Precision *Standard Model* tools: electroweak Sudakovs and multi-jet processes

Timea Vitos

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Today's talk

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Today's talk

EW sector

QCD sector

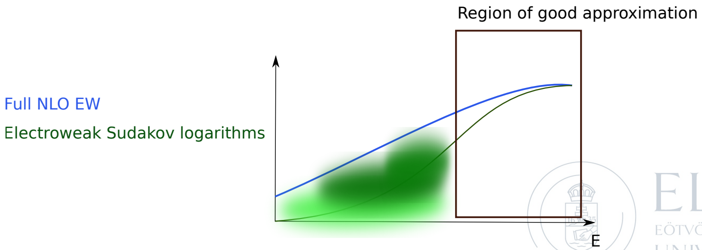
Today's talk

Capturing the high-energy chunk of NLO EW corrections

- Focus on **electroweak corrections**: automations of NLO EW corrections
- Current problems:
  1. time-consuming computations
  2. no automated matching to PS
- Alternative: capture the dominant part of it!

↓

Electroweak Sudakov logarithms (EWSL)



The Denner-Pozzorini algorithm

- In usual notation of mixed-(α_S, α)-NLO expansion:

$$\mathrm{LO} = \mathrm{LO}_1 + \mathrm{LO}_2 + \ldots + \mathrm{LO}_k$$
$$\mathrm{NLO} = \mathrm{NLO}_1 + \underbrace{\mathrm{NLO}_2}_{\text{NLO EW}} + \ldots + \mathrm{NLO}_{k+1}$$

- **One-loop leading approximation of NLO EW**:

$$\mathcal{O}(\mathrm{EWSL}) \sim \left(\alpha \log^k \left(\frac{s}{M_W^2}\right)\right) \times \mathcal{O}(\mathrm{LO}) \tag{1}$$

$$\text{double logarithms (DL)}: \quad \sim \alpha \log^2 \left(\frac{s}{M_W^2}\right)$$
$$\text{single logarithms (SL)}: \quad \sim \alpha \log \left(\frac{s}{M_W^2}\right)$$

- EWSL are universal (with simple, closed expressions): worked out originally by **Denner and Pozzorini**[1]

[1] A. Denner, S. Pozzorini, arXiv:hep-ph/0010201

The Denner-Pozzorini algorithm and MG5_aMC@NLO

- Arise as corrections to the Born-level matrix-element as

$$\mathcal{M}^{\text{LO+EWSL}} = \mathcal{M}_0 + \mathcal{M}_0 \times \delta^{\text{EWSL}} \tag{2}$$

- Denner-Pozzorini algorithm **revised and implemented in MG5**[2]

[2] D. Pagani, M. Zaro arXiv:2110.03714

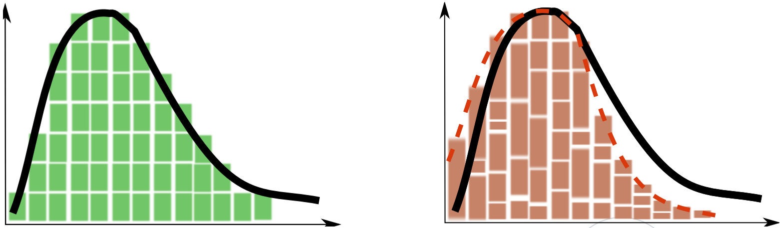
Today's talk

Reweighting

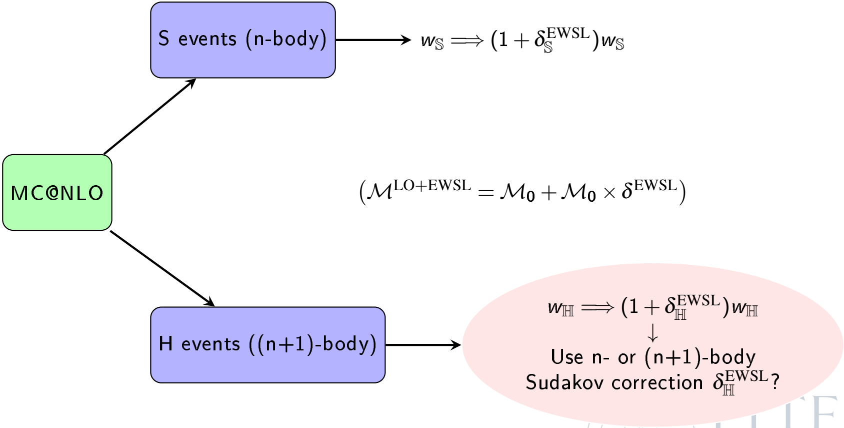
- For each event $e_i(\Phi)$ with weight w_i:

$$w_i \to R(\Phi) w_i \tag{3}$$

- When “new setup” does not depend on initial conditions/generation step



Reweighting NLO events with EWSL



Reweighting NLO events with EWSL

Problem 1

- Sudakov logarithm expressions not valid in the soft/collinear regions!

Problem 2

- IR cancellation not secured anymore!

Proposed procedure:[3]

1. Check all $r_{kl} = (p_k \pm p_l)^2$
2. If all $|r_{kl}| > c_{\mathbb{H}\to\mathbb{S}} M_W^2$: use (n+1)-body Sudakov
3. If any $|r_{kl}| < c_{\mathbb{H}\to\mathbb{S}} M_W^2$: merge particles k, l
4. If reasonable merged process: use n-body Sudakov of the mapped kinematics, **else** use the (n+1)-body Sudakov and replace $|r_{kl}| \to M_W^2$
5. Vary $c_{\mathbb{H}\to\mathbb{S}}$ to assess "Sudakov-cut" dependence

[3] D. Pagani, T. Vitos, M. Zaro arXiv:2309.00452

Summary of the implementation

1. **Reweight all events** in the described procedure and shower them:

$$\mathrm{NLO}_{\mathrm{QCD}} \otimes \mathrm{EWSL}{+}\mathrm{PS}$$

2. Assign **EWSL only to Born events** and shower the events:

$$\mathrm{NLO}_{\mathrm{QCD+EWSL}}{+}\mathrm{PS}$$

- Difference between two is roughly:

$$(\mathrm{NLO}_{\mathrm{QCD}} \otimes \mathrm{EWSL}{+}\mathrm{PS}) - (\mathrm{NLO}_{\mathrm{QCD+EWSL}}{+}\mathrm{PS}) \sim \mathrm{EWSL} \times (K_{\mathrm{QCD}} - 1) \quad (4)$$

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Today's talk

Numerical setup

Input

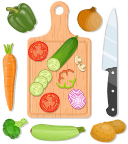
- Focus on **LHC:** $\sqrt{s} = 13$ **TeV**
- Defining jets: **anti-k_T algorithm** with

$$p_T^{\min} = 10 \text{ GeV and } R = 0.4$$

- Use **PYTHIA8** for the parton shower, with hadronization **off**

No cuts (inclusive)

- Note: Sudakov not valid in all regions here!

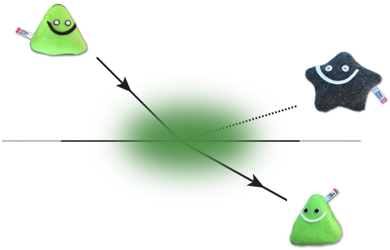


Hard cuts

- $p_T > 400$ GeV cut on all heavy final particles
- $\Delta R > 0.5$ for any two final particles

- Gray curve: NLO_{QCD}+PS
  Blue curve: EWSL with reweighting
  Red curve: EWSL only on Born events

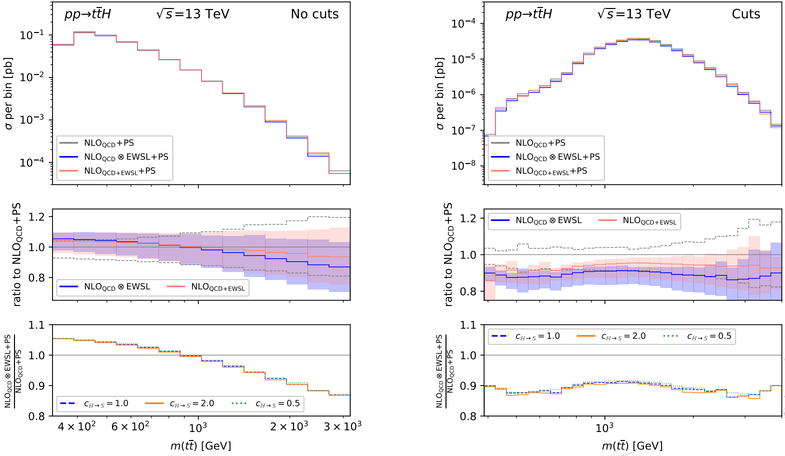
Example results: $t\bar{t}H$



$$pp \to t\bar{t}H$$

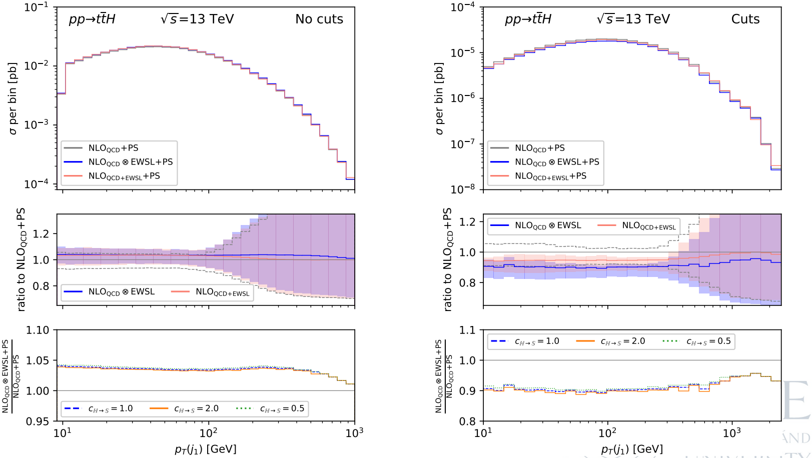
Results for $t\bar{t}H$: $m(t\bar{t})$

- Almost completely independent of $c_{\mathbb{H}\to\mathbb{S}}$
- Scale band differences: EWSL on top of NLO events (blue) and LO events (red)



Results for $t\bar{t}H$: $p_T(j_1)$

- High-p_T range: additive approach converges to NLO_{QCD}+PS



Conclusions and outlook

Summary

✓ Combined EWSL implementation and reweight module in MG5_aMC@NLO for obtaining $\mathrm{NLO}_{\mathrm{QCD}} \otimes \mathrm{EWSL}$+PS precision

Outlook: phenomenological analyses

- Processes of interest: jet-associated processes, $t\bar{t} + X$
- Quantitative comparison to fixed-order NLO EW
- Comparison to data: include hadronization

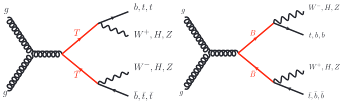
Outlook: merging

- Combine with multi-jet merging in the FxFx formalism

Today's talk

High-multiplicity processes: why bother?

- Hadronic decays of Standard Model background: leads to **multi-jet signatures** $\rightarrow \geq 5$ well-separated final-state jets in the detector
- (Almost) any new physics search includes decays leading to such signatures:



- These are the processes

$$pp \rightarrow \text{gluons} + \text{quarks}\ (+\ \text{colour singlets}) \tag{5}$$

- Colour singlets contribute to phase space but not to colour interaction

High-multiplicity processes in matrix element generators

- Current colour treatment: **colour decomposition** of amplitudes:

$$\langle|\mathcal{M}|^2\rangle \propto \sum_{\sigma_k,\sigma_l} \underbrace{C(\sigma_k,\sigma_l)}_{\text{colour matrix}} \underbrace{\mathcal{A}(\sigma_k)(\mathcal{A}(\sigma_l))^*}_{\text{dual amplitudes}} \tag{6}$$

  where $\sigma_{k,l}$ are some permutations of external particles (each dual amplitude corresponds to a **colour order**)

- Colour matrix size for n final state particles: $\sim n! \times n!$
  $\rightarrow$ We hit a wall for high-multiplicity QCD processes!
  Up to $2 \rightarrow 5(6)$ in MadGraph5 currently
- Bottleneck already at LO

Today's talk

Reducing the colour matrix complexity

- Recall colour decomposition

$$\langle|\mathcal{M}|^2\rangle \propto \sum_{\sigma_k,\sigma_l} C(\sigma_k,\sigma_l)\mathcal{A}(\sigma_k)(\mathcal{A}(\sigma_l))^* \tag{7}$$

- Expand in N_C^{-2} (large-N_C limit)

$$C(\sigma_k,\sigma_l) = \underbrace{a_0 N_C^x}_{\text{Leading colour (LC)}} + \underbrace{a_1 N_C^{x-2}}_{\text{Next-to-leading colour (NLC)}} + \mathcal{O}(N_C^{x-4}) \quad \forall k,l \tag{8}$$

keeping all terms yields full colour (FC)

$$C(\sigma_k,\sigma_l) = \begin{pmatrix} \text{LC} & 0 & 0 & 0 & 0 & \text{NLC} \\ 0 & \text{LC} & 0 & \text{NLC} & 0 & 0 \\ 0 & 0 & \text{LC} & 0 & 0 & 0 \\ 0 & \text{NLC} & 0 & \text{LC} & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{LC} & 0 \\ \text{NLC} & 0 & 0 & 0 & 0 & \text{NLC} \end{pmatrix}$$

- As we will see later: we will propose a NLC truncation[4]

[4] R. Frederix, T. Vitos, arXiv:2109.10377

Proposal for a fully fledged integration in MadGraph5

! In the following: only all-(n)-gluon case !

- The colour-summed and averaged matrix element using the colour decomposition

$$\langle|\mathcal{M}|^2\rangle \propto \sum_{\{c\},\{h\}} \sum_{\sigma_k,\sigma_l} C(\sigma_k,\sigma_l)\mathcal{A}(\sigma_k,\{h\})(\mathcal{A}(\sigma_l,\{h\}))^* \tag{9}$$

- Expansion in N_C^{-2}:

$$\langle|\mathcal{M}|^2\rangle = \frac{N_C^{n-2}(N_C^2-1)}{2^n} \sum_{\sigma_k} \left(\underbrace{|\mathcal{A}(\sigma_k,\{h\})|^2}_{\text{LC}} + \mathcal{O}(N_C^{-2}) \right) \tag{10}$$

- LC-amplitude interferences of MHV-type have peak structure

$$\mathcal{A}(\sigma_k,h)(\mathcal{A}(\sigma_k,h))^* \sim \frac{1}{s_{\sigma_k(1)\sigma_k(2)} s_{\sigma_k(2)\sigma_k(3)} \cdots s_{\sigma_k(n)\sigma_k(1)}} \tag{11}$$

with $s_{\sigma(i)\sigma(j)} = (p_{\sigma(i)} + p_{\sigma(j)})^2$

Proposal for a fully fledged integration in MadGraph5

- Re-write the integrated cross section as:

$$\int \sum_{\{c\}} |\mathcal{M}|^2 \mathrm{d}\Phi_n = \sum_{\sigma_k} \int |\mathcal{A}_{\sigma_k}|^2 \underbrace{\frac{\sum_{\{c\}} \sum_{\sigma_k,\sigma_l} \mathcal{A}_{\sigma_k} C_{\sigma_k \sigma_l} \mathcal{A}^*_{\sigma_l}}{\sum_{\sigma_k} |\mathcal{A}_{\sigma_k}|^2}}_{\text{reweight factor}} \mathrm{d}\Phi_n \tag{12}$$

- Generate FC-accurate events in two steps:
  1. Integration: generate LC events according to known peak structure
  2. Reweighting: reweight LC events to FC with reweight factor

Today's talk

Integration

Phase-space integration

- Use an integration technique based on this LC-interference peak structure:

$$\mathrm{d}\Phi_n = \prod_{i=1}^{n} \frac{\mathrm{d}^3 p}{(2\pi)^3 2E} \tag{13}$$

various options for variable choice in integration:

HAAG (hierarchical antenna generation)[5],
$2 \to 3$-integration[6]

- Combination with VEGAS optimization

[5] A. van Hameren, C.G. Papadopoulos arXiv:hep-ph/0204055
[6] E. Byckling, K. Kajantie 10.1103/PhysRev.187.2008

Reweighting

Reweighting the LC events to FC

$$\sum_{\sigma_k} \int |\mathcal{A}_{\sigma_k}|^2 \underbrace{\frac{\sum_{\{c\}} \sum_{\sigma_k,\sigma_l} \mathcal{A}_{\sigma_k} C_{\sigma_k \sigma_l} \mathcal{A}^*_{\sigma_l}}{\sum_{\sigma_k} |\mathcal{A}_{\sigma_k}|^2}}_{\text{reweight factor}} \mathrm{d}\Phi_n$$

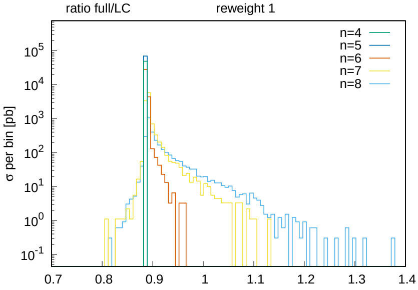
- Obtaining NLC- and FC-accurate predictions from the LC events can be done in several ways:
  1. Perform FC sum (sum over colour configurations)
  2. Pick subset of permutations (using phase space symmetry)
  3. a) Pick specific colour configuration randomly (consistent with colour ordering) or b) use an average of all colour configurations
- Reweight time for FC reweighting of (10^4) LC events:

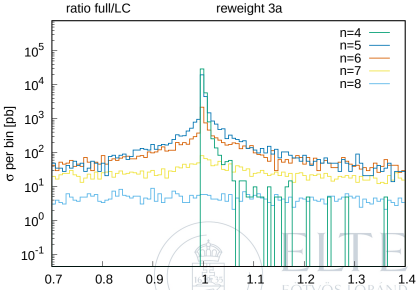
n	**reweight 1**	**reweight 3a**
4	0.4	0.4
5	0.6	0.6
6	1.7	1.0
7	24	4.3
8	1040	273

Reweighting

Reweighting the LC events to FC

- Two main aspects:
  how fast is the reweighting?
  how spread are the new weights?
- Secondary unweighting efficiency of a LC sample to FC reweighting revealed by the FC/LC ratio plot





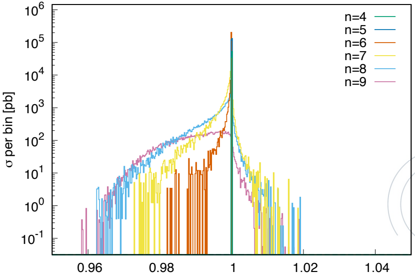
Reweighting

Reweighting the LC events to NLC

- Reweight with reweight approach 1 using the NLC-accurate colour sum of the reweight factor

$$\frac{\sum_{\{c\}}\sum_{\sigma_k,\sigma_l}\mathcal{A}_{\sigma_k}C_{\sigma_k\sigma_l}\mathcal{A}^*_{\sigma_l}}{\sum_{\sigma_k}|\mathcal{A}_{\sigma_k}|^2} = \frac{1}{\sum_{\sigma_k}|\mathcal{A}_{\sigma_k}|^2}\left(\sum_{\{c\}}\sum_{\sigma_k}\mathcal{A}_{\sigma_k}\times\left[\sum_{w}C_w\mathcal{A}^*_w\right]\right) \tag{14}$$

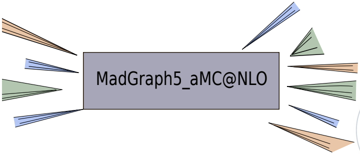
  for a subset of permutations w that have C_w of NLC accuracy (N_C^{n-2})
- Choice of colour basis might play a role:
  → colour-flow produces monomial colour factors (N_C^x)
- Ratio full/NLC:



Summary and outlook

NLC approximation in ME generators

- Finalize implementation of gluon+quarks+singlets and interface to MG5
  → In the coming ∼half year
- Extend to NLO computations (colour treatment in loops)
  → Project with Zoltán Trócsányi
- Combining with beyond-LC parton shower evolution
  → Project with Simon Plätzer
- Extend beyond NLC in colour expansion? NNLO colour treatment?
  → Future plans



Thank you for listening!

Appendix

Adding parton showers

NLO QCD: $\mathcal{O}(\alpha_S) \xrightarrow{\text{QCD PS}} \mathcal{O}(\alpha_S^n)$ $\quad n > 1$

$\rightarrow$ matching needed!

NLO QCD+EWSL: $\mathcal{O}(\alpha_S \alpha) \xrightarrow{\text{QCD PS}} \mathcal{O}(\alpha_S^n \alpha)$ $\quad n > 1$

$\rightarrow$ no additional matching needed!

NLO QCD+EWSL: $\mathcal{O}(\alpha_S \alpha) \xrightarrow{\text{QCD PS+QED PS}} \mathcal{O}(\alpha_S^n \alpha_{\text{(QED)}}^m)$ $\quad n > 1, m > 1$

$\rightarrow$ matching needed!

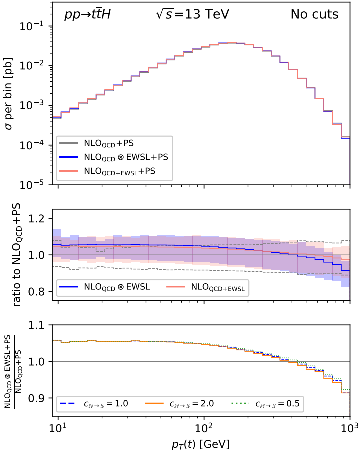
Turn off QED in the Sudakov! (SDK_{weak})

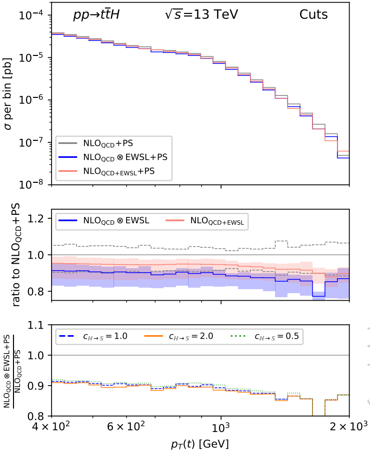
- NLO QCD+EWSL:

$$\mathcal{O}(\alpha_S \alpha_{\text{(weak)}}) \xrightarrow{\text{QCD PS+QED PS}} \mathcal{O}(\alpha_S^n \alpha_{\text{(weak)}} \alpha_{\text{(QED)}}^m), \quad n > 1, m > 0$$

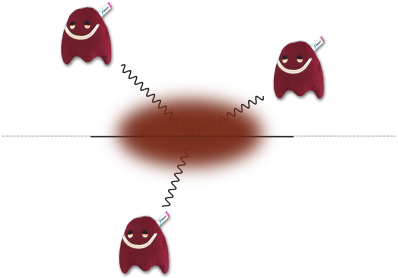
Results for $t\bar{t}H$: $p_T(t)$

- Similar positive corrections in low-p_T range
- Stable agreement to FO NLO EW
- Again, small difference between multiplicative and additive approaches



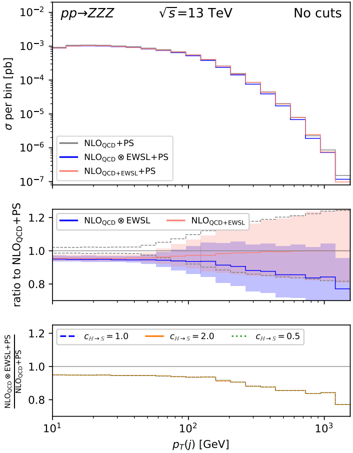


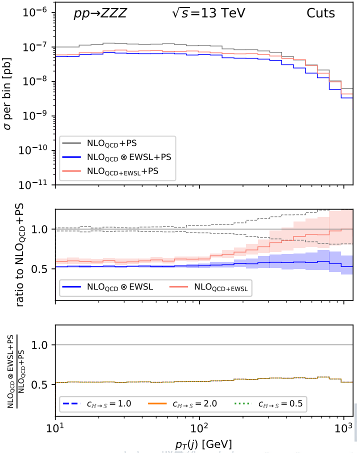
Example results: ZZZ



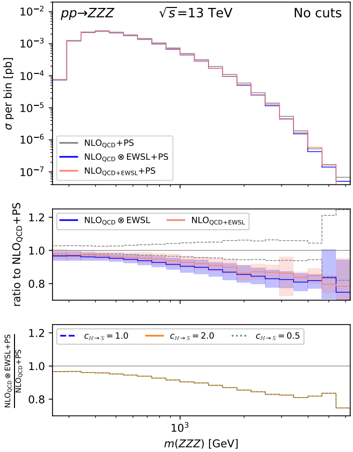
$$pp \rightarrow ZZZ$$

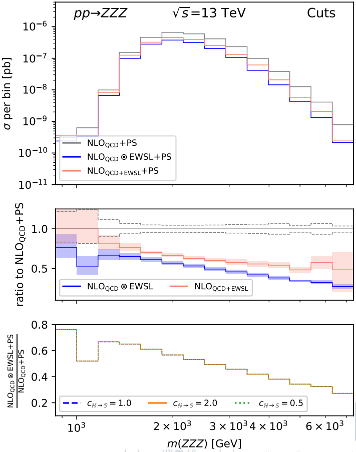
Results for ZZZ: $p_T(j_1)$





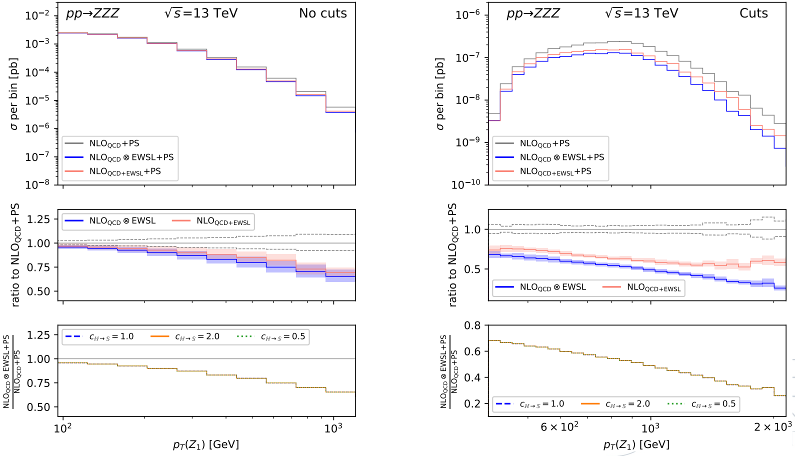
Results for ZZZ: $m(Z_1, Z_2, Z_3)$





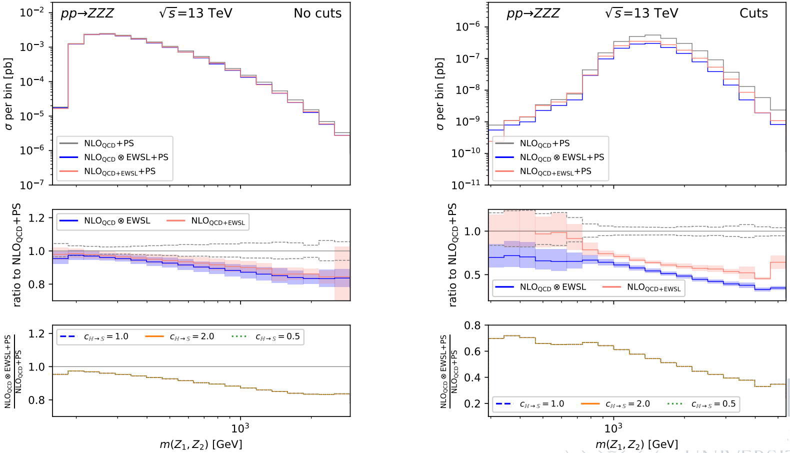
Results for ZZZ: $p_T(Z_1)$

- Smaller scale uncertainty bands: no LO $\sim \alpha_S$
- Larger EWSL effects + larger QCD K-factor

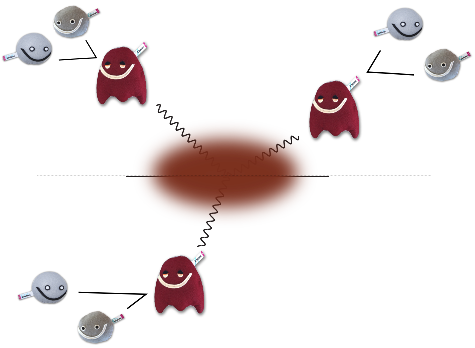


Results for ZZZ: $m(Z_1, Z_2)$

- At $\lesssim 700$ GeV, dominated by hard events $\rightarrow$ red converges to grey



Example results: ZZZ with decays



$$pp \to ZZZ \to e^+e^-e^+e^-e^+e^-$$

Example results: ZZZ with decays

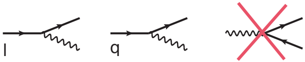
- First perform EWSL reweighting on ZZZ sample, then **decay with `MadSpin`**
- Lepton classification with jet algorithm (accepted event if 6 charged jets found):

$$p_T(\text{lepton}) > 25 \text{ GeV} \tag{15}$$

- To catch correct BW shapes: **label positrons** e_i^+ such that they **minimize**

$$\sum_i |m(e_i^- e_i^+) - M_Z|^2 \tag{16}$$

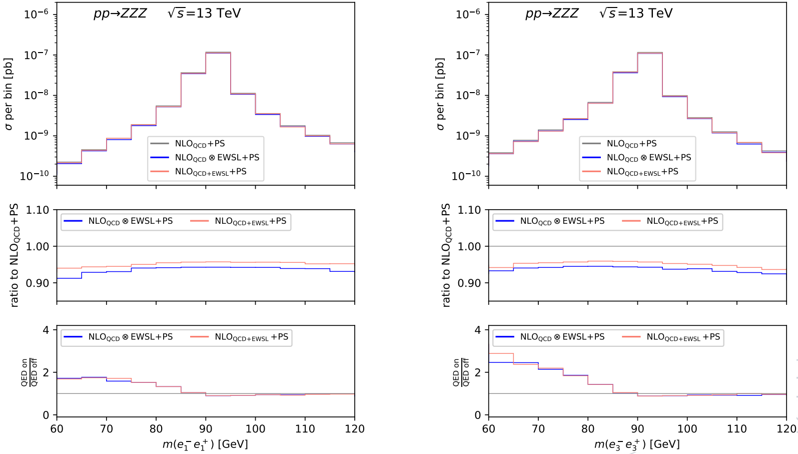
- **Final-state QED radiation**: investigate its effect by turning it off/on, including only photon radiation:



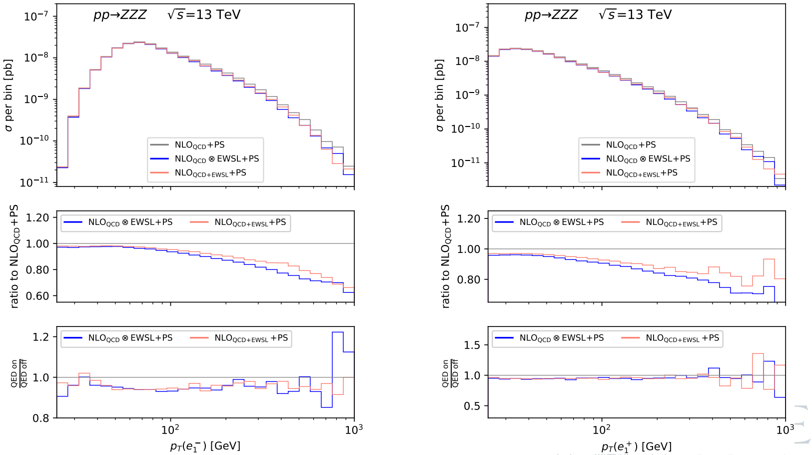
(17)

Results for ZZZ with decays: $m(e_i^- e_i^+)$

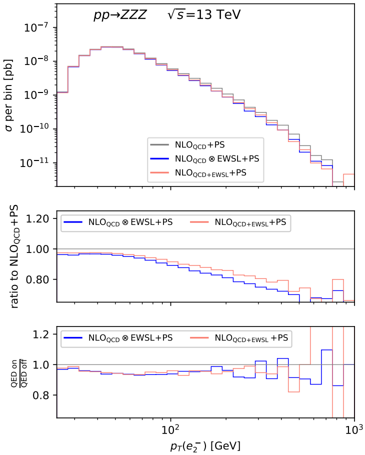
- EWSL: red and blue $\sim -5\%$
- Assess QED radiation effects: around peak region only!

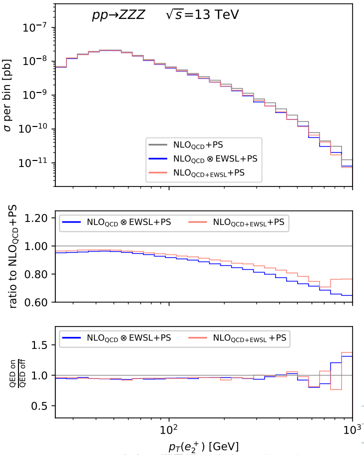


Results for ZZZ with decays: $p_T(e_1)$

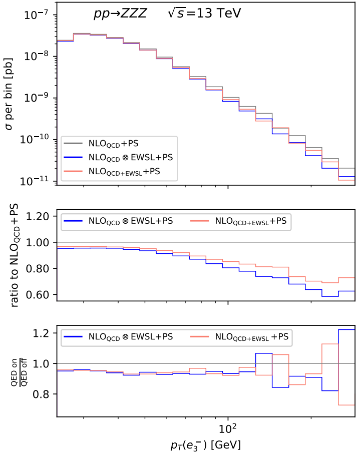


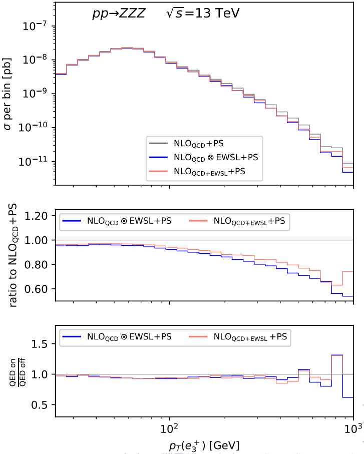
Results for ZZZ with decays: $p_T(e_2)$





Results for ZZZ with decays: $p_T(e_3)$





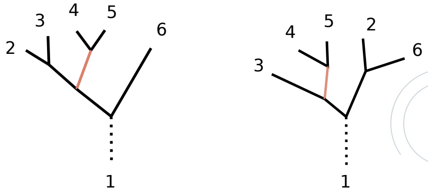
Integration

Amplitude evaluation

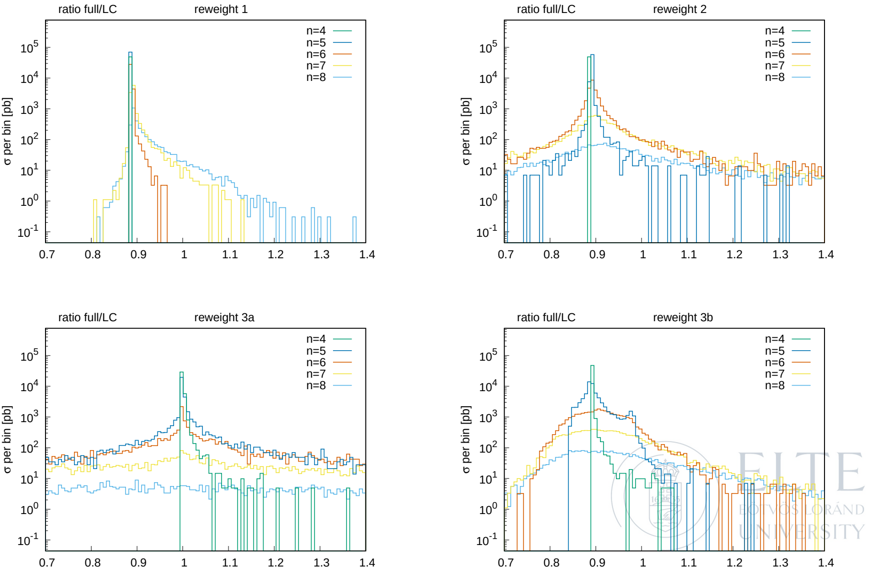
- Each of the $\mathcal{A}_{\sigma_k}$ computed in a recursive fashion
  $\rightarrow$ recycle parts of the amplitudes
- For a fixed colour-ordering (example for $n = 6$):

$$\begin{array}{c} \{1,2\} \quad \{2,3\} \quad \{3,4\} \quad \{4,5\} \\ \{(1,2),(2,3)\} \quad \{(2,3),(3,4)\} \quad \{(3,4),(4,5)\} \\ \{((1,2),(2,3)),((2,3),(3,4))\} \quad \{((2,3),(3,4)),((3,4),(4,5))\} \\ \{(((1,2),(2,3)),((2,3),(3,4))),(((2,3),(3,4)),((3,4),(4,5)))\} \end{array} \tag{18}$$

- Example: same current (4,5) appears in the two colour-orderings $\sigma = (1,2,3,4,5,6)$ and $\sigma = (1,3,4,5,2,6)$



Reweighting the LC events to FC



Colour matrix: useful identities

- Recall: Fierz identity

$$(T^a)_{ij}(T^a)_{kl} = T_R\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right) \tag{19}$$

set group index $T_R = 1$

Notation

Use $\mathcal{R},\mathcal{Q},\mathcal{S},\mathcal{P}...$ to denote strings of fundamental generators

$$\mathcal{R} = T^{a_1}T^{a_2}...T^{a_r} \quad , \quad \tilde{\mathcal{R}} = T^{a_r}T^{a_{r-1}}...T^{a_1} \quad , \quad \mathrm{len}(\mathcal{R}) = r \tag{20}$$

- Variations of the Fierz identity:

$$\text{Rule I:} \qquad \mathrm{Tr}[T^a\mathcal{R}]\mathrm{Tr}[T^a\mathcal{S}] = \mathrm{Tr}[\mathcal{R}\mathcal{S}] - \frac{1}{N_c}\mathrm{Tr}[\mathcal{R}]\mathrm{Tr}[\mathcal{S}], \tag{21}$$

$$\text{Rule II:} \qquad \mathrm{Tr}[\mathcal{R}T^a\mathcal{Q}T^a\mathcal{S}] = \mathrm{Tr}[\mathcal{Q}]\mathrm{Tr}[\mathcal{R}\mathcal{S}] - \frac{1}{N_c}\mathrm{Tr}[\mathcal{R}\mathcal{Q}\mathcal{S}], \tag{22}$$

$$\text{Rule IIb:} \qquad \mathrm{Tr}[\mathcal{R}T^aT^a\mathcal{S}] = N_c\mathrm{Tr}[\mathcal{R}\mathcal{S}] + \mathcal{O}(1/N_c) \tag{23}$$

Colour matrix: fundamental decomposition

For n-gluon amplitudes

- Matrix-element

$$\mathcal{M} = g^{n-2} \sum_{\sigma \in S_{n-1}} \mathrm{Tr}[T^{a_1} T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n-1)}}] \mathcal{A}(1, \sigma(1), ..., \sigma(n-1)) \tag{24}$$

Not a minimal set

Dual amplitudes are related by the Kleiss-Kuijf relation (dual Ward identity)

$$\mathcal{A}(1,2,3,4....,n) + \mathcal{A}(2,1,3,4,...,n) + ... + \mathcal{A}(2,3,4,...,1,n) = 0 \tag{25}$$

- Squared matrix-element

$$|\mathcal{M}|^2 = (g^2)^{n-2} \sum_{k,l=1}^{(n-1)!} C_{kl} \mathcal{A}(1, \sigma_k(1), \ldots, \sigma_k(n-1)) (\mathcal{A}(1, \sigma_l(1), \ldots, \sigma_l(n-1)))^*$$

- Colour matrix (size $(n-1)! \times (n-1)!$):

$$C_{kl} = \sum_{\text{col.}} \mathrm{Tr}[T^{a_1} T^{a_{\sigma_k(1)}} \ldots T^{a_{\sigma_k(n-1)}}] \left(\mathrm{Tr}[T^{a_1} T^{a_{\sigma_l(1)}} \ldots T^{a_{\sigma_l(n-1)}}] \right)^* \tag{26}$$

Colour matrix: fundamental decomposition

For n-gluon amplitudes

- Colour matrix

$$C_{kl} = \sum_{\text{col.}} \mathrm{Tr}[T^{a_1} T^{a_{\sigma_k(1)}} \dots T^{a_{\sigma_k(n-1)}}] \left(\mathrm{Tr}[T^{a_1} T^{a_{\sigma_l(1)}} \dots T^{a_{\sigma_l(n-1)}}]\right)^* \tag{27}$$

- LC: $\mathcal{O}(N_C^n)$, NLC: $\mathcal{O}(N_C^{n-2})$
- Leading-colour in all diagonal elements $\sigma_k = \sigma_l$

$$\left(N_C - \frac{1}{N_C}\right)^n + (N_C^2 - 1)\left(\frac{-1}{N_C}\right)^n = N_C^n + \mathcal{O}(N_C^{n-2}) \tag{28}$$

- Diagonal colour factors contain also NLC contribution

Colour matrix: fundamental decomposition

For n-gluon amplitudes

- NLC: for permutations which are related by a block interchange:[7]

$$\sigma_k \sim \mathcal{R}\mathcal{Q}_1\mathcal{S}\mathcal{Q}_2\mathcal{P} \quad , \quad \sigma_l \sim \mathcal{R}\mathcal{Q}_2\mathcal{S}\mathcal{Q}_1\mathcal{P} \tag{29}$$

with special cases if $\mathcal{S} = \not\Vdash$

- If $\mathrm{len}(\mathcal{Q}_1) = \mathrm{len}(\mathcal{Q}_2) = 1$: $-N_C^{n-2} + \mathcal{O}(N_C^{n-4})$.
- If $\mathrm{len}(\mathcal{Q}_1) = 1$ *or* $\mathrm{len}(\mathcal{Q}_2) = 1$: $-N_C^{n-2} + \mathcal{O}(N_C^{n-4})$ if $\mathrm{len}(\mathcal{R}) = 1$ and $\mathrm{len}(\mathcal{P}) = 0$, otherwise not NLC
- If $\mathrm{len}(\mathcal{Q}_{1,2}) > 1$: $+N_C^{n-2} + \mathcal{O}(N_C^{n-4})$ if $\mathrm{len}(\mathcal{R}) \neq 1$ and $\mathrm{len}(\mathcal{P}) \neq 0$, otherwise not NLC

[7] A. Labane, aXiv:2008.13640

Colour matrix: results

For *n*-gluon amplitudes

Including the adjoint decomposition: matrix size $(n-2)! \times (n-2)!$

all-gluon			
n	Fundamental	Colour-flow	Adjoint
4	6 (6)	6 (6)	2 (2)
5	11 (24)	16 (24)	5 (6)
6	24 (120)	36 (120)	18 (24)
7	50 (720)	71 (720)	93 (120)
8	95 (5040)	127 (5040)	583 (720)
9	166 (40320)	211 (40320)	4162 (5040)
10	271 (362880)	331 (362880)	31649 (40320)
11	419 (3628800)	496 (3628800)	-
12	620 (39916800)	716 (39916800)	-
13	885 (479001600)	1002 (479001600)	-
14	1226 (6227020800)	1366 (6227020800)	-

Adjoint decomposition

For n-gluon amplitudes

- The amplitude is now

$$\mathcal{M} = \sum_{\sigma \in S_{n-2}} \left(F^{a_{\sigma(2)}} \ldots F^{a_{\sigma(n-1)}} \right)_{a_1 a_n} \mathcal{A}(1, \sigma(1), \ldots, \sigma(n), n), \tag{30}$$

  with $(F^a)_{bc} = i f^{abc}$
- Minimal basis: $(n-2)!$ independent dual amplitudes
- Smaller colour matrix: but **LC not only on diagonal!**
- No found algorithm (yet) to get NLC elements

Colour-flow decompositions

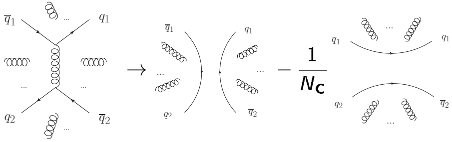
For one quark line plus n-gluon amplitudes: the full projection of U(1) gluons

$$\mathcal{M}_{1qq} = \sum_{\sigma\in S_n} \delta^{i_q}_{j_{\sigma(1)}} \delta^{i_{\sigma(1)}}_{j_{\sigma(2)}} \dots \delta^{i_{\sigma(n-1)}}_{j_{\sigma(n)}} \delta^{i_{\sigma(n)}}_{j_q} \mathcal{A}_{1qq}(q,\sigma(1),\dots,\sigma(n),\overline{q})$$
$$+ \left(\frac{-1}{N}\right) \sum_{\sigma\in S_n} \delta^{i_q}_{j_{\sigma(1)}} \delta^{i_{\sigma(1)}}_{j_{\sigma(2)}} \dots \delta^{i_{\sigma(n-1)}}_{j_q} \delta^{i_{\sigma(n)}}_{j_{\sigma(n)}} \mathcal{A}_{1qq}(q,\sigma(1),\dots,\sigma(n-1),\overline{q},\sigma(n))$$
$$+ \left(\frac{-1}{N}\right)^2 \frac{1}{2!} \sum_{\sigma\in S_n} \delta^{i_q}_{j_{\sigma(1)}} \delta^{i_{\sigma(1)}}_{j_{\sigma(2)}} \dots \delta^{i_{\sigma(n-2)}}_{j_q} \delta^{i_{\sigma(n-1)}}_{j_{\sigma(n-1)}} \delta^{i_{\sigma(n)}}_{j_{\sigma(n)}}$$
$$\mathcal{A}_{1qq}(q,\sigma(1),\dots,\sigma(n-2),\overline{q},\sigma(n-1),\sigma(n))$$
$$+ \dots$$
$$+ \left(\frac{-1}{N}\right)^n \frac{1}{n!} \sum_{\sigma\in S_n} \delta^{i_q}_{j_q} \delta^{i_{\sigma(1)}}_{j_{\sigma(1)}} \dots \delta^{i_{\sigma(n-1)}}_{j_{\sigma(n-1)}} \delta^{i_{\sigma(n)}}_{j_{\sigma(n)}} \mathcal{A}_{1qq}(q,\overline{q},\sigma(1),\dots,\sigma(n)).$$

Fundamental decompositions

For two distinct flavour quark pairs plus n-gluon amplitudes

- Now we have *two single colour lines* $\to$ internal U(1) gluon
- The internal gluon is decomposed into U(N_C) and U(1) part



(31)

- The two "quark-ordered" amplitudes

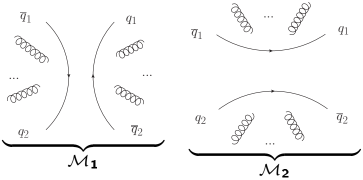
$$\mathcal{M}_{2qq} = \mathcal{M}_1 - \frac{1}{N_C}\mathcal{M}_2 \qquad (32)$$

- Decomposed as

$$\mathcal{M}_1 = \sum_{\sigma\in S_n}\sum_{n_1=0}^{n} c_1(\sigma,n_1)\mathcal{A}_1(\sigma,n_1) \quad , \quad \mathcal{M}_2 = \sum_{\sigma\in S_n}\sum_{n_1=0}^{n} c_2(\sigma,n_1)\mathcal{A}_2(\sigma,n_1)$$

Fundamental decompositions

For two **distinct flavour** quark pairs plus n-gluon amplitudes



(33)

- The colour factors

$$c_1(\sigma) = \left(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n_1)}}\right)_{i_1 j_2} \left(T^{a_{\sigma(n_1+1)}} \dots T^{a_{\sigma(n)}}\right)_{i_2 j_1} \tag{34}$$

$$c_2(\sigma) = \left(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n_1)}}\right)_{i_1 j_1} \left(T^{a_{\sigma(n_1+1)}} \dots T^{a_{\sigma(n)}}\right)_{i_2 j_2} \tag{35}$$

- The squared amplitude

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2} \sum_{\sigma_k, \sigma_l \in S_{n+1}} \begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) \end{pmatrix} \begin{pmatrix} c_1(\sigma_k) c_1(\sigma_l)^* & -c_1(\sigma_k) c_2(\sigma_l)^*/N_C \\ -c_2(\sigma_k) c_1(\sigma_l)^*/N_C & c_2(\sigma_k) c_2(\sigma_l)^*/N_C^2 \end{pmatrix} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \end{pmatrix} \tag{36}$$

Fundamental decompositions

For two **distinct flavour** quark pairs plus n-gluon amplitudes

- Note: **not all diagonal elements the same type now!**
- Leading-colour $\mathcal{O}(N_C^{n+2})$:

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2} \sum_{\sigma_k, \sigma_l \in S_{n+1}}$$

$$\begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) \end{pmatrix} \begin{pmatrix} c_1(\sigma_k) c_1(\sigma_l)^* & -c_1(\sigma_k) c_2(\sigma_l)^*/N_C \\ -c_2(\sigma_k) c_1(\sigma_l)^*/N_C & c_2(\sigma_k) c_2(\sigma_l)^*/N_C^2 \end{pmatrix} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \end{pmatrix}$$

if $\sigma_k = \sigma_l$

- NLC terms $\mathcal{O}(N_C^n)$, investigate block-by-block: appears in each block

Fundamental decompositions

For two same flavour quark pairs plus n-gluon amplitudes

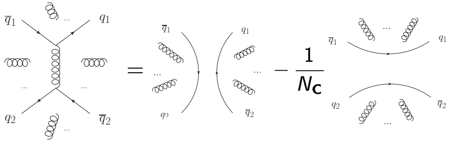
- Both a t- and s-channel contribution

$$\mathcal{M}_{2qq}(\overline{q}q\overline{q}q + ng) = \hat{\mathcal{M}}(\overline{q}_1 q_1 \overline{q}_2 q_2 + ng) - \hat{\mathcal{M}}(\overline{q}_1 q_2 \overline{q}_2 q_1 + ng) \quad (37)$$

(minus sign from Fermi statistics)

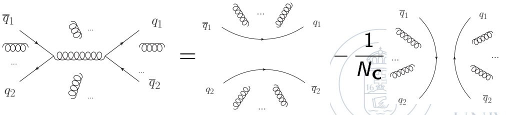
- Decomposed

$$\hat{\mathcal{M}}(\overline{q}_1 q_1 \overline{q}_2 q_2 + ng) =$$



(38)

$$\hat{\mathcal{M}}(\overline{q}_1 q_2 \overline{q}_2 q_1 + ng) =$$



Fundamental decompositions

For two same flavour quark pairs plus n-gluon amplitudes

- So then

$$\mathcal{M}_{2qq} = \left(1 + \frac{1}{N_C}\right)(\mathcal{M}_1 - \mathcal{M}_2). \tag{39}$$

- Squared-matrix:

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2}\left(1 + \frac{1}{N_C}\right)^2 \sum_{\sigma_k,\sigma_l \in S_{n+1}} \begin{pmatrix}\mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k)\end{pmatrix} \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^* & c_1(\sigma_k)c_2(\sigma_l)^* \\ c_2(\sigma_k)c_1(\sigma_l)^* & c_2(\sigma_k)c_2(\sigma_l)^* \end{pmatrix} \begin{pmatrix}\mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^*\end{pmatrix} \tag{40}$$

- Colour factors include an extra factor $\left(1 + \frac{1}{N_C}\right)^2$ here
  $\rightarrow$ LC: $\mathcal{O}(N_C^{n+2})$, non-zero $\mathcal{O}(N_C^{n+1})$

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Fundamental decompositions

For two same flavour quark pairs plus n-gluon amplitudes

- Note: **diagonal elements symmetrized now!**
- Leading-colour $\mathcal{O}(N_c^{n+2})$:

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2}\left(1+\frac{1}{N_c}\right)^2 \sum_{\sigma_k,\sigma_l\in S_{n+1}} \begin{pmatrix}\mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k)\end{pmatrix}\begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^* & c_1(\sigma_k)c_2(\sigma_l)^* \\ c_2(\sigma_k)c_1(\sigma_l)^* & c_2(\sigma_k)c_2(\sigma_l)^*\end{pmatrix}\begin{pmatrix}\mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^*\end{pmatrix} \quad (41)$$

  if $\sigma_k = \sigma_l$
- NLC terms $\mathcal{O}(N_c^{n+1}) + \mathcal{O}(N_c^{n})$, investigate block-by-block: appears in every block

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Colour decompositions

For two distinct flavour quark pairs plus n-gluon amplitudes

- Same set of dual amplitudes as for fundamental decomposition

$$\mathcal{M}_{2qq} = \mathcal{M}_1 - \frac{1}{N_C}\mathcal{M}_2 \tag{42}$$

- Once again, external gluons are projected out

$$\mathcal{M}_1 \to \mathcal{M}_1 - \frac{1}{N_C}\sum_{\bar{\sigma}\in\bar{S}_{n+1}} c_1^1(\bar{\sigma})\mathcal{A}_1^1(\bar{\sigma}), \tag{43}$$

- For NLC, it turns out that a single U(1) projection is enough
- Colour factor for this dual amplitude

$$c_1^1(\bar{\sigma}) = \delta_{j_{\sigma(1)}}^{i_{q_1}} \dots \delta_{j_{q_1}}^{i_{\sigma(n)}} \delta_{j_{\sigma(n+1)}}^{i_{\sigma(n+1)}}, \tag{44}$$

with colourless external U(1) indices

Colour decompositions

For two distinct flavour quark line plus n-gluon amplitudes

- Matrix element

$$\mathcal{M}_{2qq} = \sum_{\sigma\in S_{n+1}} c_1(\sigma)\mathcal{A}_1(\sigma) - \frac{1}{N_C}\sum_{\sigma\in S_{n+1}} c_2(\sigma)\mathcal{A}_2(\sigma) - \frac{1}{N_C}\sum_{\bar\sigma\in \bar S_{n+1}} c_1^1(\bar\sigma)\mathcal{A}_1^1(\bar\sigma),$$

- Squared matrix-element

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n-2}\sum_{\sigma_k,\sigma_l}\begin{pmatrix}\mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) & \mathcal{A}_1^1(\bar\sigma_k)\end{pmatrix}$$

$$\begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^\dagger & -c_1(\sigma_k)c_2(\sigma_l)^\dagger/N_C & -c_1(\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_C \\ -c_2(\sigma_k)c_1(\sigma_l)^\dagger/N_C & c_2(\sigma_k)c_2(\sigma_l)^\dagger/N_C^2 & c_2(\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_C^2 \\ -c_1^1(\bar\sigma_k)c_1(\sigma_l)^\dagger/N_C & c_1^1(\bar\sigma_k)c_2(\sigma_l)^\dagger/N_C^2 & c_1^1(\bar\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_C^2 \end{pmatrix}\begin{pmatrix}\mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \\ \mathcal{A}_1^1(\bar\sigma_l)^*\end{pmatrix},$$

- Leading-colour (N_C^{n+2}) for $\sigma_k = \sigma_l$
- NLC (N_C^n) needs a careful analysis block-by-block

Colour decompositions

For two same flavour quark line plus n-gluon amplitudes

- Very similar to the distinct flavour case, but we also need to U(1) project the $\mathcal{M}_2$ amplitude

$$\mathcal{M}_2 \to \mathcal{M}_2 - \frac{1}{N_C} \sum_{\bar{\sigma} \in \bar{S}_{n+1}} c_2^1(\bar{\sigma}) \mathcal{A}_2^1(\bar{\sigma}), \tag{45}$$

- 2: Squared matrix-element

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n-2} \left(1 + \frac{1}{N_C}\right)^2 \sum_{\sigma_k, \sigma_l} \left(\mathcal{A}_1(\sigma_k) \quad \mathcal{A}_2(\sigma_k) \quad \mathcal{A}_1^1(\bar{\sigma}_k) \quad \mathcal{A}_2^1(\bar{\sigma}_k)\right) \mathbb{C} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \\ \mathcal{A}_1^1(\bar{\sigma}_l)^* \\ \mathcal{A}_2^1(\bar{\sigma}_l)^* \end{pmatrix} \tag{46}$$

(Again, colourr factors no longer monomials in N_C)

Colour decompositions

For two same flavour quark line plus n-gluon amplitudes

- Colour matrix

$$\mathbb{C} = \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^\dagger & -c_1(\sigma_k)c_2(\sigma_l)^\dagger & -c_1(\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_C & c_1(\sigma_k)c_2^1(\bar\sigma_l)^\dagger/N_C \\ -c_2(\sigma_k)c_1(\sigma_l)^\dagger & c_2(\sigma_k)c_2(\sigma_l)^\dagger & -c_2(\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_C & -c_2(\sigma_k)c_2^1(\bar\sigma_l)^\dagger/N_C \\ -c_1^1(\bar\sigma_k)c_1(\sigma_l)^\dagger/N_C & c_1^1(\bar\sigma_k)c_2(\sigma_l)^\dagger/N_C & c_1^1(\bar\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_C^2 & -c_1^1(\bar\sigma_k)c_2^1(\bar\sigma_l)^\dagger/N_C^2 \\ c_2^1(\bar\sigma_k)c_1(\sigma_l)^\dagger/N_C & -c_2^1(\bar\sigma_k)c_2(\sigma_l)^\dagger/N_C & -c_2^1(\bar\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_C^2 & c_2^1(\bar\sigma_k)c_2^1(\bar\sigma_l)^\dagger/N_C^2 \end{pmatrix}. \quad (47)$$

- Leading-colour: $\mathcal{O}(N_C^{n+1})$ on first two block diagonal elements
- NLC $\mathcal{O}(N_C^n)$ is examined block-by-block

Non-zero elements without phase-space symmetrisation

$q\bar{q}\,Q\overline{Q} + n\,g$									Fundamental:		$\mathcal{A}_1$ | $\mathcal{A}_2$ types		
	min(n_1, $n - n_1$)												
n	0 ($\mathcal{A}_1$)	0 ($\mathcal{A}_2$)	1 ($\mathcal{A}_1$)	1 ($\mathcal{A}_2$)	2 ($\mathcal{A}_1$)	2 ($\mathcal{A}_2$)	3 ($\mathcal{A}_1$)	3 ($\mathcal{A}_2$)	4 ($\mathcal{A}_1$)	4 ($\mathcal{A}_2$)	5 ($\mathcal{A}_1$)	5 ($\mathcal{A}_2$)	
0	2	2											(2)
1	3	3											(4)
2	7	4	6	5									(12)
3	15	5	15	7									(48)
4	31	6	32	9	33	10							(240)
5	60	7	62	11	64	13							(1440)
6	108	8	111	13	114	16	115	17					(10080)
7	182	9	186	15	190	19	192	21					(80640)
8	290	10	295	17	300	22	303	25	304	26			(725760)
9	441	11	447	19	453	25	457	29	459	31			(7257600)
10	645	12	652	21	659	28	664	33	667	36	668	37	(79833600)

Table: Number of non-zero elements in a single row of the colour matrix for $q\bar{q}\,Q\overline{Q} + n\,g$ (distinct flavours) up to NLC accuracy, $\mathcal{O}(N_c^n)$ in the fundamental representation.

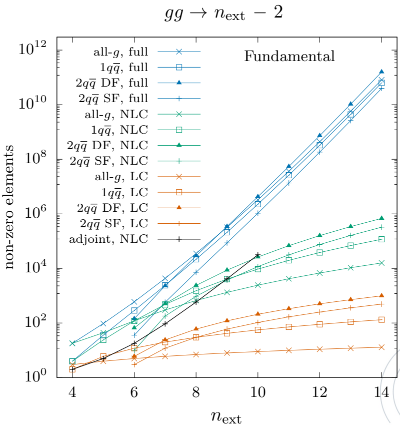
Non-zero elements without phase-space symmetrisation

$q\bar{q}\,Q\bar{Q} + ng$	Colour-flow: $\mathcal{A}_1$, $\mathcal{A}_1^1$ | $\mathcal{A}_2$ types						
	min(n_1, $n - n_1$)						
n	0	1	2	3	4	5	
0	2 , - | 2						(2)
1	5 , 3 | 3						(6)
2	11 , 4 | 4	12 , - | 5					(22)
3	23 , 5 | 5	25 , 5 | 7					(98)
4	45 , 6 | 6	48 , 6 | 9	49 , - | 10				(522)
5	82 , 7 | 7	86 , 7 | 11	88 , 7 | 13				(3262)
6	140 , 8 | 8	145 , 8 | 13	148 , 8 | 16	149 , - | 17			(23486)
7	226 , 9 | 9	232 , 9 | 15	236 , 9 | 19	238 , 9 | 21			(191802)
8	348 , 10 | 10	355 , 10 | 17	360 , 10 | 22	363 , 10 | 25	364 , - | 25		(1753618)
9	515 , 11 | 11	523 , 11 | 19	529 , 11 | 25	533 , 11 | 29	535 , 11 | 31		(17755382)
10	737 , 12 | 12	746 , 12 | 21	753 , 12 | 28	758 , 12 | 33	761 , 12 | 36	762 , - | 37	(197282022)

Table: Number of non-zero elements in a single row of the colour matrix for $q\bar{q}\,Q\bar{Q} + n\,g$ (distinct flavours) up to NLC accuracy in the colour-flow representation

Results for gg initiated processes[8]

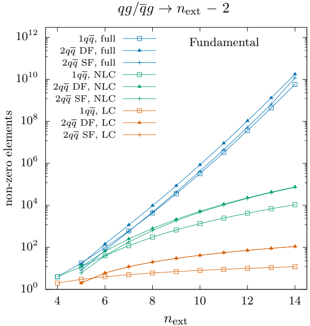
- For external particles $n_{\text{ext}} \in [4, 14]$
- Blue: full colour   Green: NLC   Red: LC

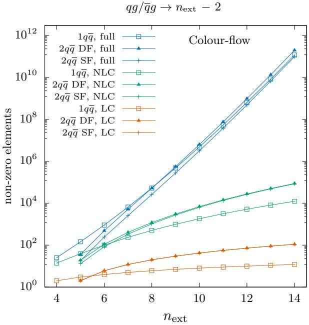


[8] R. Frederix, T. Vitos arXiv:2109.10377

Colour matrix: results for qg initiated

- Blue: full colour Green: NLC Red: LC

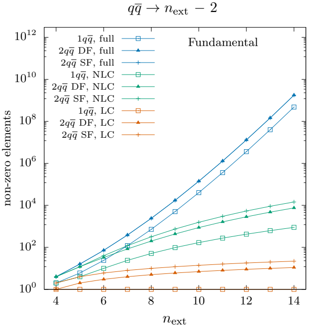


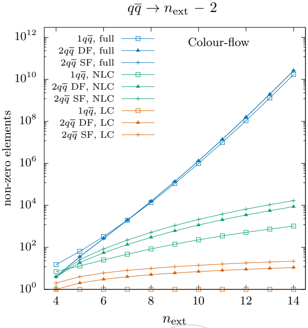


- Factorial growth for full-colour
- Polynomial scaling with n_{ext} for both LC and NLC ($\sim n_{\text{ext}}^4$)

Colour matrix: results for $q\bar{q}$ initiated

- Blue: full colour   Green: NLC   Red: LC

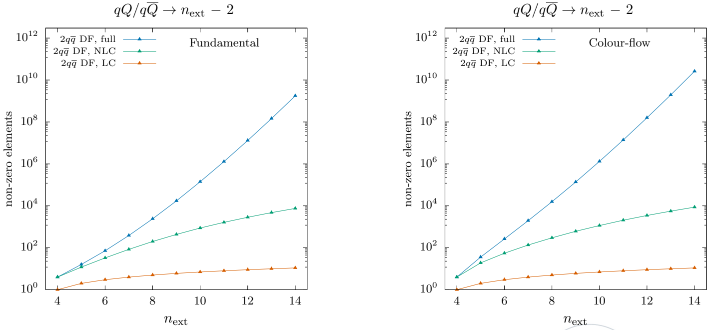




- Colour-flow very slightly less efficient than fundamental decomposition

Colour matrix: results for $qQ/q\overline{Q}$ initiated

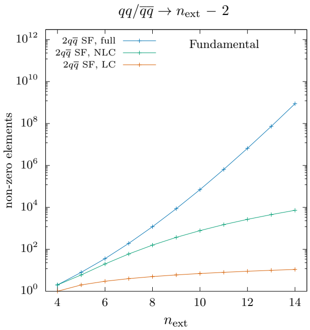
- Blue: full colour Green: NLC Red: LC

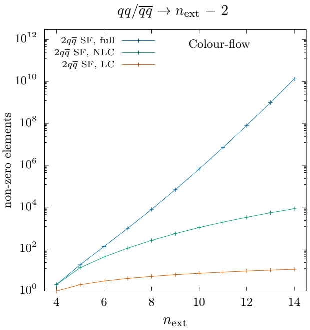


- Already a good efficiency improvement for NLC at $n_{\text{ext}} \sim 6$

Colour matrix: results for $qq/\overline{qq}$ initiated

- Blue: full colour   Green: NLC   Red: LC





- Same-flavour case has slightly more elements to consider because of symmetrisation of $\mathcal{M}_{1,2}$ amplitudes