LATTICE FIELD THEORY

Why learn LFT?

* How are QFTs defined?
* Understand connections between stat. mech. & QFT
* How to turn the Feynman path integral into a practical calculation method?
* Pert. thy doesn't work in many cases
* Even if you don't work in LFT, in many fields you have to read/interface w. results from LFT

THIS COURSE:

* doesn't assume knowledge of QFT
* builds LFT from scratch

Some books

Montvay, Münster

* Gattringer, Lang

Rothe

Smit

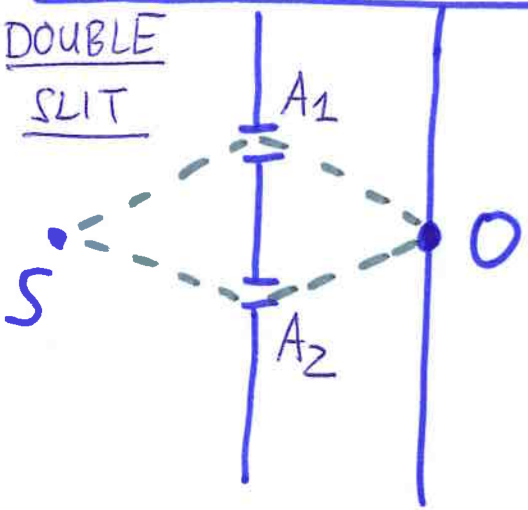
Creutz

Itzykson, Drouffe — Stat. field thy V1

THE PATH INTEGRAL IN QM

HANDWAVING STORY

DOUBLE SLIT



QM

⇓

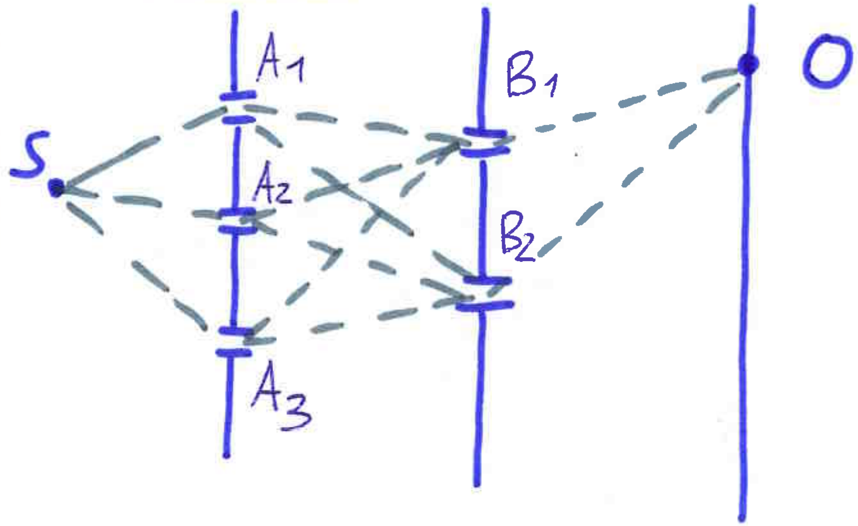
superposition principle

$$\mathcal{A}(\text{detected at } O) = \mathcal{A}(S \to A_1 \to O) + \mathcal{A}(S \to A_2 \to O)$$

MORE SLITS

$$\mathcal{A}(\text{detected at } O) = \sum_i \mathcal{A}(S \to A_i \to O)$$

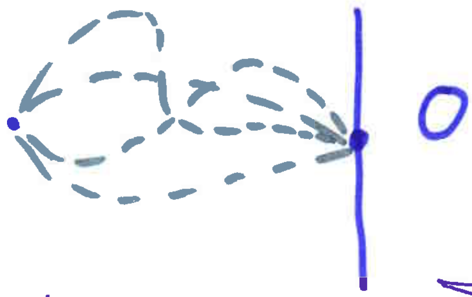
MORE SCREENS



$$\mathcal{A}(\text{detected at } O) = \sum_i \sum_j \mathcal{A}(S \to A_i \to B_j \to O)$$

∞ # OF SCREENS W. AN ∞ # HOLES

(so the screen is not there)



$$\mathcal{A}(S \to O \text{ in time } t) = \sum_{\text{paths}} \mathcal{A}(S \to O \text{ along a path in time } t)$$

HOW TO MAKE IT CONCRETE? $(\hbar = 1)$

$\langle x | e^{-iHt} | y \rangle$ the amplitude we want to calculate

for a free particle: can do it exactly

$$\langle x | e^{-i\frac{p^2}{2m}t} | y \rangle = \int dp \underbrace{\langle x | p \rangle}_{\frac{1}{\sqrt{2\pi}} e^{ipx}} e^{-i\frac{p^2}{2m}t} \underbrace{\langle p | y \rangle}_{\frac{1}{\sqrt{2\pi}} e^{-ipy}}$$

$$= \int \frac{dp}{2\pi} e^{ip(x-y)} \exp\left(-i \frac{p^2}{2m} t\right) = \sqrt{\frac{m}{2\pi i t}} \exp\left(i \frac{m}{2t}(x-y)^2\right)$$

GAUSSIAN INTEGRAL

$$\int_{-\infty}^{+\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right)$$

PARTICLE MOVING IN A POTENTIAL

$$H = H_0 + V = \frac{p^2}{2m} + V(x)$$

to turn the handwaving story into formulas we introduce a lattice (in time)

$$t = \underbrace{\epsilon}_{\text{lattice spacing}} N$$

In general $[H_0, V] \neq 0 \Rightarrow$ NON-TRIVIAL

LOOK AT A SMALL STEP 1ST: $U_\epsilon = e^{-\epsilon H} = e^{-\epsilon\left(\frac{p^2}{2m} + V\right)}$

$$U_\epsilon = \underbrace{e^{-i\epsilon V/2}\, e^{-i\epsilon p^2/2m}\, e^{-i\epsilon V/2}}_{W_\epsilon} + O(\epsilon^3)$$

can calculate:

$$\langle x|W_\epsilon|y\rangle = e^{-i\epsilon V(x)/2} \langle x|e^{-\epsilon p^2/2m}|y\rangle e^{-i\epsilon V(y)/2} = \sqrt{\frac{m}{2\pi i\epsilon}}\, \exp\left[i\frac{m}{2\epsilon}(x-y)^2 - \frac{i\epsilon}{2}\left(V(x)+V(y)\right)\right]$$

$$\boxed{e^{-i(H_0+V)t} = \lim_{N\to\infty} W_\epsilon^N}$$

$$\|U - W_\epsilon^N\| = \|U_\epsilon^N - W_\epsilon^N\| = \Big\|\sum_{k=0}^{N-1} U_\epsilon^k (U_\epsilon - W_\epsilon) W_\epsilon^{N-1-k}\Big\|$$

$$\begin{aligned}
&= \cancel{U_\epsilon W_\epsilon^{N-1}} - W_\epsilon^N \\
&+ \cancel{U_\epsilon^2 W_\epsilon^{N-2}} - \cancel{U_\epsilon W_\epsilon^{N-1}} \\
&+ \cancel{U_\epsilon^3 W_\epsilon^{N-3}} - \cancel{U_\epsilon^2 W_\epsilon^{N-2}} \\
&+ \quad \cdots \\
&+ U_\epsilon^N - \cancel{U_\epsilon^{N-1} W_\epsilon}
\end{aligned}$$

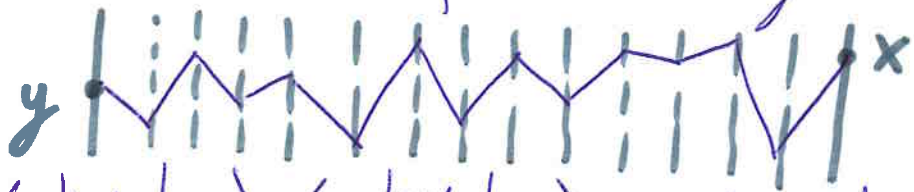
$$\leq \sum_{k=0}^{N-1} \|U_\epsilon\|^k \|W_\epsilon\|^{N-1-k} \, \|U_\epsilon - W_\epsilon\|$$

$$= N \underbrace{\|U_\epsilon - W_\epsilon\|}_{\sim \frac{\text{const.}}{N^3}}$$

$$\Rightarrow \lim_{N\to\infty} \|U_\epsilon^N - W_\epsilon^N\| = 0$$

$\square$

$\boxed{t = N\epsilon}$ insert $N-1$ complete sets of position eigenstates



$$\langle x | e^{-iHt} | y \rangle = \lim_{N\to\infty} \int dx_1 \cdots dx_{N-1} \langle x | W_\epsilon | x_1 \rangle \langle x_1 | W_\epsilon | x_2 \rangle \cdots \langle x_{N-1} | W_\epsilon | y \rangle$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi i \epsilon}\right)^{\frac{N}{2}} \int dx_1 \cdots dx_{N-1} \exp\left[i \frac{m}{2\epsilon} \left((x-x_1)^2 + (x_1-x_2)^2 + \ldots + (x_{N-1}-y)^2 \right) - i\epsilon \left(\frac{V(x)}{2} + V(x_1) + \ldots + V(x_{N-1}) + \frac{V(y)}{2} \right) \right]$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi i \epsilon}\right)^{\frac{N}{2}} \int dx_1 \cdots dx_{N-1} \; e^{iS_\epsilon}$$

$S_\epsilon = S + O(\epsilon^2)$ where $S = \int_0^t \left[\frac{m}{2} \dot{x}^2 - V(x) \right] dt'$

classical action

$$\boxed{\langle x | e^{-iHt} | y \rangle = \int \mathcal{D}x \; e^{iS}}$$

$x(0) = y$
$x(t) = x$

COMMENTS

- $$\mathcal{D}x = \lim_{N\to\infty} \left(\frac{m}{2\pi i \epsilon} \right)^{N/2} dx_1 \dots dx_{N-1}$$

  independent of V but diverges as $N \to \infty$
  we usually divide by an other path int.
  e.g. the one at $V=0$

- Unlike the general Hilbert-space formalism / abstract construction of QM, the path integral formalism relies on a specific basis: <u>the basis in which the pos. op. is diag.</u>

- Hard to give the above path integral formula math meaning: <u>integrand strongly oscillates</u>

EUCLIDEAN PATH INTEGRAL

$t = -i\tau \qquad \tau > 0 \qquad e^{-iHt} = e^{-H\tau}$ $\boxed{\text{Euclidean time} = \tau}$

repeating the same steps as before

$$\langle x | e^{-H\tau} | y \rangle = \lim_{N\to\infty} \left(\frac{m}{2\pi\epsilon}\right)^{\frac{N}{2}} \int dx_1 \dots dx_{N-1} \exp\left[-\frac{m}{2\epsilon}\left((x-x_1)^2 + \dots + (x_{N-1}-y)^2\right) - \epsilon\left(\frac{V(x)}{2} + V(x_1) + \dots + V(x_{N-1}) + \frac{V(y)}{2}\right)\right]$$

$$\langle x | e^{-H\tau} | y \rangle = \int Dx \, e^{-S_E}$$

NOW INTEGRAND > 0
AND EXP. SUPPRESSED
FOR LARGE S_E

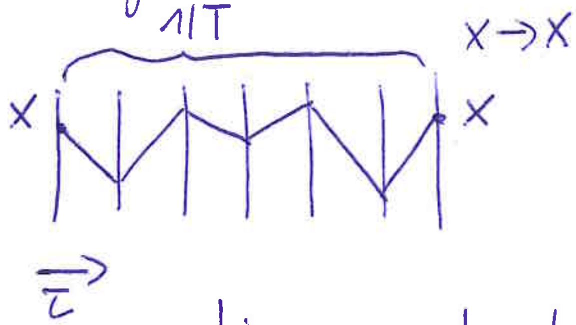
$$S_E = \int_0^\tau d\tau' \left[\frac{m}{2}\left(\frac{dx}{d\tau'}\right)^2 + V(x)\right]$$ Euclidean action

related to original
via
$S = i S_E$
if $t \to \tau$

QUANTUM STAT. MECH. $\quad (k_B = 1)$

$$Z = \mathrm{Tr}\left(e^{-H/T}\right) = \int dx\, \underbrace{\langle x | e^{-H/T} | x \rangle} = \int_{x(0)=x(1/T)} \mathcal{D}x\; e^{-S_E}$$

we just saw: $\int \mathcal{D}x\; e^{-S_E}$



$$\boxed{S_E = \int_0^{1/T} d\tau \left[\left(\frac{dx}{d\tau}\right)^2 + V(x) \right]}$$

$$\mathcal{D}x \propto dx\, dx_1 \cdots dx_{N-1}$$

time extent $= N\epsilon = 1/T$

$$\boxed{T = \frac{1}{N\epsilon} = \frac{1}{N a_t}}$$

EXP. VALUES

$$\langle A \rangle_T = \frac{\mathrm{Tr}\left(e^{-H/T} A\right)}{\mathrm{Tr}\left(e^{-H/T}\right)}$$

as $\boxed{T \to 0}$ $\quad Z \sim e^{-E_0/T} \qquad \mathrm{Tr}\left(e^{-H/T} A\right) \sim e^{-E_0/T} \langle 0 | A | 0 \rangle$

$\langle A \rangle_T \to \langle 0 | A | 0 \rangle$ can calculate expectation values in the ground state

CORRELATION FUNCTIONS

real times $\langle 0|x(t_1)\dots x(t_n)|0\rangle \qquad x(t) = e^{iHt}xe^{-iHt}$

$$\begin{aligned}&\langle 0|x(t_1)\dots x(t_n)|0\rangle \\ &= \langle 0|e^{iE_0t_1}x\,e^{-iH(t_1-t_2)}x\,e^{-iH(t_2-t_3)}\dots e^{-iH(t_{n-1}-t_n)}xe^{-iE_0t_n}|0\rangle\end{aligned}$$

Euclidean times $\tau_k = it_k$

$$\langle 0|x(\tau_1)\dots x(\tau_n)|0\rangle = \langle 0|e^{E_0\tau_1}xe^{-H(\tau_1-\tau_2)}x\,e^{-H(\tau_2-\tau_3)}\dots e^{-H(\tau_{n-1}-\tau_n)}xe^{-E_0\tau_n}|0\rangle$$

if ordered $\tau_1 > \tau_2 > \dots > \tau_n$

$$\frac{1}{Z(\frac{1}{T}=\tau)}\mathrm{Tr}\left[e^{-H\tau}x(\tau_1)\dots x(\tau_n)\right] = \frac{1}{Z(\tau)}\mathrm{Tr}\left[e^{-H\tau}e^{-H\tau_1}xe^{-H(\tau_1-\tau_2)}\dots e^{-H(\tau_{n-1}-\tau_n)}xe^{-H\tau_n}\right]$$

$$= \frac{1}{Z(\tau)}\int_{x(0)=x(\tau)} Dx\; x(\tau_1)\dots x(\tau_n)\,e^{-S_E} \xrightarrow[\tau\to 0]{} \langle 0|x(\tau_1)\dots x(\tau_n)|0\rangle$$

$\boxed{n=1}$ $$\int Dx\, x(\tau_1)\, e^{-S_E(0\to\tau)} = \int dx_0 \int_{x_0\to x_0} Dx\, x(\tau_1)\, e^{-S_E(0\to\tau)}$$

$$= \int dx_0 \int dx_1\, x_1 \int_{x_0\to x(\tau_1)} Dx\, e^{-S_E(0\to\tau_1)}\, \delta(x_1 - x(\tau_1)) \int_{x(\tau_1)\to x_0} Dx\, e^{-S_E(\tau_1\to\tau)}$$

$$= \int dx_0 \int dx_1\, x_1\, \langle x_1 | e^{-H\tau_1} | x_0\rangle \langle x_0 | e^{-H(\tau-\tau_1)} | x_1\rangle$$

$$= \int dx_0 \int dx_1\, \langle x_1 | e^{-H\tau_1} | x_0\rangle \langle x_0 | e^{-H(\tau-\tau_1)}\, x | x_1\rangle$$

$$\underset{\int dx_0 |x_0\rangle\langle x_0| = \mathbb{1}}{=} \mathrm{Tr}\left[e^{-H\tau_1} e^{-H\tau} e^{H\tau_1} x\right] = \mathrm{Tr}\left[e^{-H\tau} e^{H\tau_1} x e^{-H\tau_1}\right]$$

similarly for larger n we use:

$$\prod_{i=1}^{n} q(t_i) = \int \prod_{i=1}^{n} dq_i\, \delta(q(t_i) - q_i)\, q_i$$ an make similar manipulations

generalization to other operators is straightforward

e.g. $\langle x^2(\tau_1)\, x^3(\tau_2)\rangle = \frac{1}{Z}\int \mathcal{D}x \;\; x^2(\tau_1)\, x^3(\tau_2)\, e^{-S_E}$ etc.

CORREL. FN. $\longrightarrow$ ENERGIES

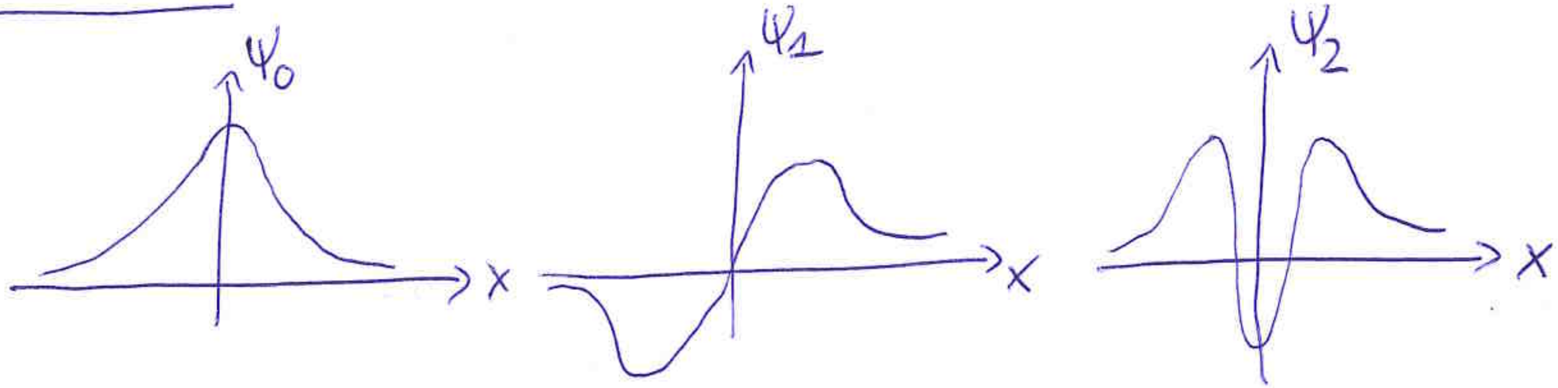
$$\langle 0|\, O(\tau_1)\, O(\tau_2)\,|0\rangle = \sum_n \langle 0|\, e^{H\tau_1}\, O e^{-H\tau_1}|n\rangle \langle n|\, e^{H\tau_2}\, O e^{-H\tau_2}|0\rangle$$

$$= \sum_n e^{-(E_n - E_0)(\tau_1-\tau_2)}\, |\langle 0|O|n\rangle|^2 \underset{\tau_1-\tau_2\to\infty}{\sim} e^{-(E_k - E_0)(\tau_1-\tau_2)}$$

where $|k\rangle$ is the lowest energy state s.t. $\langle 0|O|k\rangle \neq 0$

THIS IS HOW YOU CALCULATE THE SPECTRUM OF THE THY FROM THE EUCLIDEAN PATH INTEGRAL

EXAMPLE HARMONIC OR ANHARMONIC OSCILLATOR $V = ax^2 + bx^4$



$$\boxed{O = x} \quad \langle 0 | x(\tau_1)\, x(\tau_2) | 0 \rangle \underset{\tau_1 - \tau_2 \gg 1}{\sim} e^{-(E_1 - E_0)(\tau_1 - \tau_2)} \, |\langle 0 | x | 1 \rangle|^2$$

since $\langle 0 | x | 0 \rangle = 0$

use quantum numbers to choose correct operator: here parity

$$\boxed{O = x^2} \quad \langle 0 | x^2(\tau_1)\, x^2(\tau_2) | 0 \rangle \underset{\tau_1 - \tau_2 \gg 1}{\sim} e^{-(E_2 - E_0)(\tau_1 - \tau_2)} \, |\langle 0 | x^2 | 2 \rangle|^2 + |\langle 0 | x^2 | 0 \rangle|^2$$

use the connected correlation function

$$\langle 0 | x^2(\tau_1)\, x^2(\tau_2) | 0 \rangle - |\langle 0 | x^2(\tau_1) | 0 \rangle|^2 \underset{\tau_1 - \tau_2 \gg 1}{\sim} e^{-(E_2 - E_0)(\tau_1 - \tau_2)}$$

COMMENTS

(1) generalization to any finite num. of coords is straightforward

$$L = \frac{1}{2}\sum_{m,n} \dot{q}_m M_{mn} \dot{q}_n - V(q_1, q_2, \ldots, q_N) \Rightarrow H = \frac{1}{2}\sum_{mn} p_m M^{-1}_{mn} p_n + V(q_1, \ldots, q_N)$$

$$\langle \vec{q} | e^{-H\tau} | \vec{q}\,' \rangle = \int_{\substack{\vec{q}(0)=\vec{q} \\ \vec{q}(\tau)=\vec{q}\,'}} D\vec{q}\; e^{-S_E}$$

where

$$S_E = \int_0^\tau \left[\frac{1}{2} \left(\frac{dq_m}{d\tau'} M_{mn} \frac{dq_n}{d\tau'} \right) + V(q_1, \ldots, q_N) \right]$$

(2)

$$i\frac{\partial \Psi}{\partial t} = H\Psi \xrightarrow[t=-i\tau]{} \frac{\partial \Psi_E}{\partial \tau} + H\Psi_E = 0$$

Schrödinger eq. — diffusion eq.

EUCLIDEAN PATH INTERAL = SUM OVER RANDOM PATHS IN A RANDOM WALK

(a lot about this in Izykson, Drouffe)

TRANSFER MATRIX

SO FAR: HAMILTONIAN $\to$ PATH INT.

OTHER WAY? PATH INT. $\to$ H YES.

$$\langle x | e^{-H\tau} | y \rangle = \lim_{N\to\infty} \left(\frac{m}{2\pi\epsilon}\right)^{\frac{N}{2}} \int dx_1 \ldots dx_{N-1} \exp\left[-\frac{m}{2\epsilon}\left[(x-x_1)^2 + \ldots\right] - \epsilon\left[\frac{V(x)}{2} + V(x_1) + \ldots\right]\right]$$

$$= \lim_{N\to\infty} \int dx_1 \ldots dx_{N-1} \langle x | T | x_1 \rangle \langle x_1 | T | x_2 \rangle \ldots \langle x_{N-1} | T | y \rangle$$

$$= \lim_{N\to\infty} \langle x | T^N | y \rangle$$

where $\langle x | T | y \rangle = \sqrt{\frac{m}{2\pi\epsilon}} \exp\left[-\frac{m}{2\epsilon}(x-y)^2 - \frac{\epsilon}{2}(V(x)+V(y))\right]$

$$W_\epsilon = T = \exp(-\epsilon H_\epsilon) \qquad H_\epsilon \to H \text{ as } \epsilon \to 0$$

If we started w. a path integral repr. this is how we could define the Hamiltonian at finite ϵ