LECTURE 2:

SCALAR FIELD THY ON THE LATTICE

REAL SCALAR FIELD

$$g^{(M)}_{\mu\nu} x^\mu x^\nu = -x^0 y^0 + \vec{x}\cdot\vec{y}$$

$$S_M = \int d^4x \, \frac{1}{2}\left(-\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4\right)$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi \implies H = \int d^3x\,(\pi\dot\phi - \mathcal{L}) = \int d^3x \left[\frac{\pi^2}{2} + \frac{1}{2}(\vec\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4\right]$$

EUCLIDEAN:

$x^0 = -i x^4 \qquad x^4 \in \mathbb{R}$ will also call $x^4 = \tau$ or t when it is clear

$$x^{(E)}_\mu x^{(E)}_\mu = \vec{x}\cdot\vec{y} + x^4 y^4$$ [no need for lower and upper indices]

$$S_E = -i S_M = \int d^4x^{(E)} \left[\frac{1}{2}\partial^{(E)}_\mu\phi\,\partial^{(E)}_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4\right]$$

DIMENSIONS:

$4 \to d$ spacetime-dims

$$\left[\int d^dx\,(\partial\phi)^2\right] = E^0 = E^{-d+2}[\phi]^2 \implies [\phi] = E^{\frac{d-2}{2}}$$

$$\left[\int d^dx\, g_n\phi^n\right] = E^0 = E^{-d} E^{\frac{d-2}{2}n}[g_n] \implies [g_n] = E^{n-(\frac{n}{2}-1)d} \overset{n=4}{=} E^{4-d}$$

WE WILL QUANTIZE THIS THY
WITH A PATH INTEGRAL

LIKE LAST TIME, EXCPEPT

$$\boxed{q_n(t) \rightarrow \phi(\vec{x},t)}$$

there is a q for every $\vec{x}$

COORD. REPR.

$$\hat{\phi}(\vec{x})|\phi\rangle = \phi(\vec{x})|\phi\rangle \quad \text{where} \quad |\phi\rangle = \prod_{\vec{x}} |\phi_{\vec{x}}\rangle$$

$$\langle \phi' | \phi \rangle = \prod_{\vec{x}} \delta(\phi'(\vec{x}) - \phi(\vec{x}))$$

$$\prod_{\vec{x}} \int_{-\infty}^{+\infty} d\phi(\vec{x})\, |\phi\rangle\langle\phi| = \mathbb{1}$$

LATTICE ACTION

$$\boxed{\Delta_\mu^f f(x) = \frac{f(x+a\hat{\mu}) - f(x)}{a}}$$

bare mass

bare coupling

$$S = \frac{1}{2}\sum_x a^4 \left\{ \Delta_\mu^f \phi_0(x)\, \Delta_\mu^f \phi_0(x) + m_0^2 \phi_0(x)^2 \right\} + \frac{g_0}{4!}\sum_x a^4 \phi_0(x)^4$$

FUNCTIONAL INTEGRAL

$$Z = \int \prod_x d\phi(x)\, e^{-S}$$

This is the partition fn. of a stat. mech. system

REWRITING THE ACTION

$$S = \frac{1}{2}\sum_x a^d \left\{ (\Delta_\mu^f \phi_0)(\Delta_\mu^f \phi_0) + m_0^2 \phi_0^2 \right\} + \frac{g_0}{4!}\sum_x a^d \phi_0^4$$

$$\Downarrow \quad \boxed{a^{\frac{d-2}{2}} \phi_0 \equiv \sqrt{2K}\,\phi} \qquad \boxed{g_0 \equiv \frac{6\lambda}{K^2} a^{d-4}} \qquad \boxed{a^2 m_0^2 = \frac{1-2\lambda}{K} - 2d}$$

$$S = \sum_x \left\{ -2K \sum_{\mu=1}^{4} \phi(x)\,\phi(x+a\hat{\mu}) + \phi(x)^2 + \lambda\left(\phi(x)^2 - 1\right)^2 + \text{const.} \right\}$$

- the new fields ϕ and the new parameters K, λ are <u>dimensionless</u>
- as $\lambda \to \infty$ only $\phi(x) = \pm 1$ config.s contribute: <u>Ising model</u>
- this form is convenient for numerical simulation
- $K > 0$ ferromagnetic coupling $\qquad K < 0$ antiferromagnetic coupling

FINITE DIFFERENCES

Notation:

$\Lambda = \{(\vec{x}, \overset{x_4}{\overset{\shortparallel}{t}}) \in \mathbb{Z}^4\}$

$$(f,g) = \sum_{x\in\Lambda} a^d f(x) g(x)$$

$$\Delta_\mu^f f \equiv \frac{1}{a}\left[f(x+a\hat{\mu}) - f(x)\right]$$

$$\Delta_\mu^b f \equiv \frac{1}{a}\left[f(x) - f(x-a\hat{\mu})\right]$$

$$\Delta_\mu^s f \equiv \frac{1}{2a}\left[f(x+a\hat{\mu}) - f(x-a\hat{\mu})\right]$$

Matrix notation:

$$(\Delta_\mu^f)_{xy} = \frac{1}{a}\left(\delta_{x+a\hat{\mu},y} - \delta_{xy}\right)$$

$$(\Delta_\mu^b)_{xy} = \frac{1}{a}\left(\delta_{xy} - \delta_{x-a\hat{\mu},y}\right) = -(\Delta_\mu^f)_{yx} = -\left((\Delta_\mu^f)^T\right)_{xy}$$

$$\boxed{\Delta_\mu^b = -(\Delta_\mu^f)^T}$$

$$\boxed{\Delta_\mu^s = \frac{1}{2}(\Delta_\mu^f + \Delta_\mu^b)}$$

Discretization error:

$$\Delta_\mu^f f(x) = \partial_\mu f(x) + O(a)$$

$$\Delta_\mu^b f(x) = \partial_\mu f(x) + O(a)$$

$$\Delta_\mu^s f(x) = \partial_\mu f(x) + O(a^2)$$

$f(x\pm a\hat{\mu}) \simeq f(x) \pm (\partial_\mu f)\cdot a + \frac{1}{2}(\partial_\mu^2 f)a^2 + \dots$

$\Delta_\mu^f(x) = \partial_\mu f(x) + \frac{1}{2}(\partial_\mu^2 f)a + \dots$

$\Delta_\mu^b(x) = \partial_\mu f(x) - \frac{1}{2}(\partial_\mu^2 f)a + \dots$

"Integration by parts"

$$(\Delta_\mu^f f, g) = -(f, \Delta_\mu^b g)$$

with periodic BCs or an ∞ lattice

$$(\Delta_\mu^f f, \Delta_\mu^f f) = -(f, \Delta_\mu^b \Delta_\mu^f f) \equiv (f, \Box f)$$

Lattice d'Alembert operator

$$\Box \equiv -\Delta_\mu^b \Delta_\mu^f$$

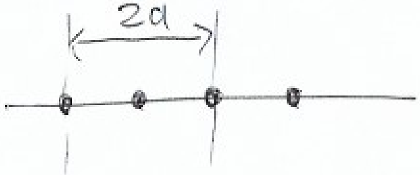
$$\Box f(x) = \sum_{\mu=1}^{d} \frac{1}{a^2}\left[2f(x) - f(x+a\hat{\mu}) - f(x-a\hat{\mu})\right]$$

LATTICE MOMENTA

$$p_\mu = \frac{2\pi}{L_\mu} n_\mu \quad \text{(no sum)} \qquad n_\mu = -\frac{N_\mu}{2}+1, \ -\frac{N_\mu}{2}+2, \dots, \frac{N_\mu}{2}$$

$\mathcal{B}$ Brillouin zone

$$p_{max} = \frac{\pi}{a} \qquad \lambda_{min} = \frac{2\pi}{p_{max}} = 2a$$



FOURIER TRANSFORM

$$\tilde{f}(p) = \sum_{x \in \Lambda} a^4 e^{-ipx} f(x)$$

$$f(x) = \frac{1}{L_1 L_2 L_3 L_4} \sum_{p \in \mathcal{B}} e^{ipx} \tilde{f}(p)$$

if $L_\mu \to \infty$: $\displaystyle \frac{1}{L_1 L_2 L_3 L_4} \sum_p \to \frac{1}{(2\pi)^4} \int_{-\pi/a_1}^{\pi/a_1} dp_1 \dots \int_{-\pi/a_4}^{\pi/a_4} dp_4$

DELTA FUNCTION

$$a_1 a_2 a_3 a_4 \sum_x e^{-i(p-p')x} = L_1 L_2 L_3 L_4 \, \delta^{(4)}_{p,p'}$$

$$\frac{1}{L_1 L_2 L_3 L_4} \sum_p e^{ip(x-x')} = \frac{1}{a_1 a_2 a_3 a_4} \delta^{(4)}_{x,x'}$$

TRANSFER MATRIX FOR THE SCALAR FIELD

$$S = a_t \sum_t \sum_{\vec{x}} \frac{1}{2a_t^2}\left[\phi_0(\vec{x}, t+a_t) - \phi_0(\vec{x},t)\right]^2 + a_t \sum_t V[\Phi_t]$$

$$V[\Phi_t] = \sum_{\vec{x}} \left[\frac{1}{2} \sum_{j=1}^{3} (\Delta_j^f \phi_0)(\Delta_j^f \phi_0) + \frac{m_0^2}{2}\phi_0^2 + \frac{g_0}{4!}\phi_0^4 \right]$$

$$\boxed{Z = \int D\phi\, e^{-S} = \mathrm{Tr}(T^N)}$$

$$\langle \Phi_{t+a_t} | T | \Phi_t \rangle \equiv \underset{\substack{\uparrow \\ \text{will fix later}}}{c}{}^{N^3} \exp\left[-a_t \sum_{\vec{x}} \frac{1}{2a_t^2}\left[\phi_0(\vec{x}, t+a_t) - \phi_0(\vec{x},t)\right]^2 - \frac{a_t}{2}\left(V[\phi_{t+a_t}] + V[\phi_t]\right)\right]$$

$$\hat{T} \equiv \exp\left[-a_t \frac{1}{2} V[\hat{\phi}]\right] \exp\left[-a_t \frac{1}{2} \sum_{\vec{x}} \hat{\pi}_{\vec{x}}^2\right] \exp\left[-a_t \frac{1}{2} V[\hat{\phi}]\right] \qquad (*)$$

$$e^{-a_t \frac{1}{2} V[\hat{\phi}]} |\phi_t\rangle = e^{-a_t \frac{1}{2} V[\phi_t]} |\phi_t\rangle \Rightarrow (*) \text{ correct if } \langle \Phi | e^{-\frac{a_t}{2}\sum_{\vec{x}} \hat{\pi}_x^2} | \varphi \rangle = e^{-\frac{1}{2a_t}\sum_{\vec{x}} (\Phi(\vec{x}) - \varphi(\vec{x}))^2}$$

This is just a product of one-degree-of-freedom relations

$$\langle q_1 | e^{-\hat{p}^2/2\xi} | q_2 \rangle = \sqrt{\frac{\xi}{2\pi}}\, e^{-\xi(q_1-q_2)^2/2}$$

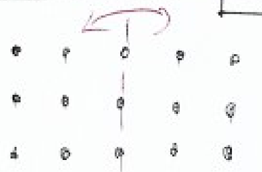
$$\frac{1}{\xi} = \frac{a_s^3}{a_t} \Rightarrow c = \sqrt{\frac{\xi}{2\pi}} = \sqrt{\frac{a_s^3}{2\pi a_t}}$$

FORMAL CONT. TIME LIMIT: $\hat{T} = \mathbb{1} - a_t \hat{H} + \ldots$ $\quad \hat{H} = a_s^3 \sum_{\vec{x}} \left[\frac{\pi_{\vec{x}}^2}{2} + \frac{1}{2}(\Delta_j^f \phi)(\Delta_j^f \phi) + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4\right] + O(a_t^2)$

REFLECTION POSITIVITY

(Given a Euclidean FT, how do you know it corresponds to a well-def. quantum theory?)

site-reflection: $\boxed{\theta_s : x_4 \to -x_4}$



$x_4=0$

link-reflection: $\boxed{\theta_\ell : x_4 \to 1-x_4}$

$x_4=0$

(Θ is roughly the Euclidean FT version of Hermitian conjugation)

On the fields $\boxed{\Theta\,\phi(x) = \overline{\phi(\theta x)}}$

On functions of the fields $\boxed{\Theta(\lambda F) = \bar{\lambda}\,\Theta F}$ and $\boxed{\Theta(FG) = (\Theta F)(\Theta G)}$

Let $F = \sum_j \int dx_1 \dots dx_j \underbrace{f_j(x_1, \dots, x_j)}_{\text{has support for}} \phi(x_1) \dots \phi(x_j)$

has support for $x_i^4 \geq 0$ (site-refl.)

$x_i^4 \geq 1$ (link-refl.)

REFLECTION POSITIVITY: $\boxed{\langle (\Theta F) F \rangle \geq 0}$

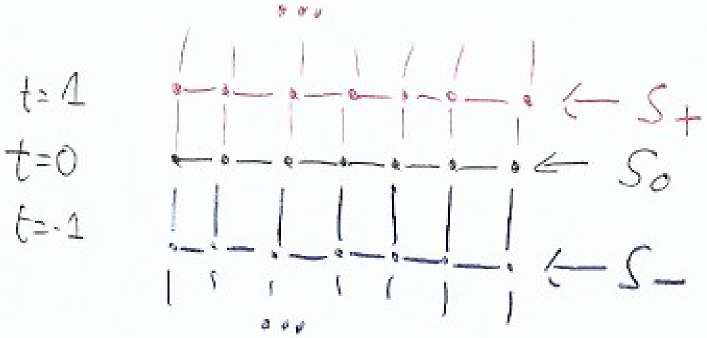
site-reflection $\Rightarrow$ $\langle \Psi | \Psi \rangle \geq 0$ $\quad$ $\langle \Psi | T^2 | \Psi \rangle \geq 0$ $\quad \dots \quad$ $\langle \Psi | T^{2n} | \Psi \rangle \geq 0$

depends on fields at $x_4 = 0$

link-reflection $\Rightarrow$ $\langle \Psi | T | \Psi \rangle \geq 0$ $\quad \dots \quad$ $\langle \Psi | T^{2n+1} | \Psi \rangle \geq 0$

REFLECTION POSITIVITY FOR A SCALAR FIELD

SITE-REFLECTION



$$S = S_+ + S_- + S_0$$

$$S_0 = L_1[\Phi_0]$$

$$S_+ = \sum_{t>0} L_1[\phi]_t + \sum_{t\geq 0} \tfrac{1}{2}\left(\phi(\vec{x},t+1) - \phi(\vec{x})\right)^2$$

$$S_- = \Theta S_+$$

$$\left\langle (\Theta F) F \right\rangle \propto \int \prod_x d\phi(x)\, e^{-S_0}\, \Theta\left(F e^{-S_+}\right) F e^{-S_+} = \ldots$$

let $\mathcal{F}[\Phi_0] = \int \prod_{x_4>0} d\phi(x)\, F e^{-S_+} \Rightarrow \overline{\mathcal{F}[\Phi_0]} = \Theta \mathcal{F}[\Phi_0]$

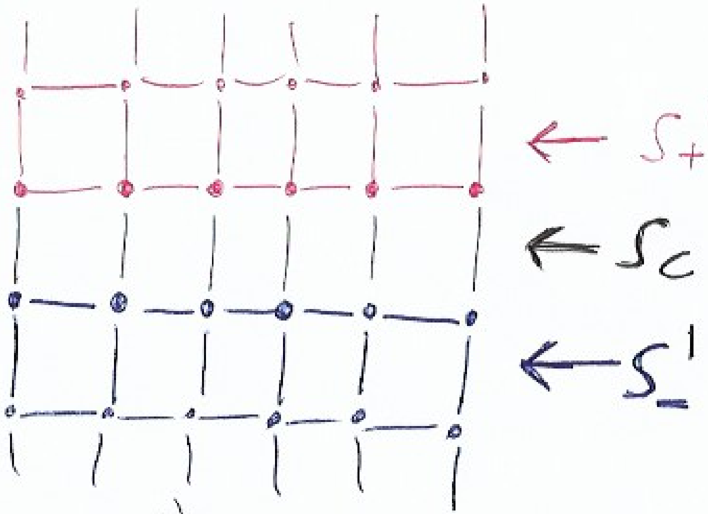
$$\Rightarrow \ldots = \int \prod_{x_4=0} d\phi(x)\, e^{-S_0} \left|\mathcal{F}[\Phi_0]\right|^2 \geq 0$$

LINK-REFLECTION

$$S = S_c + S_+' + S_-'$$

$$S_c = -2K \sum_x \phi(\vec{x},1)\,\phi(\vec{x},0)$$

$$S_-' = \Theta S_+'$$



$$\int \prod_x d\phi(x)\; e^{-S_c} \left(\Theta\left(F e^{-S_+'}\right)\right)\left(F e^{-S_+'}\right) \propto \langle (\Theta F)\, F \rangle$$

$$e^{-S_c} = \sum_{n=0}^{\infty} K^n \sum_i \underbrace{c_{ni}}_{>0}\, (\Theta s_i)\, s_i$$

$$e^{-S_c} = e^{2K\sum_{\vec{x}} \phi(\vec{x},1)\,\phi(\vec{x},0)} = \prod_{\vec{x}} \left(1 + 2K\,\phi(\vec{x},1)\,\phi(\vec{x},0) + \frac{1}{2}\left(2K\,\phi(\vec{x},1)\,\phi(\vec{x},0)\right)^2 + \dots\right)$$

$$\Rightarrow \sum_{n=0}^{\infty} K^n \sum_i c_{ni} \left| \int \prod_{x_4>0} d\phi(x)\; s_i\, F e^{-S_+'} \right|^2$$

$K > 0 \Rightarrow$ WE HAVE LINK REFLECTION POSITIVITY

$K < 0 \Rightarrow$ NO LINK REFLECTION POSITIVITY

GAUSSIAN INTEGRALS

$$\boxed{\int_{-\infty}^{+\infty} d\phi \; e^{-\frac{A}{2}\phi^2} = \sqrt{\frac{2\pi}{A}} \quad (\mathrm{Re}\,A > 0)}$$

$$\int d\phi_1 \int d\phi_2 \; e^{-\frac{1}{2}A(\phi_1^2+\phi_2^2)} = \int_0^\infty dr \int_0^{2\pi} d\varphi \, r \, e^{-\frac{Ar^2}{2}} = \pi \int_0^\infty d(r^2)\, e^{-\frac{Ar^2}{2}} = \frac{2\pi}{A} \quad \square$$

$\phi = (\phi_1, \dots, \phi_k) \in \mathbb{R}^k$

A_{ij} real, symm., pos. def.

$$Z_0 \equiv \int d^k\phi \, \exp\left[-\frac{1}{2}\phi_k A_{k\ell} \phi_\ell\right] = \frac{(2\pi)^{k/2}}{\sqrt{\det A}}$$

orthogonal

$$SAS^T = D = \mathrm{diag}(\lambda_1, \dots, \lambda_k)$$

$$y \equiv S\phi \qquad d^k\phi = |\det S|^{-1} d^k y = d^k y$$

$$\int d^k\phi \, \exp\left(-\frac{1}{2}\phi_k A_{k\ell}\phi_\ell\right) = \int d^k y \, \exp\left(-\frac{1}{2}\sum_i y_i^2 \lambda_i\right)$$

$$= \prod_{i=1}^k \sqrt{\frac{2\pi}{\lambda_i}} = \frac{(2\pi)^{k/2}}{\sqrt{\det A}} \quad \square$$

$$\boxed{Z_0(J) \equiv \frac{1}{Z_0}\int d^k\phi \, \exp\left[-\frac{1}{2}\phi_k A_{k\ell}\phi_\ell + J_k \phi_k\right] = \exp\left[\frac{1}{2} J_k A^{-1}{}_{k\ell} J_\ell\right]}$$

$$-\frac{1}{2}\phi^T A \phi + J^T\phi = -\frac{1}{2}\left(\phi + A^{-1}J\right)^T A \left(\phi + A^{-1}J\right) + \frac{1}{2} J^T A^{-1} J$$

$$Z_0(J) = \int d^k y \, \exp\left(-\frac{1}{2} y^T A y + \frac{1}{2} J^T A^{-1} J\right) / Z_0 \quad \square$$

MOMENTS OF A GAUSSIAN DISTRIBUTION

$$\langle \phi_a \phi_b \cdots \phi_c \rangle \equiv \frac{1}{Z_0} \int d^4\phi \, (\phi_a \cdots \phi_c) \, e^{-\frac{1}{2}\phi_k A_{ke} \phi_e}$$

$$\langle \phi_a \phi_b \cdots \phi_c \rangle = \frac{\partial}{\partial J_a} \frac{\partial}{\partial J_b} \cdots \frac{\partial}{\partial J_c} \int d^4\phi \, e^{-\frac{1}{2}\phi_k A_{ke}\phi_e + J_k \phi_e} \Big|_{J=0}$$

$$= \frac{\partial}{\partial J_a} \cdots \frac{\partial}{\partial J_c} Z_0(J) \Big|_{J=0} = \frac{\partial}{\partial J_a} \cdots \frac{\partial}{\partial J_c} \exp\left(\frac{1}{2} J_k \, A^{-1}{}_{ke} \, J_e\right)\Big|_{J=0}$$

$$\Rightarrow \quad \langle \phi_a \rangle = 0$$

$$\langle \phi_a \phi_b \rangle = A^{-1}{}_{ab}$$

$$\langle \phi_a \phi_b \phi_c \rangle = 0$$

$$\langle \phi_a \phi_b \phi_c \phi_d \rangle = A^{-1}{}_{ab} A^{-1}{}_{cd} + A^{-1}{}_{ac} A^{-1}{}_{bd} + A^{-1}{}_{ad} A^{-1}{}_{bc}$$

odd moments $= 0$

even moments $\langle \phi_{i_1} \cdots \phi_{i_{2n}} \rangle = \sum_{\text{pairings}} A^{-1}{}_{j_1 k_1} A^{-1}{}_{j_2 k_2} \cdots A^{-1}{}_{j_n k_n}$

FREE SCALAR FIELD - 1

$$S = \frac{1}{2}\sum_x \left[(\Delta_\mu^f \phi)(\Delta_\mu^f \phi) + m^2 \phi^2 \right] = \frac{1}{2}\sum_{xy} S_{xy}\, \phi_x \phi_y$$

$$\boxed{S_{xy} = \sum_z \left[\sum_{\mu=1}^{4} (\delta_{z+a\hat{\mu},x} - \delta_{zx})(\delta_{z+a\hat{\mu},y} - \delta_{zy}) + m^2 \delta_{zx}\delta_{zy} \right]}$$

propagator

$$\boxed{S_{xy}\, G_{yz} = \delta_{xz}}$$

$$Z(J) = \int D\varphi \, \exp\left(-S + \sum_x J_x \phi_x\right) = Z(0)\, \exp\left(\frac{1}{2} G_{xy} J_x J_y\right)$$

$$S_{p,-q} = \sum_{xy} e^{-ipx+iqy} S_{xy} = \sum_z \left[\sum_{\mu=1}^{4} (e^{-ip\hat{\mu}} - 1)(e^{iq\hat{\mu}} - 1) + m^2 \right] e^{-ipz+iqz}$$

$$= \delta_{pq} \left(m^2 + \sum_{\mu=1}^{4} \left[2 - 2\cos p_\mu \right] \right) = \delta_{pq} \left[m^2 + \sum_\mu 4 \sin^2 \frac{p_\mu}{2} \right]$$

$$\boxed{G_p = \frac{a^2}{m^2 a^2 + \sum_\mu 4 \sin^2 \frac{p_\mu a}{2}}} = \frac{1}{m^2 + p^2 + O(a^2)}$$

LATTICE PROPAGATOR

FREE SCALAR FIELD – 2

$G_{x,0} \equiv G(\vec{x},t)$

$$G(\vec{x},t) = \sum_{\vec{p}} e^{i\vec{p}\vec{x}} \underbrace{\int_{-\pi}^{+\pi} \frac{dp_4}{2\pi} \frac{e^{ip_4 t}}{2b - 2\cos p_4}}_{\equiv I}$$

where $b = 1 + \frac{1}{2}\left(m^2 + \sum_{j=1}^{3} 4\sin^2\frac{p_j}{2}\right)$

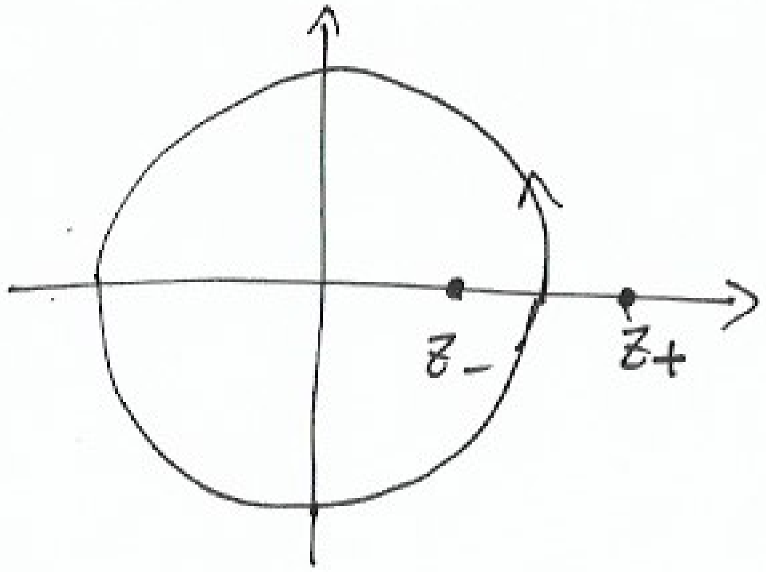
$$I = -\int \frac{dz}{2\pi i} \frac{z^t}{z^2 - 2bz + 1}$$

$z = e^{ip_4}$

$dz = z\, i\, dp_4$

poles? $(z^2 - 2bz + 1) = (z - z_+)(z - z_-)$

$z_\pm = b \pm \sqrt{b^2 - 1}$ $\quad z_+ z_- = 1$ $\quad z_+ > 1$, $z_- < 1$



Let $z_- \equiv e^{-w}$

$\cosh w = b$

$w = \ln\left[b + \sqrt{b^2 - 1}\right]$

residue at $z = z_-$ gives $\quad I = \dfrac{e^{-wt}}{2\sinh w} \Rightarrow G(\vec{x},t) = \sum_{\vec{p}} \dfrac{e^{i\vec{p}\vec{x} - wt}}{2\sinh w}$

as $\boxed{a \to 0}$ $\quad b - 1 \to \frac{1}{2}\left(m^2 a^2 + \sum_j p_j^2 a^2\right) + O(a^4)$

$$w \to \frac{1}{a} \ln\left[1 + \frac{1}{2}(m^2 + \vec{p}^2)a^2 + \ldots + \sqrt{(m^2 + \vec{p}^2)a^2 + \ldots}\right] = \sqrt{m^2 + \vec{p}^2} + O(a)$$

SPONTANOUS SYMMETRY BREAKING

ISING: $\sigma_i = \pm 1$

$$Z = \sum_{\{\sigma\}} e^{K \sum_{\langle xy \rangle} \sigma_x \sigma_y + H \sum_x \sigma_x}$$

$H = 0$: $\sigma_i \to -\sigma_i$
Z_2 symmetry

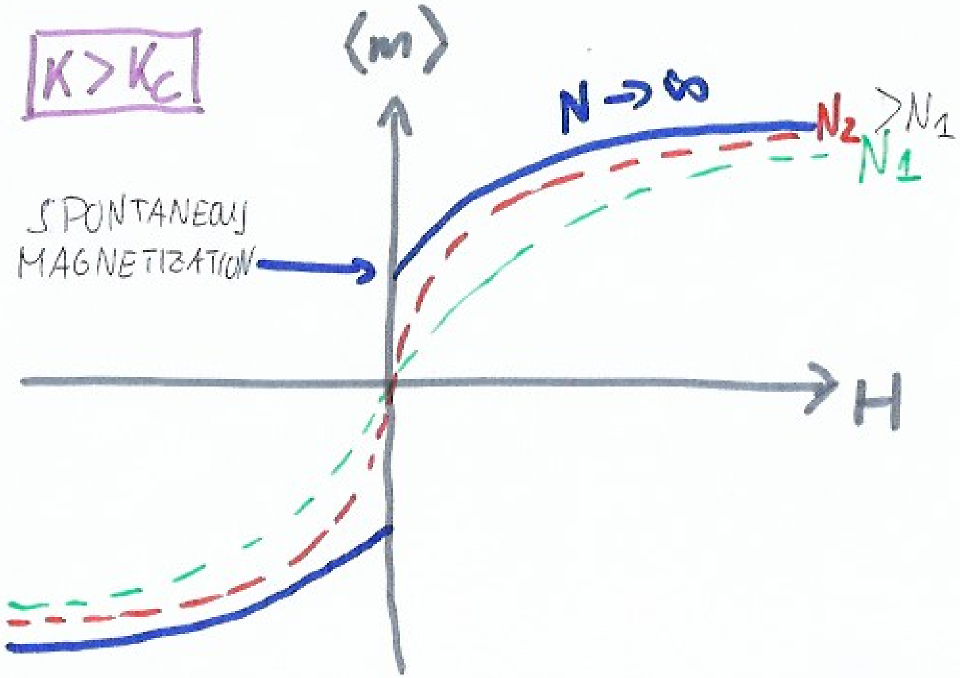
magnetization: $m = \frac{1}{N_{spins}} \sum_x \sigma_x$

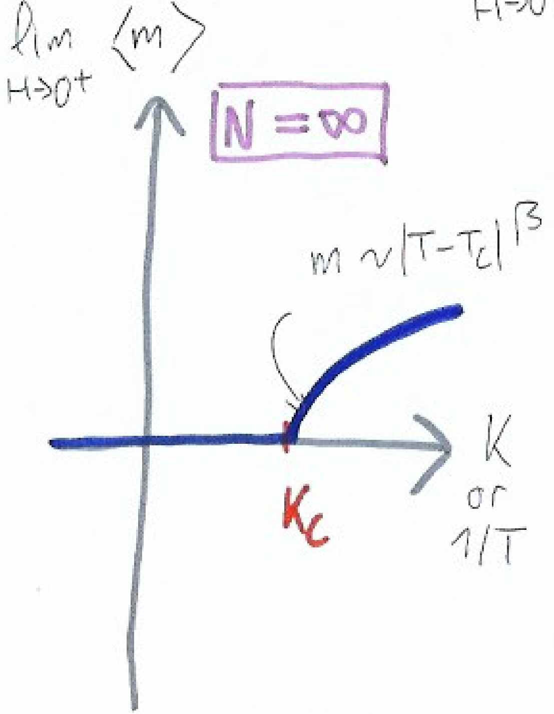
SSB: $T < T_c$ or $K > K_c$

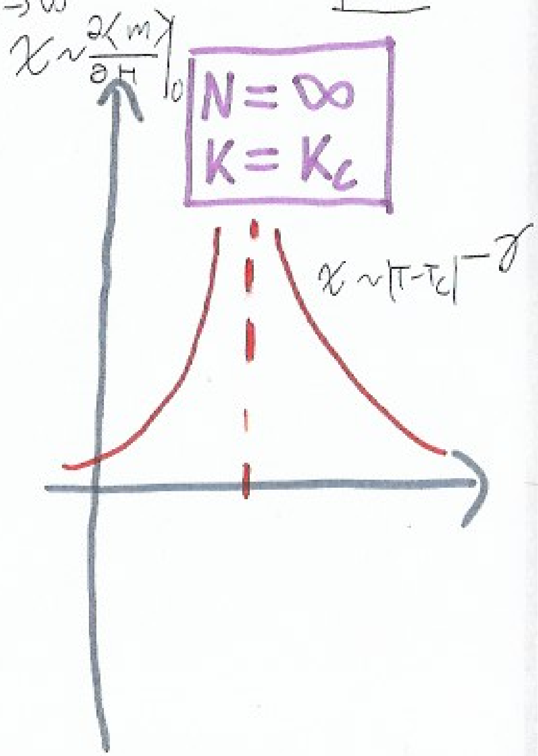
$$\lim_{H \to 0^+} \lim_{N_{spins} \to \infty} \langle m \rangle > 0$$

ordered/ ferromagnetic/ broken phase

$T > T_c$ $\lim_{H \to 0^+} \lim_{N \to \infty} \langle m \rangle = 0$ disordered phase







2ND ORDER PHASE TRANSITIONS

$$\chi = \frac{\partial\langle m\rangle}{\partial H} = \frac{1}{N}\frac{\partial\langle M\rangle}{\partial H} = \frac{1}{N}\frac{\partial}{\partial H}\left(\frac{1}{Z}\sum M e^{K\sum_{\langle xy\rangle}\sigma_x\sigma_y + HM}\right) = \frac{1}{N}\left(\langle M^2\rangle - \langle M\rangle^2\right) \sim |T-T_c|^{-\gamma}$$

susceptibility

$\rightarrow \infty$ as $T \rightarrow T_c$

$$G_{xy} = \langle\sigma(x)\sigma(y)\rangle - \langle\sigma(x)\rangle\langle\sigma(y)\rangle \sim \exp(-|x-y|/\xi)$$

correlation fn.

correlation length

$$\chi = \frac{1}{N}\left(\langle M^2\rangle - \langle M\rangle^2\right) = \frac{1}{N}\left[\langle\sum_{xy}\sigma(x)\sigma(y)\rangle - \langle\sum_x\sigma_x\rangle\langle\sum_y\sigma_y\rangle\right]$$

$$= \frac{1}{N}\sum_{x,y} G_{xy} \sim \frac{1}{N} N \underbrace{\int dr\, r^{d-1}\, G(r)}_{\text{can only diverge if } \boxed{\xi \rightarrow \infty}}$$

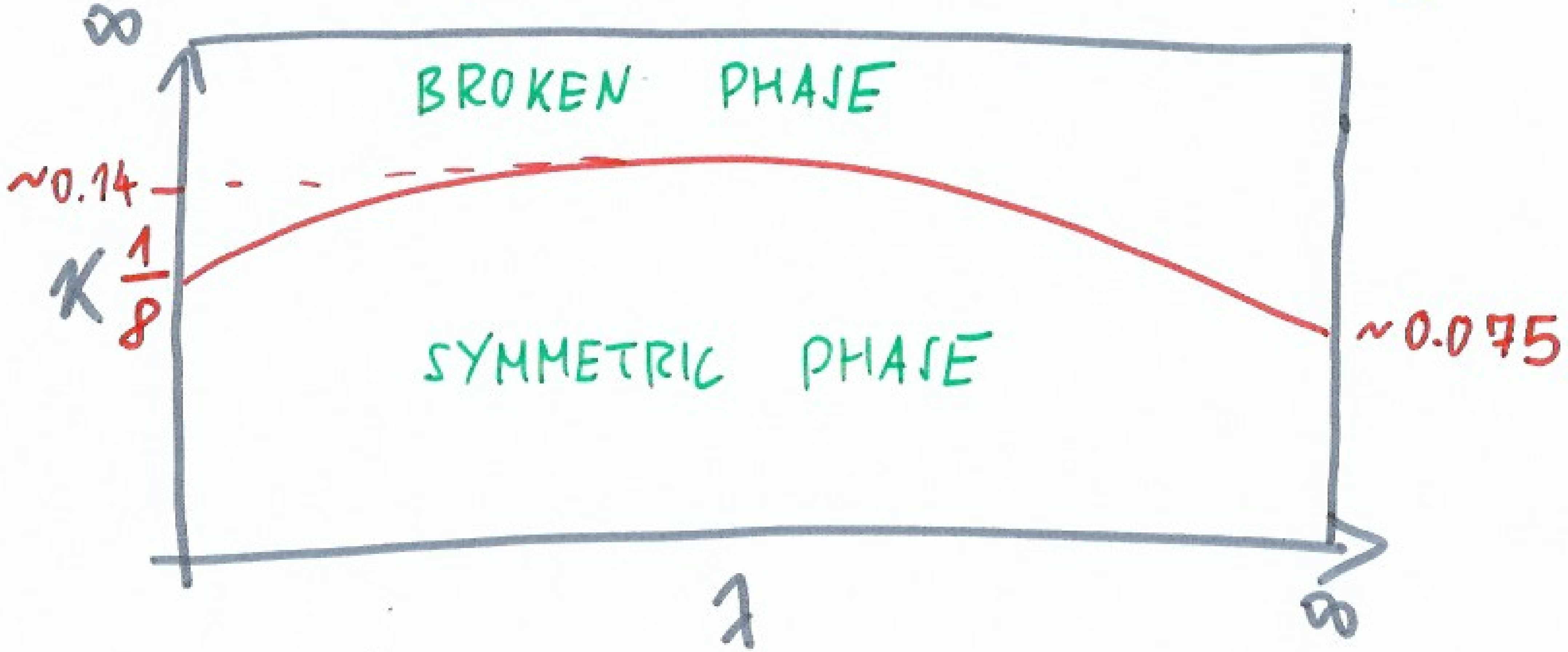
$$\boxed{\xi \sim |T-T_c|^{-\nu}}$$

NEAR A 2ND ORDER PH. TR.: LONG-RANGE CORRELATIONS

($m = \frac{1}{\xi} \rightarrow 0$ near 2nd order ph. tr.)

PHASE DIAGRAM OF SCALAR FIELD THY ON THE LATTICE

$(d=4)$ $$S=\sum_x\left[-2\kappa\sum_{\mu=1}^{4}\phi(x)\phi(x+\hat\mu)+\phi(x)^2+\lambda\left(\phi(x)^2-1\right)^2\right]$$



CRITICAL LINE

as $\lambda\to\infty$ goes to 4D Ising

as $\lambda\to 0$ path integral Gaussian / free field theory

$$m_0=0 \Rightarrow \frac{1}{\kappa}-2d=0 \Rightarrow \kappa=1/2d=1/8$$

O(N) MODELS

(LINEAR SIGMA MODEL)

$$S = \int d^d x \left[\frac{1}{2} (\partial_\mu \varphi^a)(\partial_\mu \varphi^a) + \frac{\mu^2}{2} \varphi^a \varphi^a + \frac{g}{4} (\varphi^a \varphi^a)^2 \right] \qquad a = 0, 1, 2, \dots, N-1$$

has O(N) rotation symmetry

$\boxed{D=4}$
- $N=1$ scalar thy from before
- $N=4$
  - Higgs sector of the standard model
  - low-energy effective theory for pions

$\boxed{D=3}$
- $N=1$ liquid-gas
- $N=2$ superfluid tr.
- $N=3$ isotropic ferromagnet
- ...

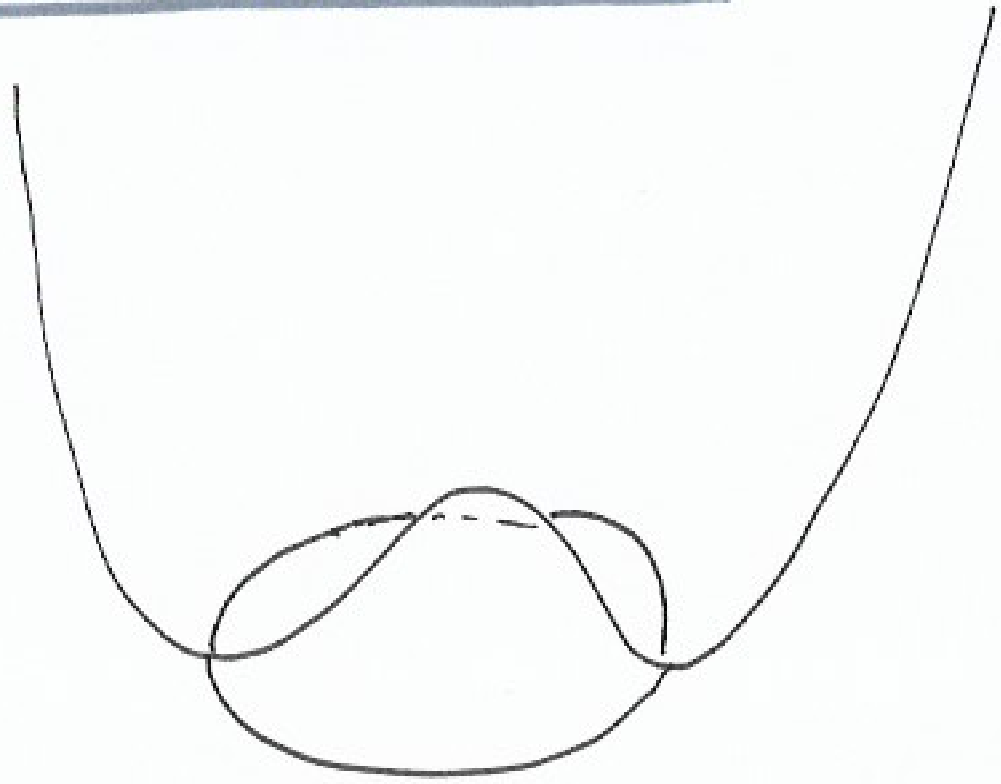
For $d \geq 3$ there are also two phases: $\exists$ SSB

(For $d=2$ the Mermin-Wagner thm prohibits SB of a cont. symmetry

MEXICAN HAT POTENTIAL

$$U = \frac{1}{2} \mu^2 \varphi^2 + \frac{1}{4} g (\varphi^2)^2$$

$\mu^2 < 0$



CAN ALSO BE REWRITTEN ON THE LATTICE AS

$$S = \sum_x \left[-2\kappa \sum_\mu \phi_x^a \phi_{x+\hat{\mu}}^a + \phi_x^a \phi_x^a + \lambda (\phi_x^a \phi_x^a - 1)^2 \right]$$

$\lambda \to \infty$: $\phi_x^a \phi_x^a = 1$

NONLINEAR SIGMA MODEL

$$S = -\beta \sum_{\langle xy \rangle} \vec{\varphi}_x \cdot \vec{\varphi}_y$$

$\vec{\varphi}_x^{\,2} = 1$

GOLDSTONE BOSONS (IN CLASSICAL FIELD THY)

infinitely degenerate ground state

$$\frac{\partial U}{\partial \varphi^k} = 0 \quad \Rightarrow (\mu^2 + g\varphi^2)\varphi^k = 0$$

has a solution: $\varphi_g^a = v\,\delta_{a0}$

$v^2 = -\mu^2/g$

an all $O(N)$ rotations are also soln-s

To force the system into a definite ground state, add symm. breaking term $\Delta S = \int d^dx\, \epsilon\, \varphi^0(x)$ $\quad \epsilon > 0$

$$\Rightarrow (\mu^2 + g\varphi^2)\varphi^a = \epsilon\,\delta_{a0} \quad \Rightarrow \varphi_g^a = v\,\delta_{a0} \quad (\mu^2 + g v^2)v = \epsilon$$

EoM: $(\partial^2 + \mu^2 + g\varphi^2)\varphi^a = \epsilon\,\delta_{a0}$

linearize around $\varphi = \varphi_g$ $\Rightarrow$ $\varphi^0 = v + \sigma$ $\quad \varphi^k = \pi_k$ $\quad k = 1, \dots, N-1$

$$\Rightarrow (\partial^2 + m_\sigma^2)\sigma = 0 \qquad \text{where } m_\sigma^2 = \mu^2 + 3gv^2 = 2gv^2 + \epsilon/v$$

$$(\partial^2 + m_\pi^2)\pi^k = 0 \qquad m_\pi^2 = \mu^2 + gv^2 = \epsilon/v$$

as $\epsilon \to 0$ $\quad m_\sigma^2 \to 2gv^2$

$m_\pi^2 \to 0$ $\quad$ GOLDSTONE BOSON

ALSO REMAIN MASSLESS IN QFT