

Exact finite volume expectation values of conserved currents

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ELTE

$\left\{ \begin{array}{l} \text{arXiv:1903.06990} \text{ [}'19 \text{ Bajnok, Smirnov]} \\ \text{arXiv:1908.07320} \text{ [}'19 \text{ Borsi, Pozsgay, Pristyák]} \end{array} \right.$

\Rightarrow arXiv:1911.08525

Contents

- ▶ Integrable QFT in finite volume/temperature
- ▶ Semi-classical picture on current expectation values
- ▶ Generalized Gibbs ensemble and vacuum expectation values
- ▶ Excited states
- ▶ Consistency and outlook

Ininitely many conserved charges

$$[\hat{Q}_s, \hat{Q}_{s'}] = 0$$

$$\hat{Q}_{\pm 1} = \hat{H} \pm \hat{p}$$

One-particle eigenvalues expressed with rapidity

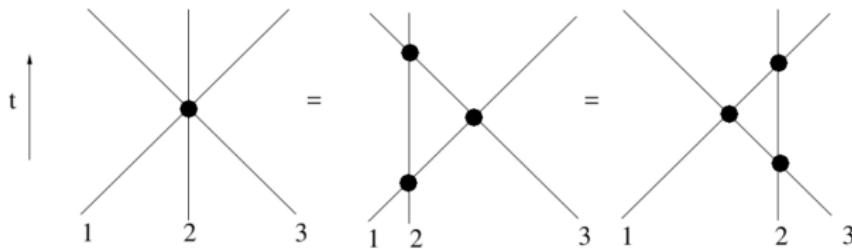
$$\hat{Q}_s |\theta\rangle = h_s(\theta) = h_s \cdot e^{s\theta} |\theta\rangle, \quad \begin{cases} \hat{H} |\theta\rangle = E(\theta) |\theta\rangle & E(\theta) = m \cosh \theta \\ \hat{p} |\theta\rangle = p(\theta) |\theta\rangle & p(\theta) = m \sinh \theta \end{cases}$$

$$\hat{Q}_s |\theta_1, \dots, \theta_N\rangle = (h_s^{(1)} e^{s\theta_1} + \dots + h_s^{(N)} e^{s\theta_N}) |\theta_1, \dots, \theta_N\rangle$$

Effect of higher charges on scattering

$$\sum_{i \in \text{initial}} h_s^{(i)} e^{s\theta_i} = \sum_{j \in \text{final}} h_s^{(j)} e^{s\theta_j} \quad \boxed{\forall s}$$

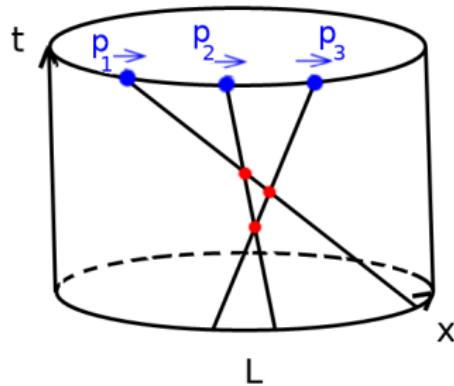
- ▶ no particle production
- ▶ fully elastic scattering $\{\theta_{i \in \text{initial}}\} = \{\theta_{j \in \text{final}}\}$
- ▶ factorization of scattering \Rightarrow two-particle S-matrix is enough: $S(\theta_1 - \theta_2) = e^{i\delta(\theta_1 - \theta_2)}$



Finite volume - Bethe Ansatz

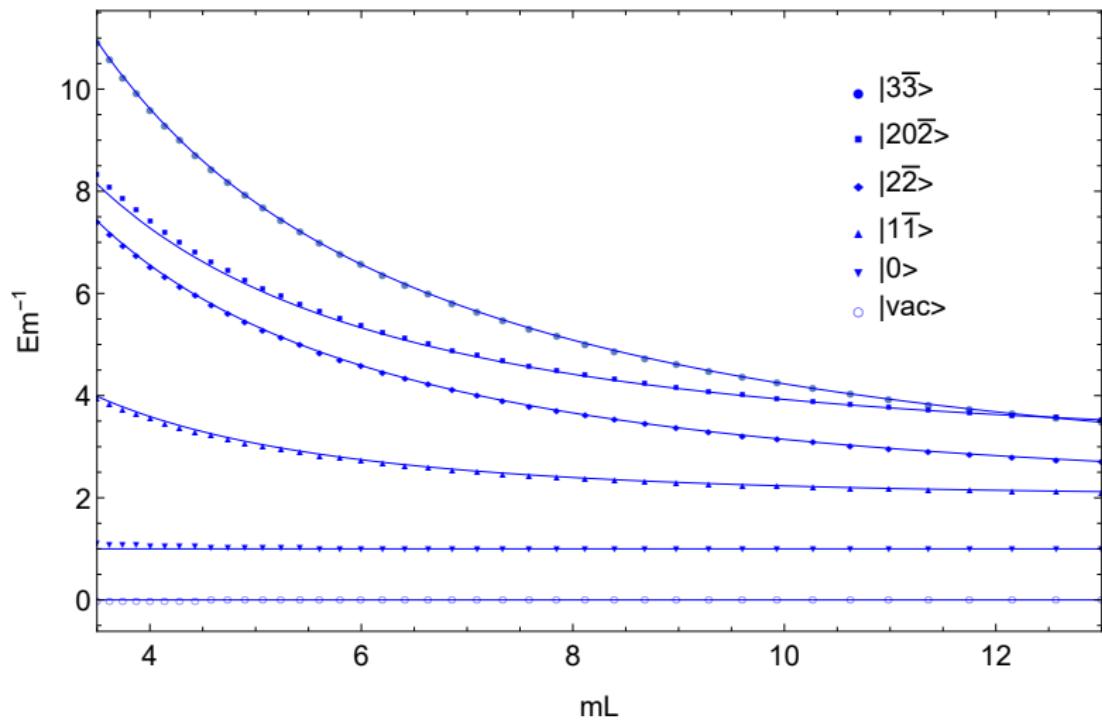
$$e^{ip_j L} \prod_{k \neq j}^N S(\theta_j - \theta_k) = 1 \quad \Rightarrow p(\theta_j)L + \sum_{k \neq j} \delta(\theta_j - \theta_k) + \dots = 2\pi n_j, \quad n_j \in \mathbb{Z}$$

$$E_{|n_1, \dots, n_N\rangle_L} = \sum_{j=1}^N E(\theta_j) + \dots$$



Spectrum

Agrees up to e^{-mL} corrections!



Gaudin-matrix

$$\hat{\mathbb{1}} = |\text{vac}\rangle\langle \text{vac}|_L + \sum_n |n\rangle\langle n|_L + \sum_{n_1 > n_2} |n_1, n_2\rangle\langle n_1, n_2|_L + \dots$$

$$\sum_n |n\rangle\langle n|_L = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \rho(\theta) |\theta\rangle\langle \theta|_L, \quad \rho(\theta) = " \frac{2\pi\Delta n}{\Delta\theta} "$$

$$\rho(\theta_1, \dots, \theta_N) = \begin{vmatrix} \frac{\partial(2\pi n_1)}{\partial\theta_1} & \dots & \frac{\partial(2\pi n_1)}{\partial\theta_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial(2\pi n_N)}{\partial\theta_1} & \dots & \frac{\partial(2\pi n_N)}{\partial\theta_N} \end{vmatrix} = \det G$$

$$G_{jk} = \frac{\partial \text{BY}(\theta_k)}{\partial \theta_j} = \delta_{jk} \left[p'(\theta_j)L + \sum_{k=1}^N \varphi(\theta_j - \theta_k) \right] - \varphi(\theta_j - \theta_k) + \dots$$

$$\varphi(\theta) = \partial_\theta \delta(\theta) = -i \partial_\theta \ln S(\theta)$$

Conservation laws

$$[\hat{Q}_\alpha, \hat{Q}_\beta] = 0, \quad \begin{cases} \hat{H} = \hat{Q}_1, & h_1(\theta) = E(\theta) = m \cosh \theta \\ \hat{p} = \hat{Q}_2, & h_2(\theta) = p(\theta) = m \sinh \theta \\ \vdots & \vdots \end{cases}$$

$$\hat{Q}_\alpha |\theta_1, \dots, \theta_N\rangle_L = \left(\sum_{j=1}^N h_\alpha(\theta_j) + \dots \right) |\theta_1, \dots, \theta_N\rangle_L$$

Charge and current densities

$$\hat{Q}_\alpha = \int_0^L \hat{q}_\alpha(t, x) dx \quad \partial_t \hat{q}_\alpha(t, x) + \partial_x \hat{j}_\alpha(t, x) = 0$$

What we want:

$$\langle \theta_1, \dots, \theta_N | \hat{q}_\alpha(0, 0) | \theta_1, \dots, \theta_N \rangle_L = L^{-1} \left(\sum_{j=1}^N h_\alpha(\theta_j) + \dots \right)$$

$$\langle \theta_1, \dots, \theta_N | \hat{j}_\alpha(0, 0) | \theta_1, \dots, \theta_N \rangle_L = ?$$

Semi-classical picture on currents

One particle

$$\langle \theta | j_\alpha | \theta \rangle_L = \frac{1}{L} v(\theta) h_\alpha(\theta), \quad v(\theta) = \frac{\partial E(p)}{\partial p} = \frac{E'(\theta)}{p'(\theta)}$$

Multi-particle

$$\langle \theta_1, \dots, \theta_N | j_\alpha | \theta_1, \dots, \theta_N \rangle_L = \frac{1}{L} \sum_{k=1}^N v^{\text{eff}}(\theta_k) h_\alpha(\theta_k)$$

Semi-classical picture on currents - effective velocity

Phase shift \Rightarrow displacement

$$\Delta s_{jk} = \frac{\partial \delta(p_j, p_k)}{\partial p_j} = \frac{\varphi(\theta_j - \theta_k)}{p'(\theta_j)}$$

Frequency of scattering

$$T_{jk} = \frac{L}{v^{\text{eff}}(\theta_j) - v^{\text{eff}}(\theta_k)} \quad \Rightarrow \quad \sum_{k \neq j}^N \frac{\Delta s_{jk}}{T_{jk}} \stackrel{!}{=} v(\theta_j) - v^{\text{eff}}(\theta_j)$$

Effective velocity

$$v(\theta_j) = v^{\text{eff}}(\theta_j) + \frac{1}{L p'(\theta_j)} \sum_{k \neq j}^N \varphi(\theta_j - \theta_k) (v^{\text{eff}}(\theta_j) - v^{\text{eff}}(\theta_k))$$

Semi-classical picture on currents - Gaudin-matrix

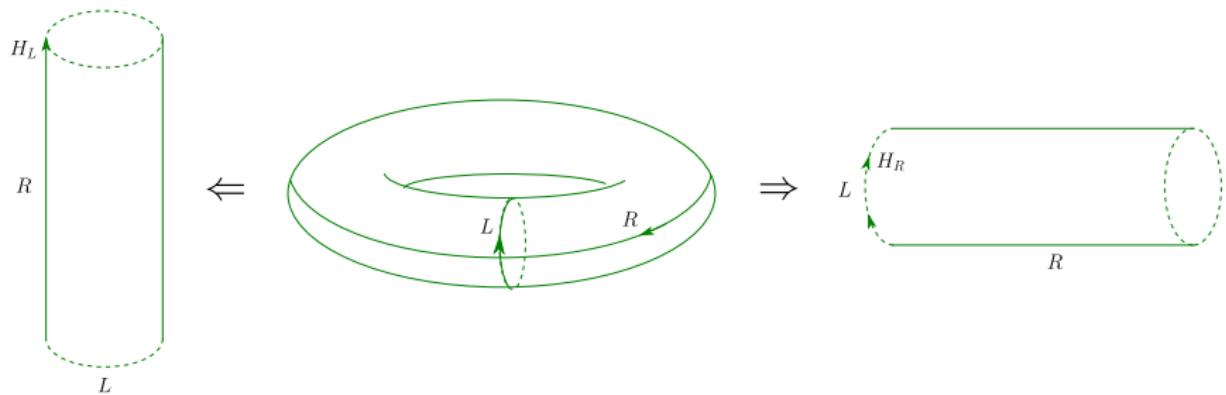
$$LE'(\theta_j) = Lp'(\theta_j)v^{\text{eff}}(\theta_j) + \sum_{k=1}^N \varphi(\theta_j - \theta_k) (v^{\text{eff}}(\theta_j) - v^{\text{eff}}(\theta_k))$$

$$L\mathbf{E}' = G \cdot \mathbf{v}^{\text{eff}}$$

$$\Rightarrow \boxed{\langle \theta_1, \dots, \theta_N | j_\alpha | \theta_1, \dots, \theta_N \rangle_L = \frac{\mathbf{v}^{\text{eff}} \cdot \mathbf{h}_\alpha}{L} + \dots = \mathbf{E}' \cdot G^{-1} \cdot \mathbf{h}_\alpha + \dots}$$

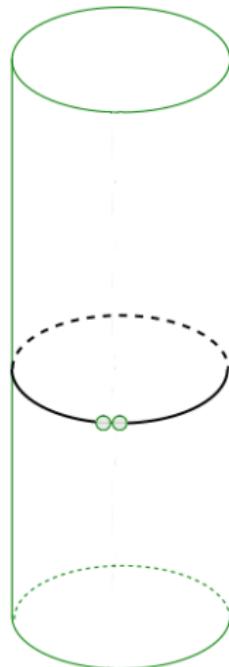
$$\mathbf{E}' = \begin{pmatrix} E'(\theta_1) \\ \vdots \\ E'(\theta_N) \end{pmatrix}, \quad \mathbf{v}^{\text{eff}} = \begin{pmatrix} v^{\text{eff}}(\theta_1) \\ \vdots \\ v^{\text{eff}}(\theta_N) \end{pmatrix}, \quad \mathbf{h}_\alpha = \begin{pmatrix} h_\alpha(\theta_1) \\ \vdots \\ h_\alpha(\theta_N) \end{pmatrix}$$

Finite volume - exponential corrections



$$\begin{aligned} Z(L, R) &= \text{Tr}_L(e^{-R\hat{H}_L}) = \text{Tr}_R(e^{-L\hat{H}_R}) = \\ &= e^{-RE_{\text{vac}}(L)} + \dots = 1 + \sum_n e^{-LE_{|n\rangle}(R)} + \sum_{n_1 > n_2} e^{-LE_{|n_1, n_2\rangle}(R)} + \dots \end{aligned}$$

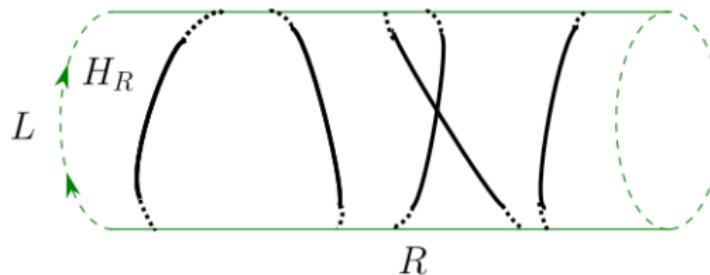
Finite volume - virtual pairs



$$\sum_n e^{-L E_{|n\rangle}(R)} \rightarrow \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \overbrace{Rp'(\theta)}^{\rho(\theta)} e^{-L E(\theta)}$$

$$E_{\text{vac}}(L) \sim - \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-L E(p)} + \mathcal{O}(e^{-2mL})$$

Thermodynamical Bethe Ansatz (TBA)



$$R \rightarrow \infty, \quad N \rightarrow \infty, \quad \frac{N}{R} = \text{const.}$$

Free energy

$$Z(L, R) \sim e^{-RE_{\text{vac}}(L)} \sim \sum_{\text{configurations}} e^{-\beta F[\rho^{(\text{particle})}, \rho^{(\text{hole})}]}, \quad \beta = T^{-1} \equiv L$$

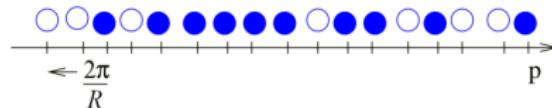
$$F[\rho^{(\text{particle})}, \rho^{(\text{hole})}] = E[\rho^{(\text{particle})}] - TS[\rho^{(\text{particle})}, \rho^{(\text{hole})}]$$

Density of Bethe roots

For large R volume

$$Rp(\theta_j) + \sum_{k \neq j} \delta(\theta_j - \theta_k) = 2\pi n_j$$

Continuous case: # Bethe roots = # particles + # holes



$$Rp'(\theta) + \int_{-\infty}^{\infty} d\theta' \varphi(\theta - \theta') \rho^{(\text{particle})}(\theta') = 2\pi (\rho^{(\text{particle})}(\theta) + \rho^{(\text{hole})}(\theta))$$

$$E[\rho^{(\text{particle})}] = \int d\theta E(\theta) \rho^{(\text{particle})}(\theta)$$

Ground state TBA equation

Pseudo energy

$$e^{-\epsilon(\theta)} = \frac{\rho^{(\text{particle})}(\theta)}{\rho^{(\text{hole})}(\theta)}$$

TBA equation

$$\epsilon(\theta) = LE(\theta) - \int_{-\infty}^{\infty} d\theta' \varphi(\theta - \theta') \ln \left(1 + e^{-\epsilon(\theta')} \right)$$

Filling fraction

$$n(\theta) = \frac{\rho^{(\text{particle})}}{\rho^{(\text{particle})} + \rho^{(\text{hole})}} = \frac{1}{1 + e^\epsilon}$$

Density again

$$\boxed{\rho^{(\text{particle})}(\theta) = \frac{1}{2\pi} n(\theta) (p')^{\text{dr}}(\theta)}$$

$$E[\rho^{(\text{particle})}] = E(\theta) \circ \rho^{(\text{particle})}(\theta)$$

Pairing, dressing

Pairing:

$$f(\theta) \circ g(\theta) = \int \frac{d\theta}{2\pi} n(\theta) f(\theta)g(\theta)$$

Dressing:

$$g^{\text{dr}}(\theta) = g(\theta) + \varphi(\theta - u) \circ g^{\text{dr}}(u)$$

Properties:

$$f(\theta) \circ g(\theta) = g(\theta) \circ f(\theta)$$

$$f(\theta) \circ g^{\text{dr}}(\theta) = f^{\text{dr}}(\theta) \circ g(\theta)$$

Generalized Gibbs ensemble

[’16 Castro-Alvaredo, Doyon, Yoshimura]

$$Z = \text{Tr} \left(e^{-\sum_{\alpha} \beta_{\alpha} \hat{Q}_{\alpha}} \right) \quad \Rightarrow \quad \langle \hat{\mathcal{O}} \rangle$$

$$\langle \hat{q}_{\alpha} \rangle = \int d\theta \ h_{\alpha}(\theta) \rho^{\text{(particle)}}(\theta) = -\frac{\partial F}{\partial \beta_{\alpha}}$$

$$\boxed{\langle \hat{q}_{\alpha} \rangle = (p')^{\text{dr}}(\theta) \circ h_{\alpha}(\theta), \quad \langle \hat{j}_{\alpha} \rangle = (E')^{\text{dr}}(\theta) \circ h_{\alpha}(\theta) = v_{\text{eff}}(\theta) \circ h_{\alpha}(\theta)}$$

$$v_{\text{eff}}(\theta) = \frac{(E')^{\text{dr}}(\theta)}{(p')^{\text{dr}}(\theta)}$$

Difference:

$$\epsilon(\theta) = \beta_{\alpha} h_{\alpha}(\theta) - \int_{-\infty}^{\infty} d\theta' \ \varphi(\theta - \theta') \ln \left(1 + e^{-\epsilon(\theta')} \right)$$

Continuation to the finite volume channel

Transformation

$$\underline{\beta} \rightarrow (L, 0, 0, 0, \dots)$$

$$\theta \rightarrow \theta^\gamma \equiv \theta + \frac{i\pi}{2}$$

$$(x, t) \rightarrow (it, ix)$$

$$(j, q) \rightarrow (iq, ij)$$

$$(p, E) \rightarrow (iE, ip)$$

VEVs

$$\langle \text{vac} | \hat{q}_\alpha | \text{vac} \rangle_L = i(E')^{\text{dr}}(\theta) \circ h_\alpha(\theta^\gamma) = i \langle \hat{j}_k^\gamma \rangle$$

$$\langle \text{vac} | \hat{j}_\alpha | \text{vac} \rangle_L = i(p')^{\text{dr}}(\theta) \circ h_\alpha(\theta^\gamma) = i \langle \hat{q}_\alpha^\gamma \rangle$$

Simplification for the charges

$$h_1(\theta) = E(\theta)$$

$$\partial_\theta \epsilon(\theta) = L(E')^{\text{dr}}(\theta)$$
$$\Rightarrow E_{\text{vac}}(L) = L \langle \text{vac} | \hat{q}_1 | \text{vac} \rangle_L = - \int \frac{d\theta}{2\pi} E(\theta) \ln(1 + e^{-\epsilon(\theta)})$$

$$\ln(1 + e^{-\epsilon(\theta)}) \sim e^{-LE(\theta)} + \mathcal{O}(e^{-2mL})$$

$$E_{\text{vac}}(L) = - \int \frac{d\theta}{2\pi} E(\theta) e^{-LE(\theta)} + \mathcal{O}(e^{-2mL}) \checkmark$$

Excited states and re-defined pairing

Excited state TBA

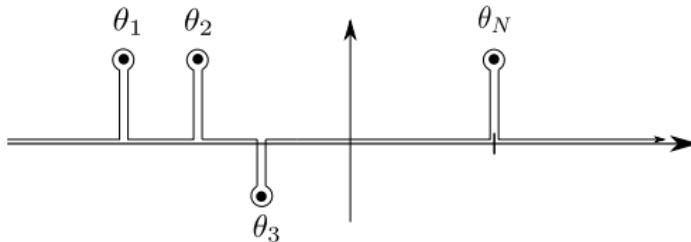
$$\epsilon(\theta) = LE(\theta) + \sum_k \eta_k \log S(\theta - \theta_k) - \int \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \ln(1 + e^{-\epsilon(\theta')})$$

Active singularities $\{\theta_j\}$

$$\epsilon(\theta_j) = i\pi(2n_j + 1)$$

Pairing and dressing for excited states: [19 Bajnok, Smirnov]

$$f(\theta) \bullet g(\theta) = \sum_j \frac{\eta_j i f(\theta_j) g(\theta_j)}{\partial_\theta \epsilon(\theta)|_{\theta=\theta_j}} + f(\theta) \circ g(\theta)$$



Conjecture

New dressing:

$$g^{\text{dr}}(\theta) = g(\theta) + \varphi(\theta - u) \bullet g^{\text{dr}}(u)$$

Keeping original formulae:

$$\langle q_k \rangle = (p')^{\text{dr}}(\theta) \bullet h_k(\theta) = ?$$

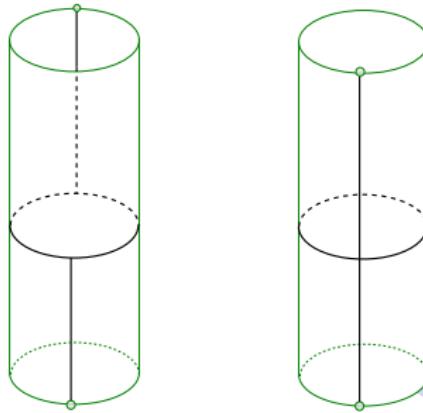
$$\langle j_j \rangle = (E')^{\text{dr}}(\theta) \bullet h_k(\theta) = ?$$

Simplification for the charges

In case of sinh-Gordon: $\theta_j \equiv \bar{\theta}_j + \frac{i\pi}{2}$, $\eta_j = 1$:

$$\begin{aligned} E_{|\bar{\theta}_1, \dots, \bar{\theta}_n\rangle}(L) &= L \langle \bar{\theta}_1, \dots, \bar{\theta}_n | q_1 | \bar{\theta}_1, \dots, \bar{\theta}_n \rangle_L \\ &= \sum_j E(\bar{\theta}_j) - \int \frac{d\theta}{2\pi} E(\theta) \ln(1 + e^{-\epsilon(\theta)}) \end{aligned}$$

$$E_{|\bar{\theta}_1, \dots, \bar{\theta}_n\rangle}(L) = \sum_j E(\bar{\theta}_j) - \int \frac{d\theta}{2\pi} E(\theta) \prod_j S(\theta - \theta_j) e^{-L E(\theta)} + \dots$$



Gaudin-matrix

$$G_{jk}^0(\theta) = \delta_{jk} \left[p'(\theta)L + \sum_{k=1}^N \varphi(\theta - \theta_k) \right] - \varphi(\theta - \theta_k)$$

Gaudin-matrix:

$$G_{jk} = -i\partial_{\theta_j}\epsilon(\theta_k) = (G_{jk}^0)^{\text{dr}}(\theta_j)$$

Relation between dressings

$$g^{\text{dr}}(\theta) = g^{\text{dr}}(\theta) + \sum_j \varphi_j^{\text{dr}}(\theta) G_{jk}^{-1} g^{\text{dr}}(\theta_k)$$

$$f(\theta) \bullet g^{\text{dr}}(\theta) = \sum_{j,k} \eta_j f^{\text{dr}}(\theta_j) G_{jk}^{-1} g^{\text{dr}}(\theta_k) + f(\theta) \circ g^{\text{dr}}(\theta)$$

Expression via Gaudin matrix

$$\langle q_\alpha \rangle = (p')^{\text{dr}}(\theta) \bullet h_\alpha(\theta) = \sum_{j,k} \eta_j(p')^{\text{dr}}(\theta_j) G^{-1} \cdot h_\alpha^{\text{dr}}(\theta_k) + p'(\theta) \circ h_\alpha^{\text{dr}}(\theta)$$

Current expectation value in the finite volume channel:

$$\langle \theta_1, \dots, \theta_N | j_\alpha | \theta_1, \dots, \theta_N \rangle_L = i \langle q_\alpha^\gamma \rangle$$

Consistency in the Bethe Ansatz regime

sinh-Gordon: $\theta_j \equiv \bar{\theta}_j + \frac{i\pi}{2}$, $\eta_j = 1$:

$$\epsilon(\bar{\theta}_j + \frac{i\pi}{2}) = i\pi(2n_j + 1)$$

Dropping all e^{-mL} -type corrections: [19 Borsi, Pozsgay, Pristyák]

$$\langle \bar{\theta}_1, \dots, \bar{\theta}_N | j_\alpha | \bar{\theta}_1, \dots, \bar{\theta}_N \rangle_L = \sum_{j,k} E'(\bar{\theta}_j) G_{jk}^{-1} h_\alpha(\bar{\theta}_k)$$

Consistency in the Lüscher regime

Excited state formula [^{'14} Pozsgay, Szécsényi, Takács]

$$\langle \theta_1, \dots, \theta_n | \mathcal{O} | \theta_1, \dots, \theta_n \rangle_L = \frac{\sum_{\{\theta_+\} \cup \{\theta_-\}} \mathcal{D}_\epsilon^{\mathcal{O}}(\{\theta_+\}) \rho(\{\theta_-\} | \{\theta_+\})}{\rho(\theta_1, \dots, \theta_n)}$$

$$\rho(\theta_1, \dots, \theta_n) = \det G \quad (\text{Gaudin-matrix})$$

$$\rho(\{\theta_-\} | \{\theta_+\}) = \det G_- \quad (\text{submatrix of } \{\theta_-\} \text{ particles})$$

$$\mathcal{D}_\epsilon^{\mathcal{O}}(\{\theta\}) = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{k=1}^n \int \frac{n(u_k) du_k}{2\pi} F_{2(m+n),c}^{\mathcal{O}}(\theta_1, \dots, \theta_m, u_1, \dots, u_n)$$

For 1 particle states, up to e^{-mL} order in the large volume expansion (Lüscher-order), this gives the same result for the current!

Outlook

- ▶ Non-diagonal matrix elements
- ▶ Generalisation for non-diagonally scattering theories
- ▶ Cumulants of charges and currents [**'19 Vu**] in excited states

Thank you for your attention!