$$\boxed{\begin{aligned} S_E^{Wilson} = {} & \sum_x a^4 \Bigg(\frac{1}{2a} \sum_{\mu=1}^{4} \bar{\Psi}(x) \gamma_\mu \left[\Psi(x+a\hat{\mu}) - \Psi(x-a\hat{\mu}) \right] + \underbrace{\frac{r}{2a} \sum_{\mu=1}^{4} \bar{\Psi}(x) \left[2\Psi(x) - \Psi(x+a\hat{\mu}) - \Psi(x-a\hat{\mu}) \right]}_{\textbf{Wilson term}} \Bigg) \\ & + \sum_x a^4\, m\, \bar{\Psi}(x) \Psi(x) \end{aligned}}$$

INVERSE PROPAGATOR

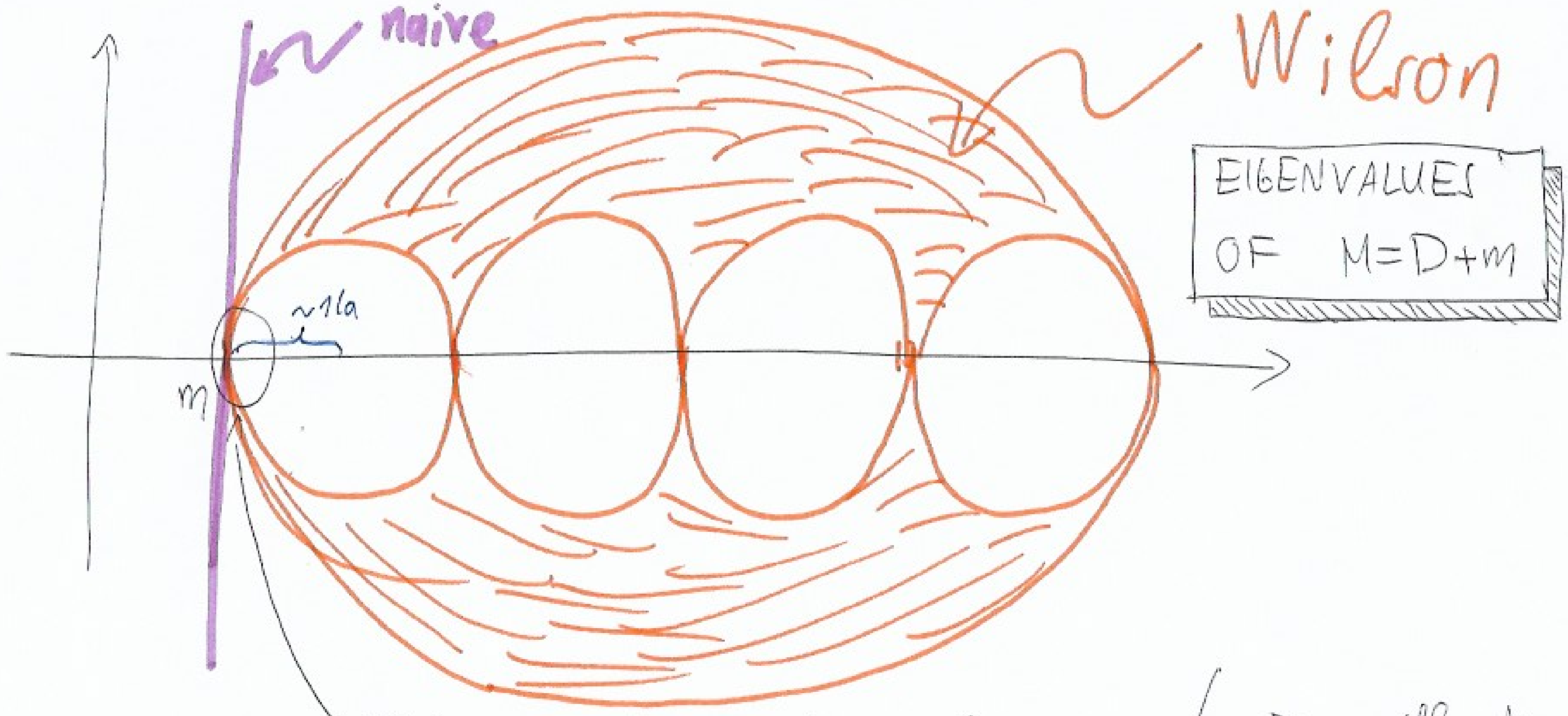
$$G^{-1}(p) = \underbrace{m + i \frac{1}{a} \sum_{\mu=1}^{4} \gamma_\mu \sin(p_\mu a)}_{\text{naive}} + \frac{r}{a} \sum_{\mu=1}^{4} \left[1 - \cos(p_\mu a) \right] \underset{a \to 0}{\sim} m_W(a) + i \frac{1}{a} \sum_\mu \sin(p_\mu a)$$

$$m_W(a) = \left(m + 2 \frac{r\, n_\pi}{a} \right) + O(a) \underset{\text{for doublers}}{\longrightarrow} \infty \qquad n_\pi = \left(\begin{array}{c} \#\ p_\mu \text{ components} \\ \text{with } p_\mu = \pi \end{array} \right)$$

WILSON FERMIONS -2

The price to pay for removing doublers is $\{\gamma_5, D^{\text{Wilson}}\} \neq 0$

$\Rightarrow$ spectrum looks very different than the continuum Dirac spectrum



Wilson operator spectrum only resembles the continuum spectrum around here

($\Rightarrow$ will have large discretization effects for many observables)

We saw w. Wilson that to remove the doublers we had to give up something (chiral symm.). This is quite general.

NIELSEN-NINOMIYA THEOREM

D = a massless lattice Dirac operator

① correct continuum limit: $\tilde{G}^{-1}(p) \sim i p^\mu \gamma_\mu$ as $a \to 0$

② locality $\tilde{G}^{-1}(p)$ continuous fn. of p or $|D_{xy}| \leq C e^{-\lambda|x-y|}$ as $|x-y| \to \infty$

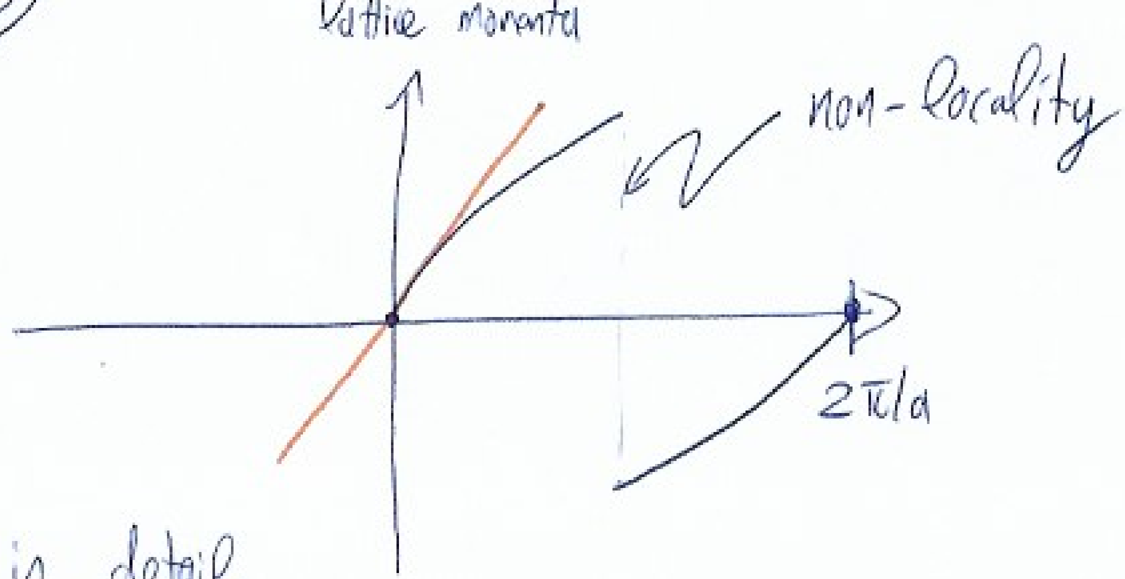
③ continuum chiral symmetry $\{\gamma_5, D\} = 0$

④ no doublers

Claim: IMPOSSIBLE TO SATISFY ①,②,③,④ SIMULTANEOSLY

e.g. Wilson fermions violate ③

an example where two is violated:



PROOF: Topological. I will only cover it next semester, when we discuss chiral symm. in detail.

STAGGERED FERMIONS - 1

we start w. the naive action: $S = \frac{1}{2}\sum_{x,\mu}\left[\bar{\Psi}(x)\gamma_\mu \Psi(x+a\hat{\mu}) - \bar{\Psi}(x)\gamma_\mu \Psi(x-a\hat{\mu})\right] + m\sum_x \bar{\Psi}(x)\Psi(x)$

let us diagonalize in spinor space/indices with x dependent matrices

$$\Psi(x)_\alpha = T(x)_{\alpha\beta}\,\chi(x)_\beta \qquad \text{and} \qquad \bar{\Psi}(x)_\alpha = \bar{\chi}(x)_\beta\, T^+(x)_{\beta\alpha}$$

$$S = \frac{1}{2}\sum_{x,\mu}\left[\bar{\chi}(x)T^+(x)\gamma_\mu T(x+a\hat{\mu})\chi(x+a\hat{\mu}) - \bar{\chi}(x)T^+(x)\gamma_\mu T(x-a\hat{\mu})\chi(x-a\hat{\mu})\right] + m\sum_x \bar{\chi}(x)\chi(x)$$

If $T^+(x)\gamma_\mu T(x+a\hat{\mu}) = \eta_\mu(x)\cdot \mathbb{1}$ $\qquad T^+(x+a\hat{\mu})\gamma_\mu T(x) = \eta^*_\mu(x)\,\mathbb{1}$

suppose we can find such matrices $T(x)$

$$T^+(x)\gamma_\mu T(x-a\hat{\mu}) = \eta^*_\mu(x-a\hat{\mu})\,\mathbb{1}$$

$$S = \frac{1}{2}\sum_{x,\mu,\alpha}\left[\eta_\mu(x)\,\bar{\chi}_\alpha(x)\chi_\alpha(x+a\hat{\mu}) - \eta^*_\mu(x-a\hat{\mu})\,\bar{\chi}_\alpha(x)\chi_\alpha(x-a\hat{\mu})\right] + \sum_{x,\alpha} m\,\bar{\chi}(x)_\alpha\,\chi(x)_\alpha$$

Now this is diagonal in the α index, so let us drop that index

STAGGERED FERMIONS - 2

E.g. $T(x) = \gamma_1^{x_1/a}\, \gamma_2^{x_2/a}\, \gamma_3^{x_3/a}\, \gamma_4^{x_4/a} \equiv \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3} \gamma_4^{n_4}$ $\quad (n_\mu = \frac{1}{a} x_\mu)$

achieves the diagonalization with

EXERCISE $\to$ $\eta_\mu(x) = (-1)^{n_1 + n_2 + \dots + n_{\mu-1}}$ $\quad \eta_\mu^* = \eta_\mu$ also $\eta_\mu(x - a\hat{\mu}) = \eta_\mu(x)$

$\eta_1(x) = 1 \quad \eta_2(x) = (-1)^{n_1} \quad \eta_3(x) = (-1)^{n_1+n_2} \quad \eta_4(x) = (-1)^{n_1+n_2+n_3}$

STAGGERED ACTION

$$S = \sum_{x,\mu} \eta_\mu(x) \left[\underbrace{\bar{\chi}}_{\text{1 component}}(x)\chi(x + a\hat{\mu}) - \bar{\chi}(x)\chi(x - a\hat{\mu}) \right] + m \sum_x \bar{\chi}(x)\chi(x)$$

these phases are the remnants of the original Dirac structure

The staggered transformation mixes spacetime and Dirac indices

$\to$ technical difficulties and conceptual difficulties

$\frac{16}{4} \to 4$ fermions

STAGGERED FERMIONS – 3

What is γ_5 in the new basis?

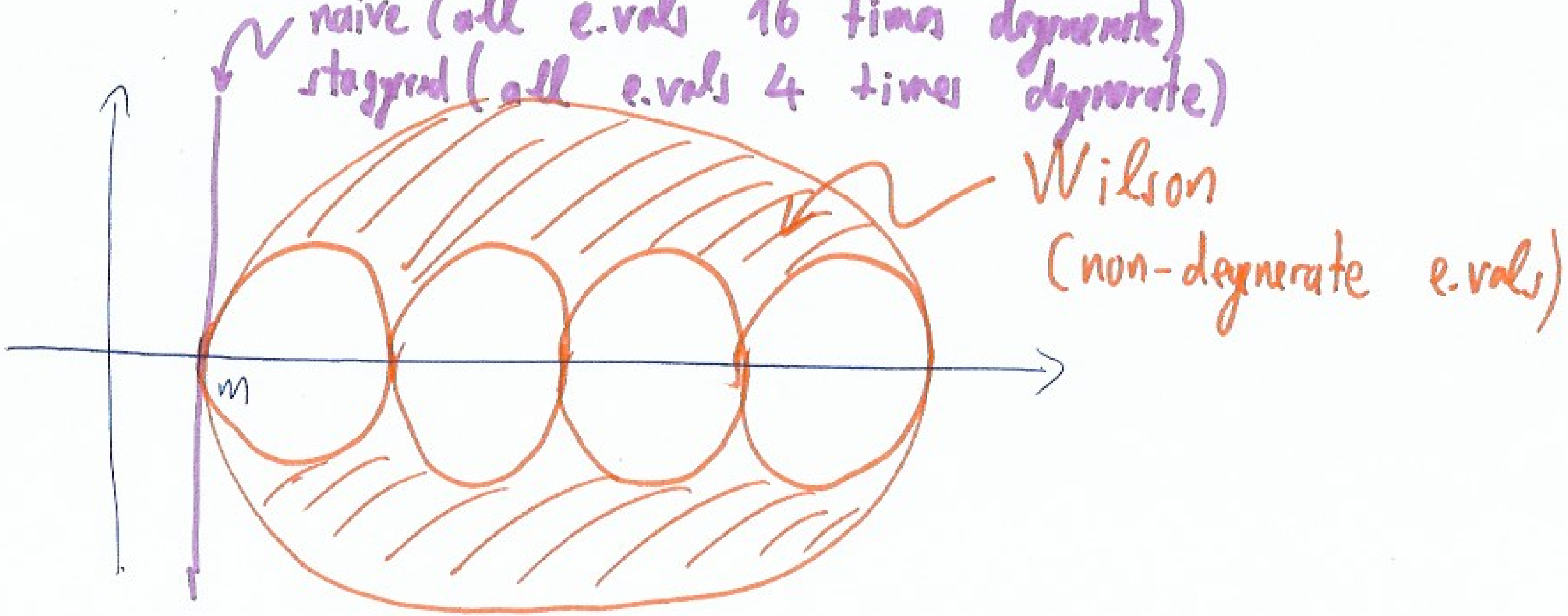
$$\bar\Psi(x)\gamma_5\Psi(x) = \bar\chi(x)\,T^+(x)\underbrace{\gamma_5}_{\gamma_1\gamma_2\gamma_3\gamma_4}T(x)\,\chi(x) = \eta_5(x)\,\bar\chi(x)\mathbb{1}\chi(x)$$

$$\eta_5(x) = (-1)^{n_1+n_2+n_3+n_4} \quad \leftarrow \underline{\text{EXERCISE}}$$

<u>a remnant of chiral symmetry:</u>

$\{D^{stagg}, \eta_5\} = 0 \Rightarrow$ spectrum of D^{stagg} still on the imaginary axis

spectrum of free Dirac matrix



GINSPARG – WILSON RELATION

suppose $\boxed{D\gamma_5 + \gamma_5 D = a D\gamma_5 D}$ (Nielsen – Ninomiya thm is avoided since $\{\gamma_5 D\} \neq 0$) (many of the features of chiral symm. remain)

History: originally written down in 1982, immediately forgotten (<10 citations in ~ 10 years) (renormalization group argument → next semester)

..., 1997, 1998 : construction of operators which satisfy the relation EXPLOSION OF INTEREST (domain wall, overlap, fixed point fermions → all 3 will be covered next semester)

$$(D\gamma_5 + \gamma_5 D)\gamma_5 = (a D\gamma_5 D)\gamma_5 \Rightarrow D + D^+ = a D D^+$$

$$\gamma_5(D\gamma_5 + \gamma_5 D) \underset{\uparrow}{=} \gamma_5(a D\gamma_5 D) \Rightarrow D^+ + D = a D^+ D$$

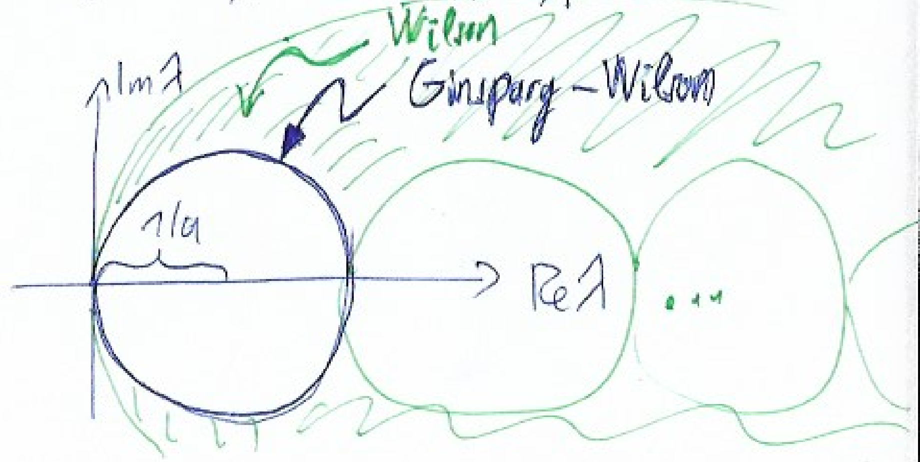
$\Rightarrow D D^+ = D^+ D$ normal matrix ($\Rightarrow$ e.vecs form an orthogonal basis)

γ_5-hermiticity

multiply from left and right with eigenvector

$$v_\lambda^+ (D^+ + D) v_\lambda = a\, v_\lambda^+ D^+ D v_\lambda \Rightarrow \lambda^* + \lambda = a \lambda^* \lambda$$

let $\lambda = x + iy \Rightarrow \left(x - \frac{1}{a}\right)^2 + y^2 = \frac{1}{a^2}$



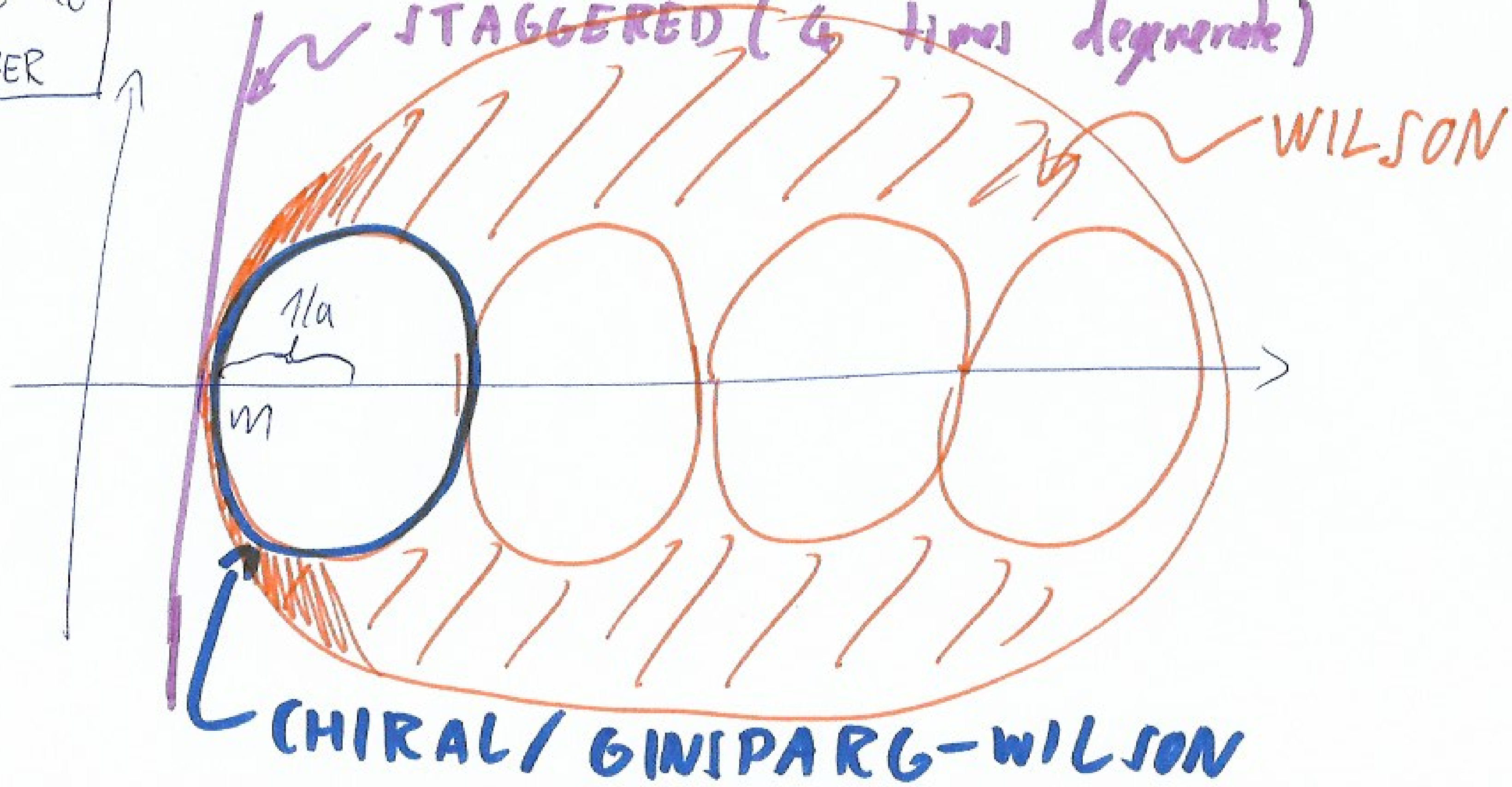
Essentially, we have 3 widely used classes of lattice fermions

LATTICE FERMIONS

WILSON	STAGGERED	CHIRAL / GINSPARG-WILSON
add 2nd deriv.	diag. in Dirac indices	$\{D, \gamma_5\} = a D \gamma_5 D$

All of these can have many variations and properties of them can be combined

PICTURE TO REMEMBER



ON THYS W. INTERACTIONS (where the fermionic part is quadratic)

e.g. $$S_E = \int d^4x \left[\bar{\Psi} (\not{\partial} + m) \Psi + \underbrace{\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2}_{\mathcal{L}_{scalar}} + g\,\phi\,\bar{\Psi}\Psi \right]$$

$$Z = \int D\bar{\Psi} D\Psi D\phi \; e^{-S_E}$$

$$= \int D\bar{\Psi} D\Psi D\phi \; e^{-\int_{x,y} \bar{\Psi}(x)(\not{\partial} + m + g\phi)_{xy}\Psi(y) - S_{scalar}[\phi]}$$

$$= \int D\phi \; \underbrace{\det(\not{\partial} + m + g\phi)}_{\text{fermion determinant}} \; e^{-S_{scalar}[\phi]} = \int e^{-S_{eff}[\phi]} D\phi$$

fermion determinant now depends on the scalar field ϕ

non-local interactions between the bosons that come from integrating out the fermions

$$S_{eff} = S_{scalar}[\phi] - \ln \det \underbrace{(\not{\partial} + m + g\phi)}_{M}$$