Improving on accuracy and efficiency for Standard Model theory predictions

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Why do Standard Model precision?

Unsolved problems in the successsful theory

- Baryon-antibaryon asymmetry $\rightarrow$ **Beyond Standard Model**
- Neutrino masses $\rightarrow$ **Beyond Standard Model**
- Dark matter $\rightarrow$ **Beyond Standard Model**
- ...

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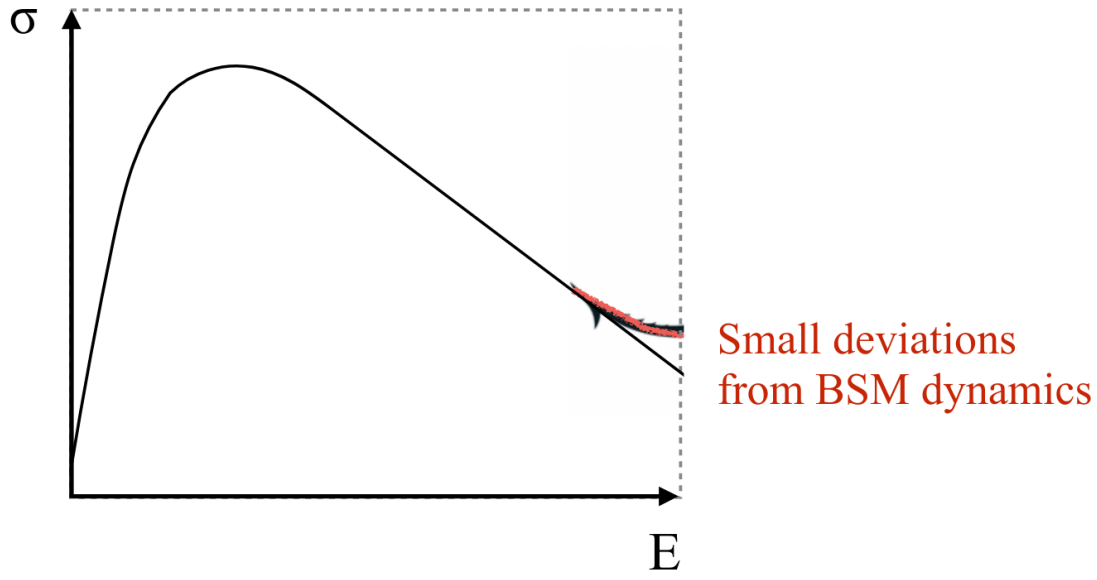

Figure: Figure from Davide Pagani.

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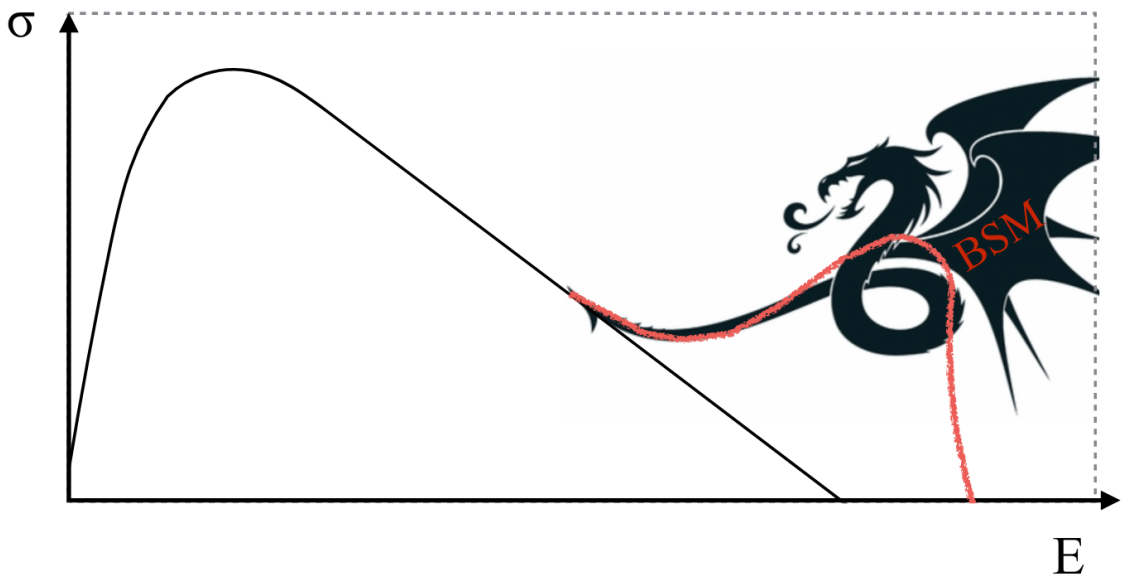

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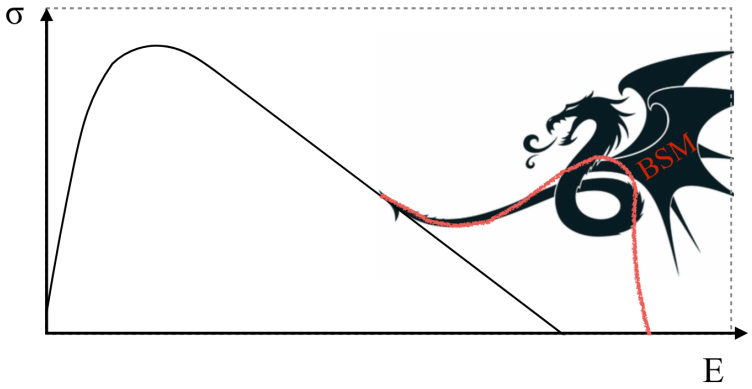

Figure: Figure from Davide Pagani.

→ **We need high-precision SM phenomenology**

Today's talk

1. Spin observables for $t\bar{t}$
2. Decay coefficients for V+jet
3. Improving on the SU(3) group treatment

Probing the spin correlation of tt production at NLO QCD+EW
arXiv:2105.11478

Electroweak corrections to the angular coefficients in finite-pT Z-boson production and dilepton decay
arXiv:2007.08867

Angular coefficients in W+j production at the LHC with high precision
arXiv:2204.12394

The colour matrix at next-to-leading-colour accuracy for tree-level multi-parton processes
arXiv:2109.10377

Precision SM phenomenology

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1. Spin observables for $t\bar{t}$
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Top quark pair production and spin correlations

- $t\bar{t}$ pair production at LHC 14 TeV: $\sim$ 950 pb [2]
- Top quarks: decay before *spin decorrelation*

[2] M. Czakon, P. Fiedler, A. Mitov. arXiv:1303.6254
[3] W. Bernreuther, A. Brandenburg, Z.G. Si, P. Uwer. arXiv:hep-ph/0304244

Top quark pair production and spin correlations

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- Top quarks: decay before *spin decorrelation*
- Measure decay products to **indirectly measure spin correlation**
- Decay $\boldsymbol{t\bar{t} \to W^+ b W^- \bar{b} \to \{e^+e^-, \mu^+\mu^-\} \nu_l \bar{\nu}_l b\bar{b}}$ (10.5%) has a **clear signature**

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- Spin correlations in the **spin-density formalism** [3]

$$|\mathcal{M}(pp \to t\bar{t})|^2 \propto \mathrm{Tr}[\rho_t(k_t, s_t)\rho_{\bar{t}}(k_{\bar{t}}, s_{\bar{t}})R] \tag{1}$$

with

$$R = A(1 \otimes 1) + B_i^+ (1 \otimes \sigma_i) + B_j^- (\sigma_j \otimes 1) + C_{ij} (\sigma_i \otimes \sigma_j) \tag{2}$$

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Top quark pair production and spin correlations

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- Expand differential cross section as

$$\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta_+^i\,\mathrm{d}\cos\theta_-^j} = \frac{1}{4}\left(1 + B_+^i\cos\theta_+^i + B_-^j\cos\theta_-^j + C_{ij}\cos\theta_+^i\cos\theta_-^j\right) \tag{3}$$

- A set of 6 B-coefficients and 9 C-coefficients (frame $\{\hat{r}, \hat{k}, \hat{n}\}$)

$$C_{ij} = -9\frac{\langle\cos\theta_+^i\cos\theta_-^j\rangle}{\sigma}, \qquad B_i^{\pm} = 3\frac{\langle\cos\theta_\pm^i\rangle}{\sigma} \tag{4}$$

[4] M. Czakon, P. Fiedler, A. Mitov. arXiv:1303.6254

Top quark pair production

- Expansion to NLO of $pp \to t\bar{t}$ ($\times\mathcal{O}(\alpha^4)$ for the leptonic decays)

$$\begin{aligned}\Sigma_{\text{LO}}(\alpha_S,\alpha) &= \underbrace{\alpha_S^2\Sigma_{2,0}}_{\text{LO}_1} + \underbrace{\alpha_S\alpha\Sigma_{1,1}}_{\text{LO}_2} + \underbrace{\alpha^2\Sigma_{0,2}}_{\text{LO}_3} \\ \Sigma_{\text{NLO}}(\alpha_S,\alpha) &= \underbrace{\alpha_S^3\Sigma_{3,0}}_{\text{NLO}_1} + \underbrace{\alpha_S^2\alpha\Sigma_{2,1}}_{\text{NLO}_2} + \underbrace{\alpha_S\alpha^2\Sigma_{1,2}}_{\text{NLO}_3} + \underbrace{\alpha^3\Sigma_{0,3}}_{\text{NLO}_4}.\end{aligned} \tag{5}$$

- Perturbation lingo:
  LO=LO_1, NLO QCD=LO_1+NLO_1, NLO EW = LO_1(+LO_2)+NLO_2

[5] M. Czakon, A. Mitov, R. Poncelet, arXiv:2008.11133
[6] W. Bernreuther, D. Heisler, Z-G. Si, arXiv:1508.05271

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- Perturbation lingo:
  LO=LO_1, NLO QCD=LO_1+NLO_1, NLO EW = LO_1(+LO_2)+NLO_2
- **Existing NNLO QCD** predictions for spin correlation coefficients [5]
- **Existing leading NLO (electro)weak corrections** for the coefficients [6]

- Consider complete-NLO (=:NLO QCD+EW)

$$\underbrace{\alpha_S^3\Sigma_{3,0}}_{\text{NLO}_1} + \underbrace{\alpha_S^2\alpha\Sigma_{2,1}}_{\text{NLO}_2} + \underbrace{\alpha_S\alpha^2\Sigma_{1,2}}_{\text{NLO}_3} + \underbrace{\alpha^3\Sigma_{0,3}}_{\text{NLO}_4}. \quad (6)$$

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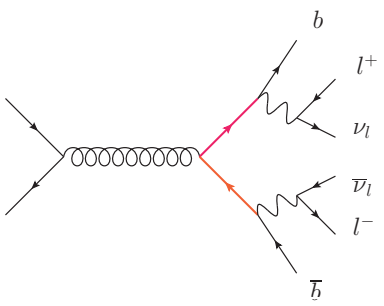
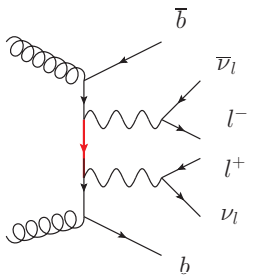
Electroweak corrections

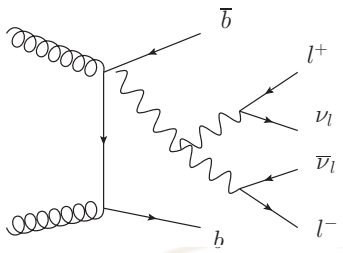
- **This project**: calculate **complete NLO QCD+EW** in production for spin correlation coefficients

$$\sigma^{\text{NLO QCD+EW}} \times \Gamma^{\text{LO}}_{t\to l^+\nu b} \times \Gamma^{\text{LO}}_{\bar{t}\to l^-\nu\bar{b}} \tag{7}$$

- Uses **narrow width approximation**: top quarks are produced **on-shell**







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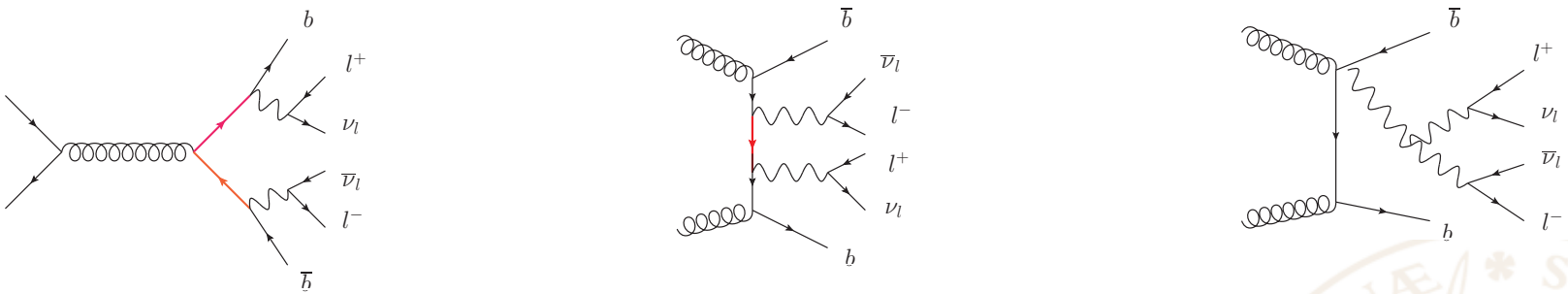
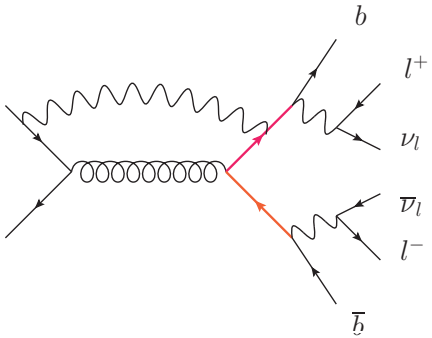
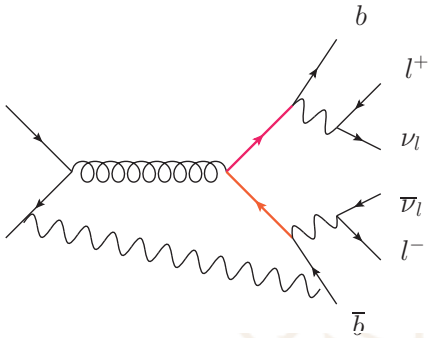


- NLO EW corrections only in production:



Included



Not included

Method and setup

- Use decay chain model: MadSpin internal to MadGraph5_aMC@NLO

[7] S. Frixione, E. Laenen, P. Motylinski, B.R. Webber, arXiv:hep-ph/0702198

Method and setup

- Use decay chain model: MadSpin internal to MadGraph5_aMC@NLO
  1. Generate production events at NLO QCD+EW (fixed-order)
  2. Generate LO decay events
  3. Attach the decays to the production events to create fully-decayed events
  4. Reweight the events to obtain **NLO spin correlation**

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Approximations!

- No off-shell effects
- LO in decay
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- Setup:

13 TeV LHC	NNPDF3.1NLOluxqed
5-flavour scheme	G_μ-scheme

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Results: C-coefficients

- All except C_{kk}, C_{nn}, C_{rr} and C_{rk+kr} vanish at NLO QCD and complete NLO

Order / [%]	C_{kk}	C_{nn}	C_{rr}	C_{rk+kr}
NLO QCD+EW × LO (our[8])	32.69(5)	31.97(3)	4.80(3)	-20.51(6)
NLO QCD × LO (our)	32.88(3)	31.89(5)	4.83(5)	-20.48(9)
NLO × NLO (Czakon [9])	33.0(3)	33.0(2)	5.8(2)	-20.3(2)
NLO × LO (MCFM)	33.04(4)	33.09(4)	5.96(4)	-20.71(7)

[8] R. Frederix, I. Tsinikos, T.Vitos, arXiv:2105.11478

[9] M. Czakon, A. Mitov, R. Poncelet, arXiv:2008.11133

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Approximations!

- (No off-shell effects)
- LO in decay
- Virtual spin correlation approximated by tree-level ones

- Compare to existing NLO × NLO and NLO × LO with virtual effects

- NLO in decay is not source of discrepancy (cf. rows 3 and 4)
- Seems to stem from *virtual spin correlation effects*

[8] R. Frederix, I. Tsinikos, T.Vitos, arXiv:2105.11478
[9] M. Czakon, A. Mitov, R. Poncelet, arXiv:2008.11133

Today's talk

1. Spin observables for $t\bar{t}$
2. Decay coefficients for V+jet
3. Improving on the SU(3) group treatment

Decay coefficients for V+jet

- One of the key processes for measuring EW parameters (m_W) at LHC:

$$pp \to V + \text{jet} \to l_1 l_2 + \text{jet}$$

  (V=Z,$W^{\pm}$)
- Total cross section at LHC 13 TeV for V+jet production: $\sim$ 12000 pb

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  (V=Z,W$^{\pm}$)
- Total cross section at LHC 13 TeV for V+jet production: $\sim$ 12000 pb
- Expansion of process to NNLO:

$$\begin{aligned}
\Sigma_{\text{LO}}(\alpha_S,\alpha) &= \underbrace{\alpha_S\alpha^2\Sigma_{1,2}}_{\text{LO}_1} + \underbrace{\alpha^3\Sigma_{0,3}}_{\text{LO}_2} \\
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\end{aligned} \tag{8}$$

Decay coefficients for V+jet

- **Differential cross section** (5-dimensional) in V-boson kinematics expanded in **real spherical harmonics**

$$\frac{d\sigma}{dp_{T,Z}dy_Z dm_{ll}d\Omega} \propto \Big((1+\cos^2\theta) + A_0 \frac{1}{2}(1-3\cos^2\theta) + A_1 \sin 2\theta \cos\phi + A_2 \frac{1}{2}\sin^2\theta \cos 2\phi + A_3 \sin\theta\cos\phi + A_4 \cos\theta + A_5 \sin^2\theta \sin 2\phi + A_6 \sin 2\theta \sin\phi + A_7 \sin\theta \sin\phi \Big) \quad (9)$$

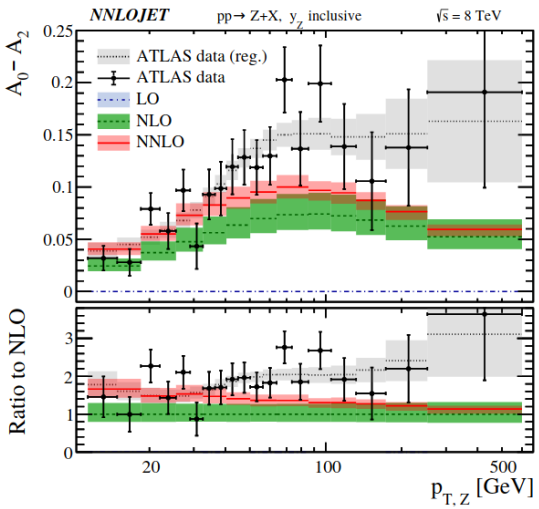
  with eight **angular/decay coefficients** $A_i(p_{T,V}, y_V, m_{ll})$
- Angles (θ, ϕ) are angles of $l^{\pm}$ in the **Collins-Soper frame**
- This decomposition separates production mechanism and decay part

Decay coefficients for Z+jet: Lam-Tung relation

- Up to order $\boldsymbol{\alpha^2\alpha_S}$ **(LO): Lam-Tung relation** $A_0 = A_2$ (if $\Phi_1 = 0$)

$$A_0 = \sin^2\theta_1 \quad , \quad A_2 = \sin^2\theta_1 \cos 2\Phi_1 \tag{10}$$

- **Lam-Tung relation** $A_0 = A_2$ holds up to order $\boldsymbol{\alpha_S\alpha^2}$ **(LO)**
- Predictions for Z+jet available at order $\boldsymbol{\alpha_S^3\alpha^2}$ **(NNLO QCD)**[10]
- ATLAS and CMS (and runs at Tevatron) all measured **higher violation** of Lam-Tung than predicted by NNLO QCD at $p_{T,Z} > 20$ GeV



[10] R. Gauld, A. Gehrmann-De Ridder, T. Gehrmann, *et al.* High Energ. Phys. 2017, 3 (2017)

Decay coefficients for Z+jet: setup

- **This project**: Calculate electroweak corrections to the dominant angular coefficients and Lam-Tung relation
- **Fixed-order**: $pp \to \{e^+e^-, \mu^+\mu^-\} + j$ at 8 TeV with MadGraph5_aMC@NLO at

$$\text{NLO QCD+EW} := \text{LO}_1 + \text{LO}_2 + \text{NLO}_1 + \text{NLO}_2$$

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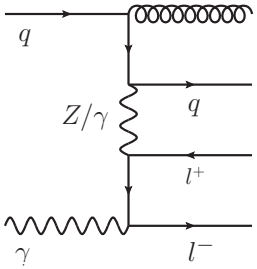
$$\text{NLO QCD+EW} := \text{LO}_1 + \text{LO}_2 + \text{NLO}_1 + \text{NLO}_2$$

- Introduce **single lepton p_T cut** to avoid double IR (2-loop) singularity $\to$ vary cut to extrapolate to the full phase space of the dilepton pair



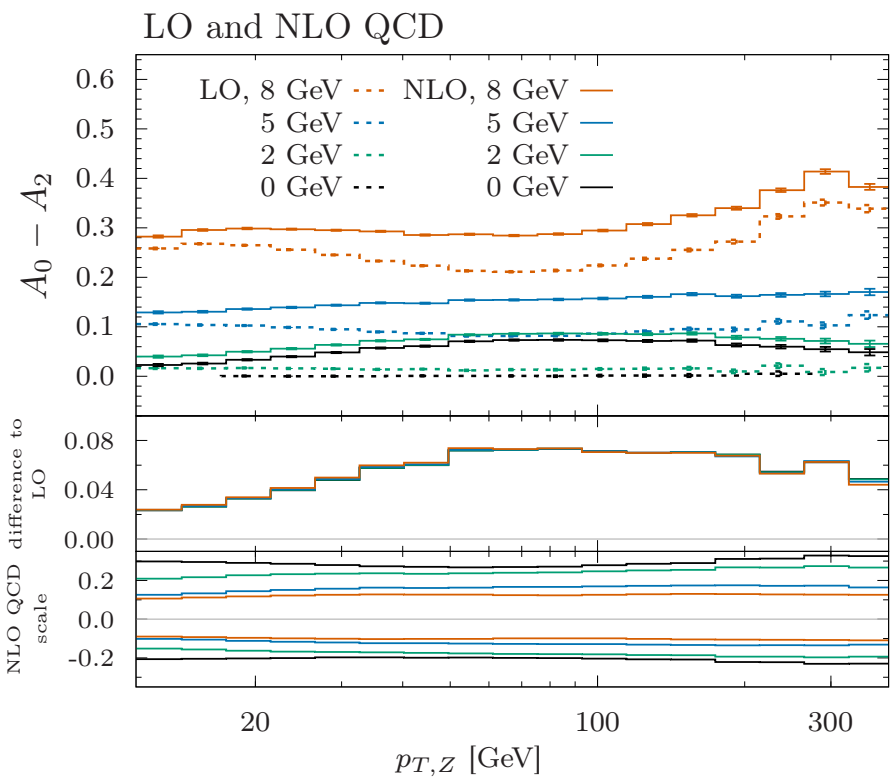
- Use moments method for each coefficient in $A_i f(\theta, \Phi)$

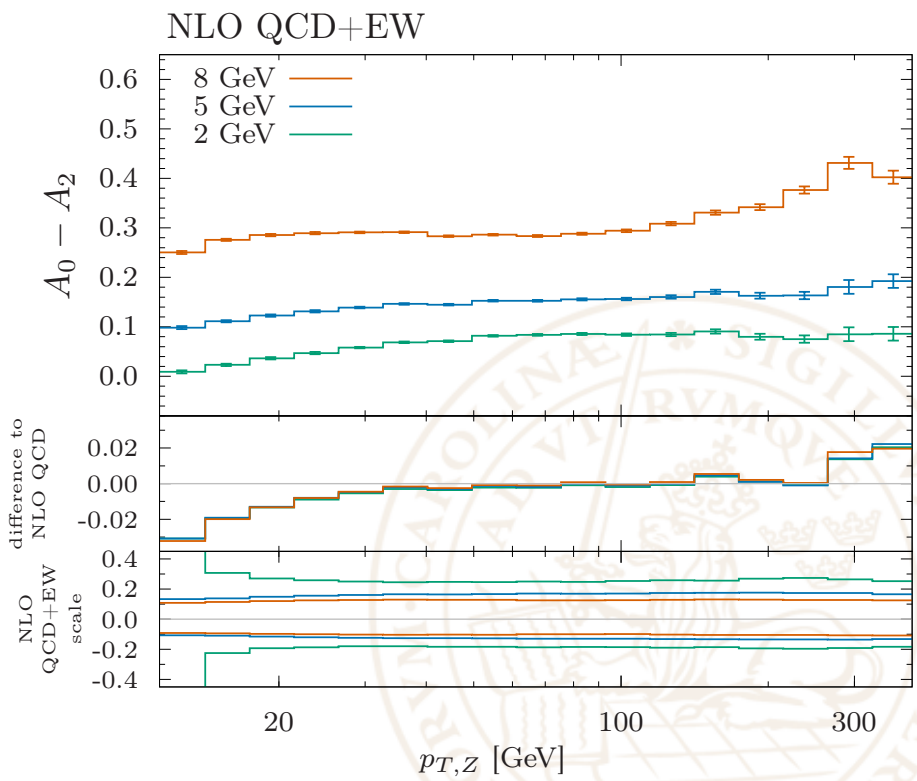
$$A_i \propto \frac{\int d\Omega d\sigma f(\theta, \Phi)}{\int d\Omega d\sigma} \tag{11}$$

- **Note!** Due to the ratio-nature of the ocefficients, EW Sudakovs are not necessarily expected to show up!

Angular coefficients for Z+jet: results

- Lam-Tung violation $A_0 - A_2$ (differentially in the Z-boson p_T) at LO and NLO QCD (left) and NLO QCD+EW (right) [11]

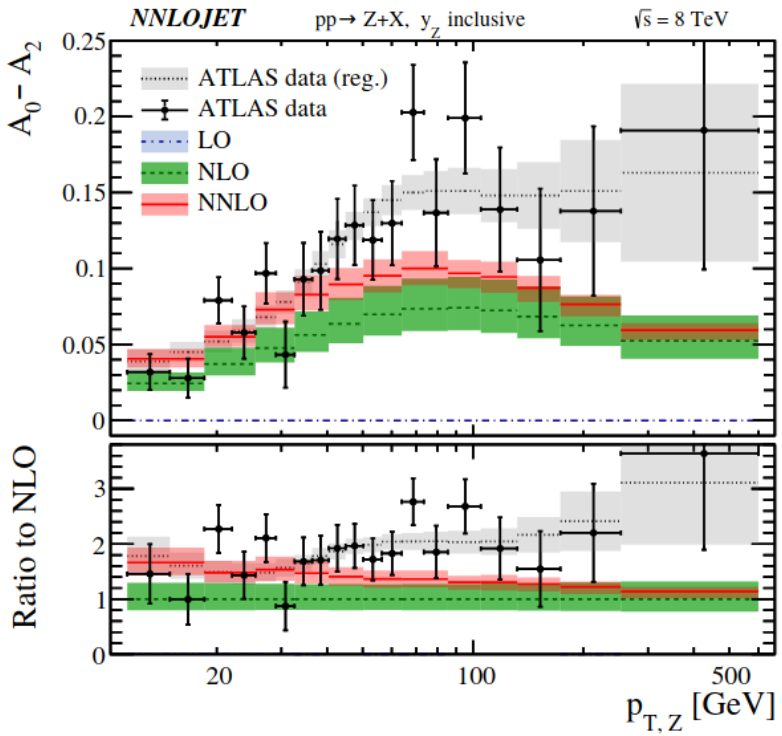


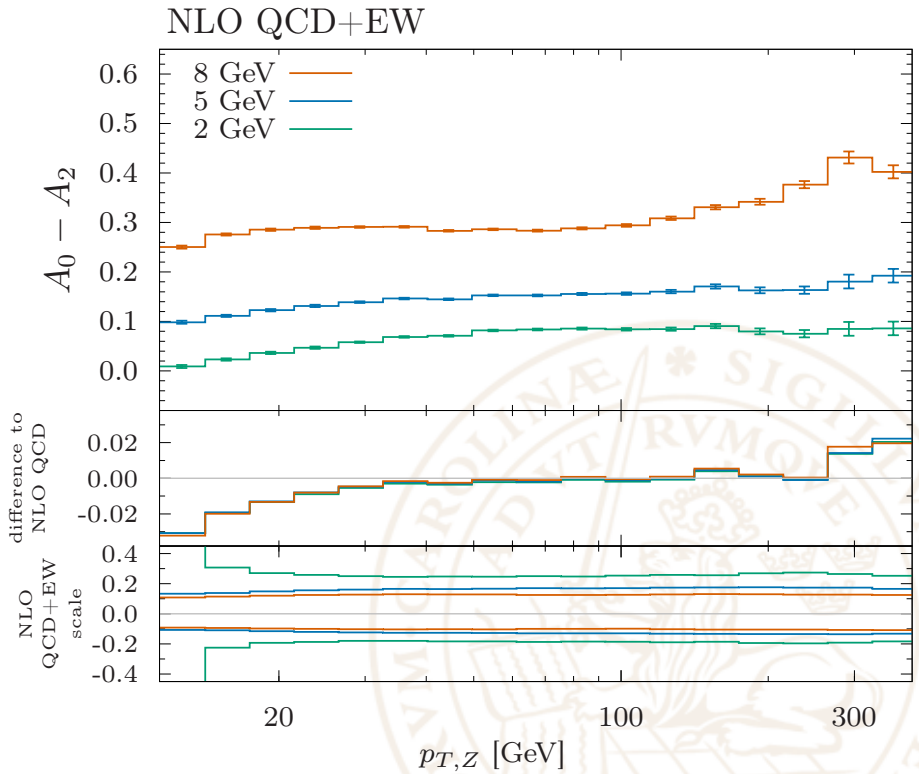


[11] R. Frederix, T. Vitos, arXiv:2007.08867

Angular coefficients for Z+jet: results

- Lam-Tung violation $A_0 - A_2$ at NNLO QCD with ATLAS data (left) [12] and NLO QCD+EW (right)

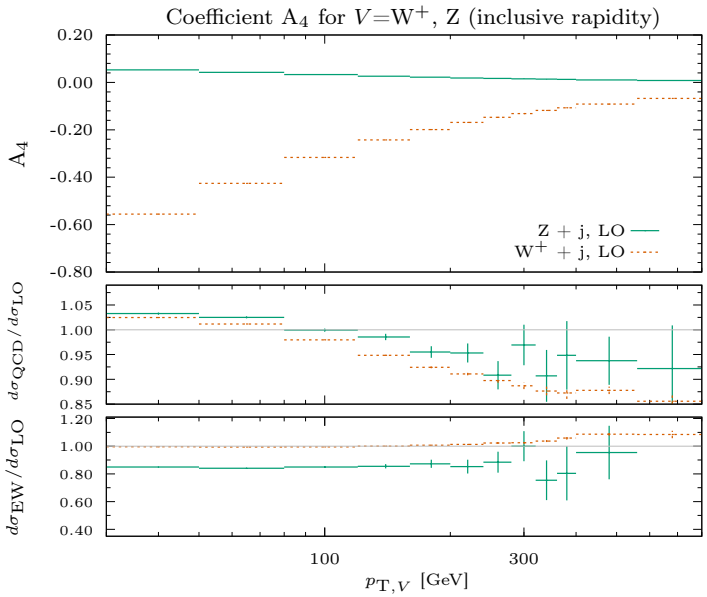




- Electroweak effects move violation **towards the data** for the **low-p_T region**

[12] R. Gauld, A. Gehrmann-De Ridder, T. Gehrmann, *et al*, High Energ. Phys. 2017, 3 (2017)

Angular coefficients for W+jet: motivation



- $W^{\pm}$+jet more difficult to measure due to the neutrino
- Direct decay coefficient measurements by CDF (1.8 TeV)[13]
- Template fits of distributions to measure W-boson mass
- Improve fluctuations by an unfolding to Z+jet [14]

[13] CDF Collaboration arXiv:hep-ex/0504020
[14] ATLAS Collaboration arXiv:1701.07240

Angular coefficients for W+jet: setup

- **This project**: Calculate and combine NNLO QCD and NLO EW corrections to the angular coefficients
- **Fixed-order**: $pp \to \{e^+\nu_e\} + j$ at 13 TeV at:

$$\text{NLO EW} := \text{LO}_1 + \text{LO}_2 + \text{NLO}_2$$
$$\text{NNLO QCD} := \text{LO}_1 + \text{NLO}_1 + \text{NNLO}_1$$

- MadGraph5_aMC@NLO (for NLO EW) and STRIPPER (for NNLO QCD) [15]

[15] M. Czakon arXiv:1005.0274

Angular coefficients for W+jet: setup

- **This project**: Calculate and combine NNLO QCD and NLO EW corrections to the angular coefficients
- **Fixed-order**: $pp \to \{e^+\nu_e\} + j$ at 13 TeV at:

$$\text{NLO EW} := \text{LO}_1 + \text{LO}_2 + \text{NLO}_2$$
$$\text{NNLO QCD} := \text{LO}_1 + \text{NLO}_1 + \text{NNLO}_1$$

- MadGraph5_aMC@NLO (for NLO EW) and STRIPPER (for NNLO QCD) [15]
- Combining NLO EW and NNLO QCD, default way (unexpanded):

$$\mathrm{A}_i^{\mathrm{default}} = \frac{N}{D}, \tag{12}$$

- Expansion in α_s:

$$\mathrm{A}_i^{\mathrm{exp}} = A + \alpha_s B + \alpha_s^2 C, \tag{13}$$

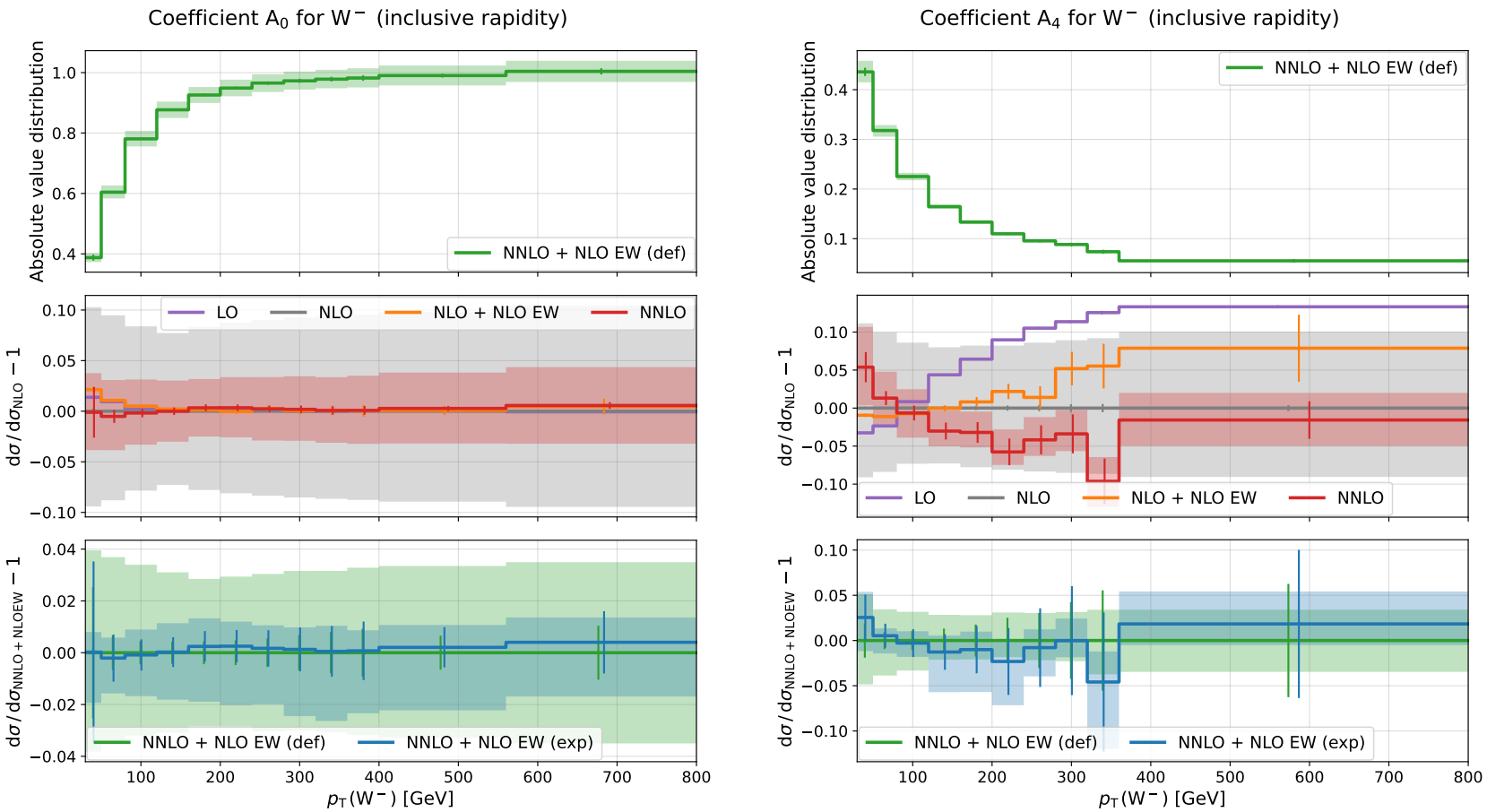
- Inclusion of NLO EW through an overall K-factor (avoids $p_T(l)$ cut dependence)

$$\mathrm{A}_{i,\mathrm{QCD+EW}} = K_{\text{NLO EW}} \times \mathrm{A}_i, \tag{14}$$

[15] M. Czakon arXiv:1005.0274

Angular coefficients for W+jet: results, inclusive rapidity

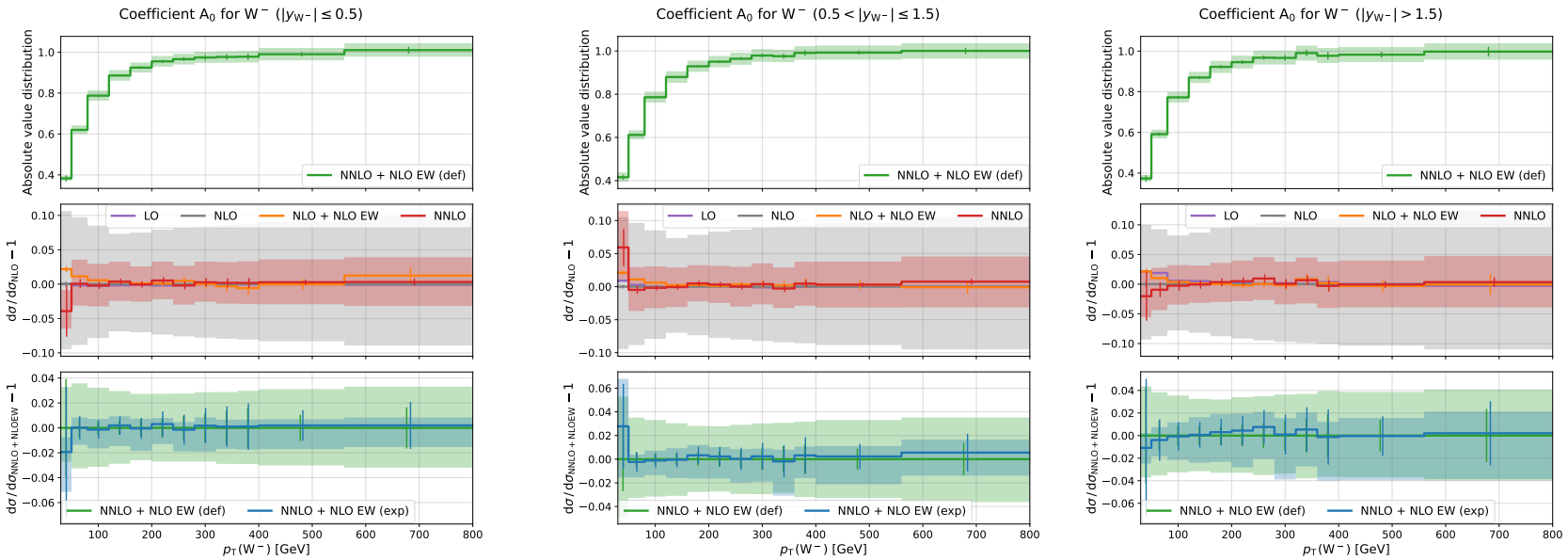
- The coefficients A_0 (left) and A_4 (right) for W^- signature, inclusive in rapidity [16]



[16] M. Pellen, R. Poncelet, A. Popescu, T. Vitos. arXiv:2204.12394

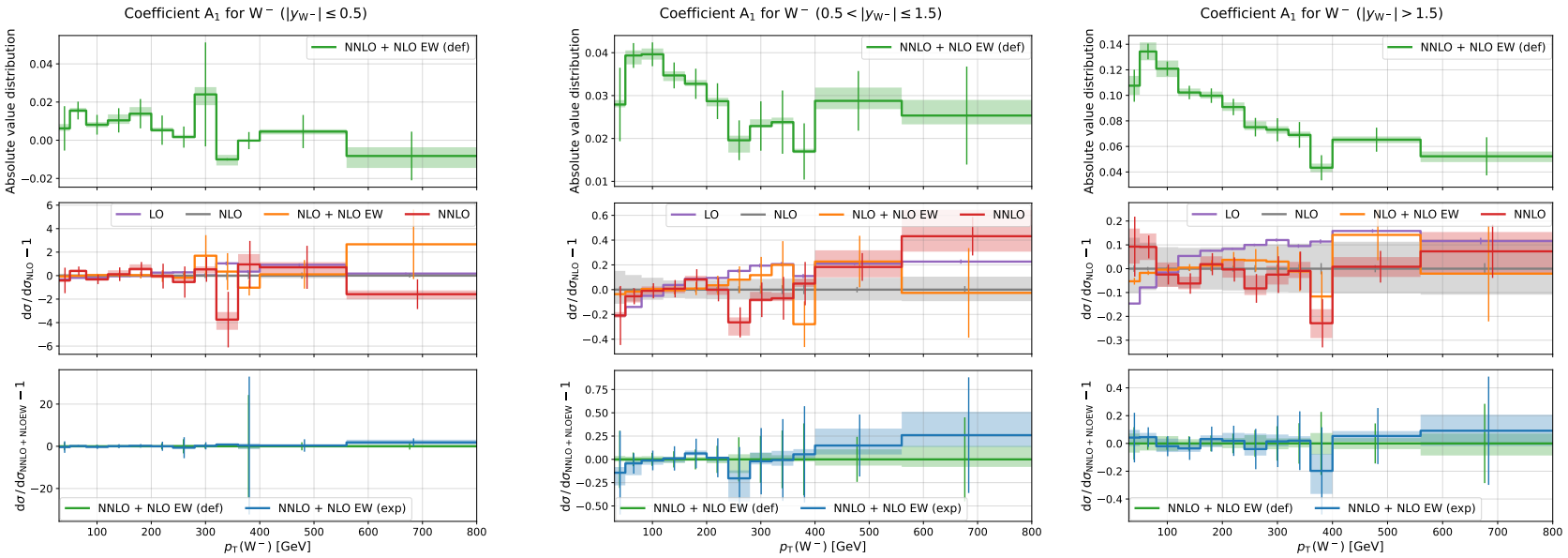
Angular coefficients for W+jet: results, A_0 rapidity dependence

- The coefficients A_0 in various rapidity bins
- No rapidity dependence (same for A_2)



Angular coefficients for W+jet: results, A_1 rapidity dependence

- The coefficients A_1 in various rapidity bins
- Note: different y-scales!
- Heavily rapidity-dependent (same for A_3 and A_4)



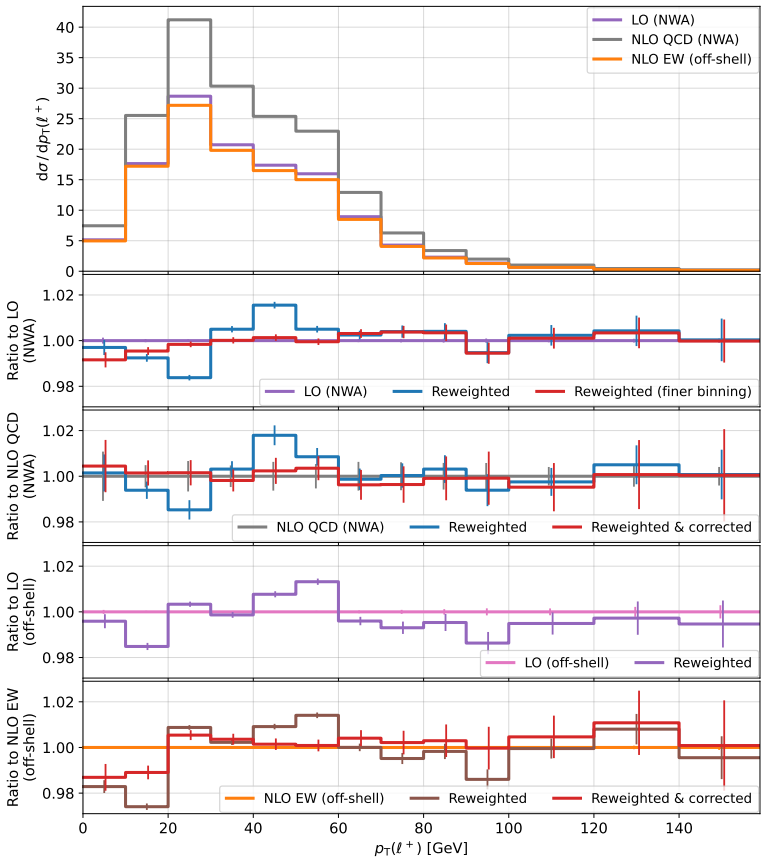
Angular coefficients for W+jet: EW non-closure effect[17]

- The expansion to spherical harmonics is no longer valid when EW splittings are allowed ($1 \to 3$ kinematics)

[17] M. A. Ebert, *et al.*. arXiv:2006.11382

Angular coefficients for W+jet: EW non-closure effect[17]

- The expansion to spherical harmonics is no longer valid when EW splittings are allowed ($1 \to 3$ kinematics)
- Reproduce lepton distribution ($p_T(l^+)$) with angular coefficients with reweighting
- Correct for binning effects
- Correct for NWA effects (from LO)
- NLO EW (off-shell) versus eweighted with A_i show good agreement (except first few beins)

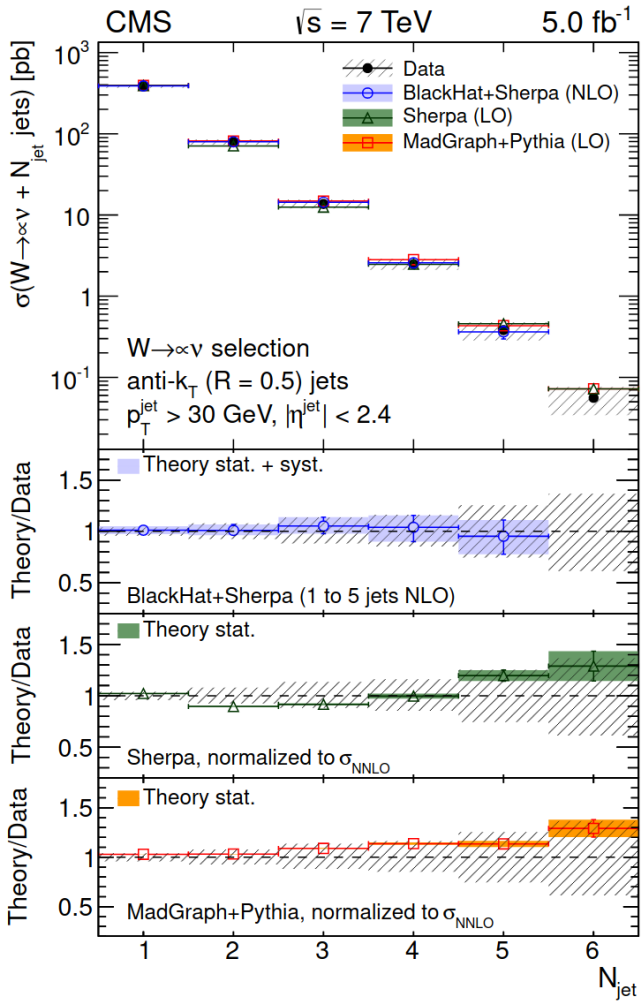


[17] M. A. Ebert, *et al.*. arXiv:2006.11382

Today's talk

1. Spin observables for $t\bar{t}$
2. Decay coefficients for V+jet
3. Improving on the SU(3) group treatment

Why do we bother about high-multiplicity?



- Measurements of multi-jet processes (ATLAS and CMS)
- Measurements of final states with >6 jets
- As detectors get better, **predictions must get better!**
- At the **future HL-LHC**, multi-jet processes will be even more abundant

Bottleneck for theory predictions

$SU(3)_C \times SU(2)_w \times U(1)_Y$

High-multiplicity processes in matrix-element generators

- Current colour treatment: **colour decomposition** of amplitudes:

$$|\mathcal{M}|^2 \propto \sum_{\sigma_k,\sigma_l} \underbrace{C(\sigma_k,\sigma_l)}_{\text{colour matrix}} \underbrace{\mathcal{A}(\sigma_k)(\mathcal{A}(\sigma_l))^*}_{\text{dual amplitudes}} \tag{15}$$

where $\sigma_{k,l}$ some permutation of final state particles

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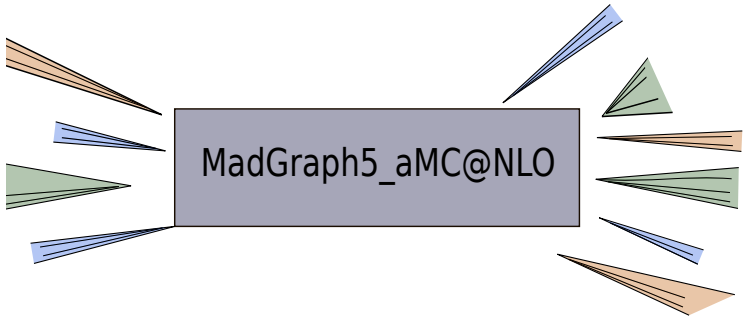
  where $\sigma_{k,l}$ some permutation of final state particles
- Colour matrix: basis-dependent (fundamental, colour-flow, multiplet basis...)
- Colour factors: either T^a (fundamental) or f^{abc} (adjoint)

$$[T^a, T^b] = if^{abc}T^c \tag{16}$$

  or δ^a_b (colour-flow)
- Colour matrix size for n final state particles: $\sim n! \times n!$
  $\rightarrow$ We hit a wall for high-multiplicity QCD processes!



The large-N_c limit

- First introduced by Gerard 't Hooft (1974) [18]
- Use the model

$$\mathrm{SU}(3)_C \to \mathrm{SU}(N_\mathbf{C})$$

  and then $N_\mathbf{C} \to \infty$
- Observables are expanded in terms of $\frac{1}{N_\mathbf{C}} \to$ colour expansion
- In the Standard Model, $N_\mathbf{C} = 3$: expansion in ~ 0.3
- Effectively, an expansion in $\left(\frac{1}{N_\mathbf{C}}\right)^2 \to\sim$ 10% accuracy at second order!

[18] G. 't Hooft. Nucl. Phys. B72 (1974) 461 - 473

Reducing the complexity: our take on it

- One possible solution: make the colour matrix sparse!
- Expand in N_c^{-2} (large-N_c limit)

$$C(\sigma_k, \sigma_l) = \underbrace{a_0 N_c^x}_{\text{Leading colour (LC)}} + \underbrace{a_1 N_c^{x-2}}_{\text{Next-to-leading colour (NLC)}} + \mathcal{O}(N_c^{x-4}) \quad \forall k, l \quad (17)$$

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$$C(\sigma_k, \sigma_l) = \begin{pmatrix} \mathrm{LC} & 0 & 0 & 0 & 0 & \mathrm{NLC} \\ 0 & \mathrm{LC} & 0 & \mathrm{NLC} & 0 & 0 \\ 0 & 0 & \mathrm{LC} & 0 & 0 & 0 \\ 0 & \mathrm{NLC} & 0 & \mathrm{LC} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{LC} & 0 \\ \mathrm{NLC} & 0 & 0 & 0 & 0 & \mathrm{NLC} \end{pmatrix}$$

- + use symmetry for identical final state particles

Reducing the complexity: our take on it

- One possible solution: make the colour matrix sparse!
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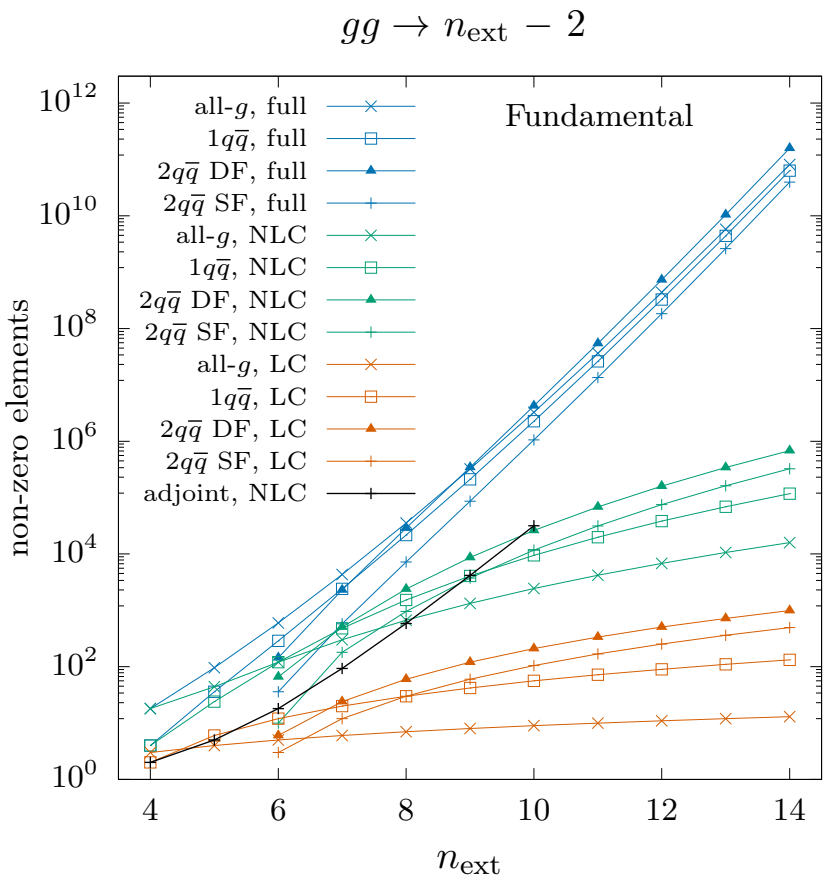
- + use symmetry for identical final state particles

Where do we find the NLC terms, how sparse does the matrix get, } In R. Frederix, T. Vitos arXiv:2109.10377.

and how accurate is this approximation? } Current work in progress

Results for gg initiated processes[19]

- For external particles $n_{\text{ext}} \in [4, 14]$
- Blue: full colour Green: NLC Red: LC



[19] R. Frederix, T. Vitos arXiv:2109.10377

Current status in MadGraph5_aMC@NLO

- Recall the exact colour decomposition

$$|\mathcal{M}|^2 \propto \sum_{\sigma_k,\sigma_l} \underbrace{C(\sigma_k,\sigma_l)}_{\text{colour matrix}} \mathcal{A}(\sigma_k)(\mathcal{A}(\sigma_l))^* \tag{18}$$

- Presently computes **all dual amplitudes** $\mathcal{A}(\sigma)$

[20] A. van Hameren, C.G. Papadopoulos. arXiv:hep-ph/0204055

Current status in MadGraph5_aMC@NLO

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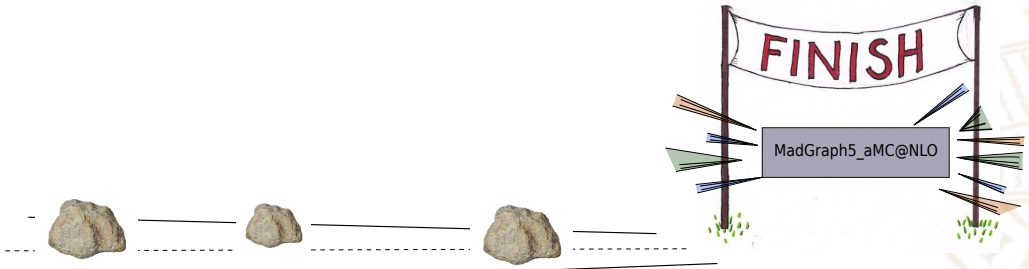
- Presently computes **all dual amplitudes** $\mathcal{A}(\sigma)$
- With a sparse matrix, only a subset of these need to be computed!
- The phase space integration

$$d\hat{\sigma}(a+b\to x) \sim \underbrace{d\Phi_n}_{\text{kinematics}} \; |\underbrace{\mathcal{M}(a+b\to x)}_{\text{amplitude}}|^2 \tag{19}$$

needs to be revised (following *antenna structure poles* [20])

- Verify accuracy of NLC expansion with recent Berends-Giele implementations



[20] A. van Hameren, C.G. Papadopoulos. arXiv:hep-ph/0204055

Summary and outlook

Top pair spin correlations

- Examine further off-shell effects + virtual spin correlations

V+jet decay coefficients

- Low-p_T predictions for the coefficients for ATLAS

NLC approximation in ME generators

- Extend beyond NLC (NNLC?)
- Extend to NLO computations (colour treatment in loops)

Thank you for listening!

Köszönöm a figyelmüket!

Top spin correlations: reweighting

- **Unweighting** procedure for all events using the **upper-boundedness**:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\Phi_{full})} < B_{max} \frac{\mathrm{d}\sigma}{\mathrm{d}(\Phi_{prod})} \tag{20}$$

  for a constant B_{max} (in principle calculable)
- Let event weights be: $d_{B/R} = r_{B/R} p_{B/R}$
- Use instead the decomposition

$$\sigma_{\mathrm{dec}} = \sum_{i=1}^{N} (d_B^i + d_R^i) = r_{\max} \sum_{i=1}^{N} \frac{r_B^i}{r_{\max}} \left(p_B^i + \frac{r_R^i}{r_B^i} p_R^i \right) \tag{21}$$

  1. Attach the decay with the **unweighting probability** $\frac{r_B^i}{r_{\max}}$ to **Born event** i
  2. **Reweight** corresponding **real event** i with

$$\frac{r_R^i}{r_B^i} \text{ with } r_{B/R}^i = \frac{|\mathcal{M}^i_{\mathbf{full},B/R}|^2}{|\mathcal{M}^i_{\mathbf{prod},B/R}|^2 |\mathcal{M}^i_{\mathrm{dec}}|^2} \tag{22}$$

- Include only **tree-level matrix-elements in the reweighting** $\rightarrow$ virtual spin correlation is approximated by tree-level

Top spin correlations: B-coefficients

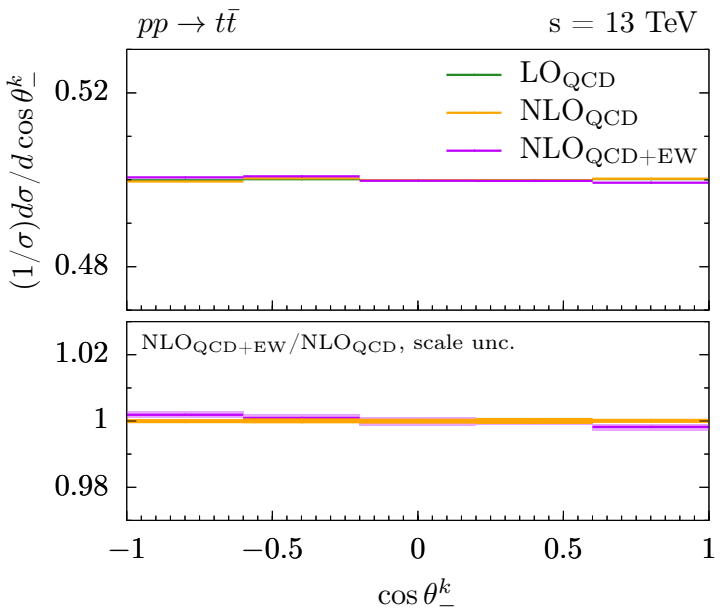
	Unexpanded	
	NLO QCD [%]	NLO QCD+EW [%]
B_k^+	$-0.001(20)^{+0\%}_{-0\%}\ ^{+0\%}_{-0\%}$	$-0.10(3)^{+12.0\%}_{-11.9\%}\ ^{+2.8\%}_{-2.8\%}$
B_n^+	$-0.03(2)^{+42.4\%}_{-31.5\%}\ ^{+1.6\%}_{-1.6\%}$	$0.07(4)^{+73.0\%}_{-47.6\%}\ ^{+2.5\%}_{-2.5\%}$
B_r^+	$-0.04(2)^{+55.4\%}_{-62.5\%}\ ^{+4.2\%}_{-4.2\%}$	$-0.11(4)^{+14.5\%}_{-18.3\%}\ ^{+2.0\%}_{-2.0\%}$
B_k^-	$0.04(2)^{+47.9\%}_{-31.5\%}\ ^{+4.7\%}_{-4.7\%}$	$-0.13(3)^{+9.4\%}_{-11.3\%}\ ^{+7.3\%}_{-7.3\%}$
B_n^-	$-0.05(2)^{+23.3\%}_{-18.6\%}\ ^{+2.5\%}_{-2.5\%}$	$-0.05(4)^{+35.9\%}_{-49.1\%}\ ^{+2.2\%}_{-2.2\%}$
B_r^-	$-0.02(2)^{+26.9\%}_{-32.5\%}\ ^{+3.0\%}_{-3.0\%}$	$-0.13(4)^{+13.5\%}_{-26.1\%}\ ^{+2.6\%}_{-2.6\%}$

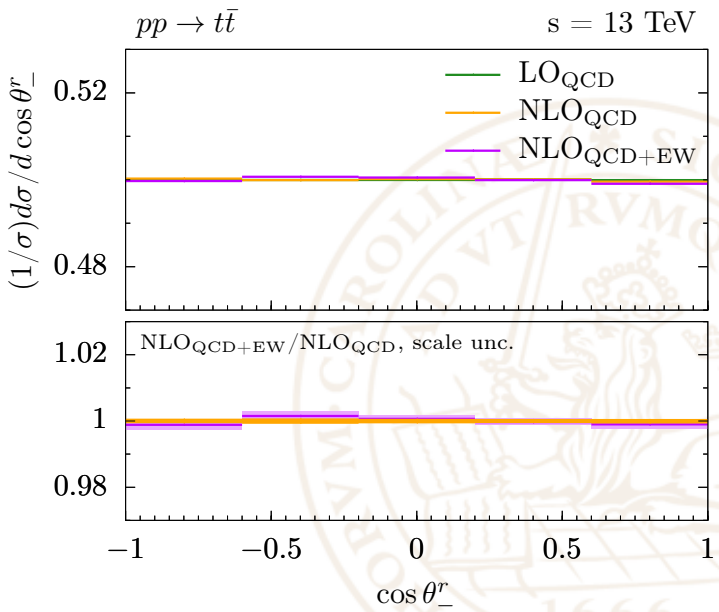
- Errors: statistical, PDF error
- All B-coefficients vanish at NLO QCD
- Some obtain non-zero corrections at complete-NLO
- Largest corrections: B_k^- and B_r^-

Top spin correlations: B-coefficients

- How are the corresponding distributions for B_k^- and B_r^-?

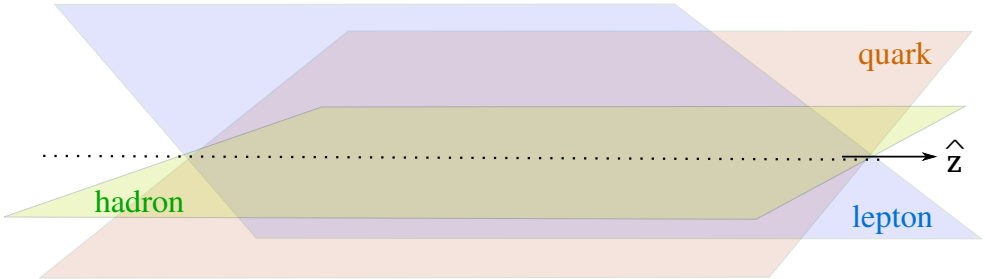
$$C_{ij} = -9\frac{\langle \cos\theta_+^i \cos\theta_-^j \rangle}{\sigma}, \qquad B_i^{\pm} = 3\frac{\langle \cos\theta_\pm^i \rangle}{\sigma}$$



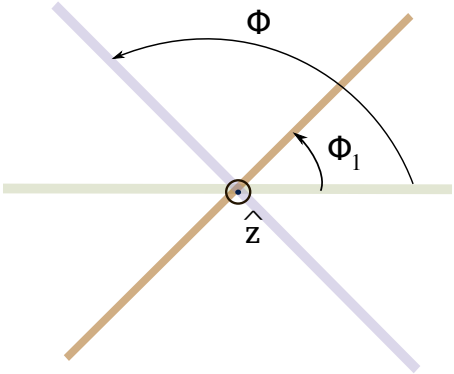


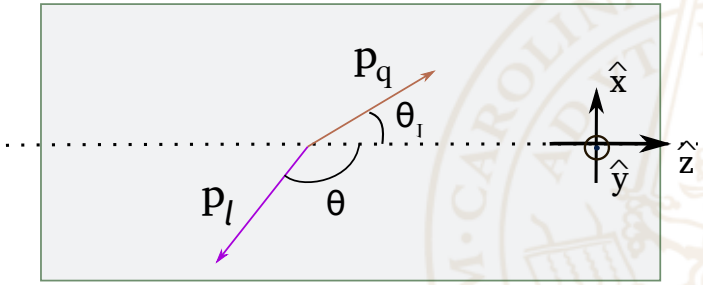
Decay coefficients: Collins-Soper frame

- $pp \to Z/\gamma + X \to l^+ l^- + X$ [7]: in Collins-Soper frame



- Introduce polar and azmuthal angles θ_1, Φ_1 of quark compared to the hadron plane



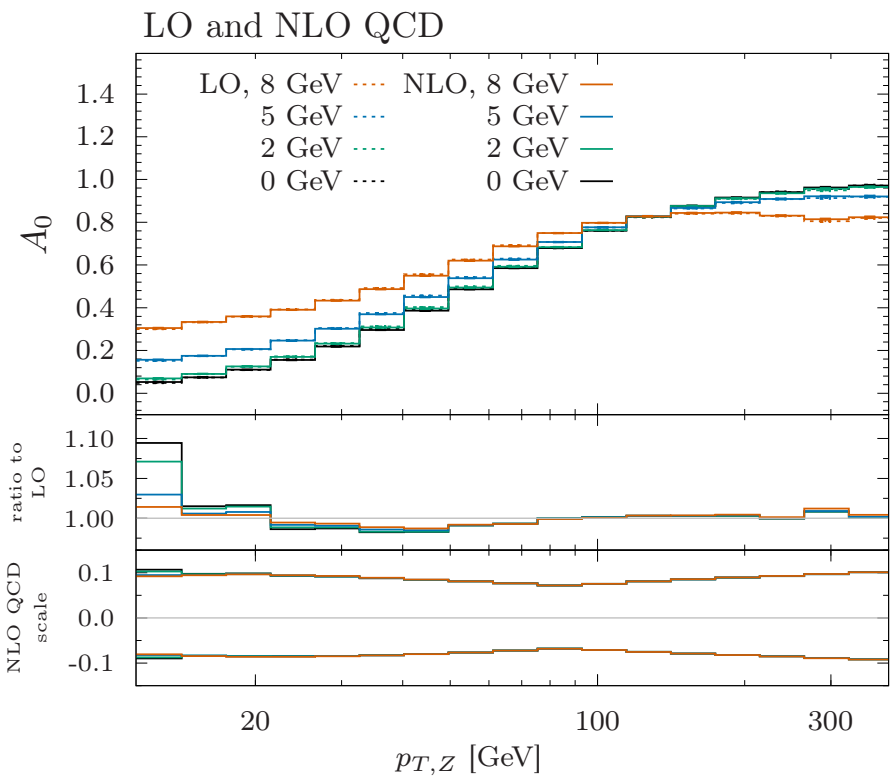


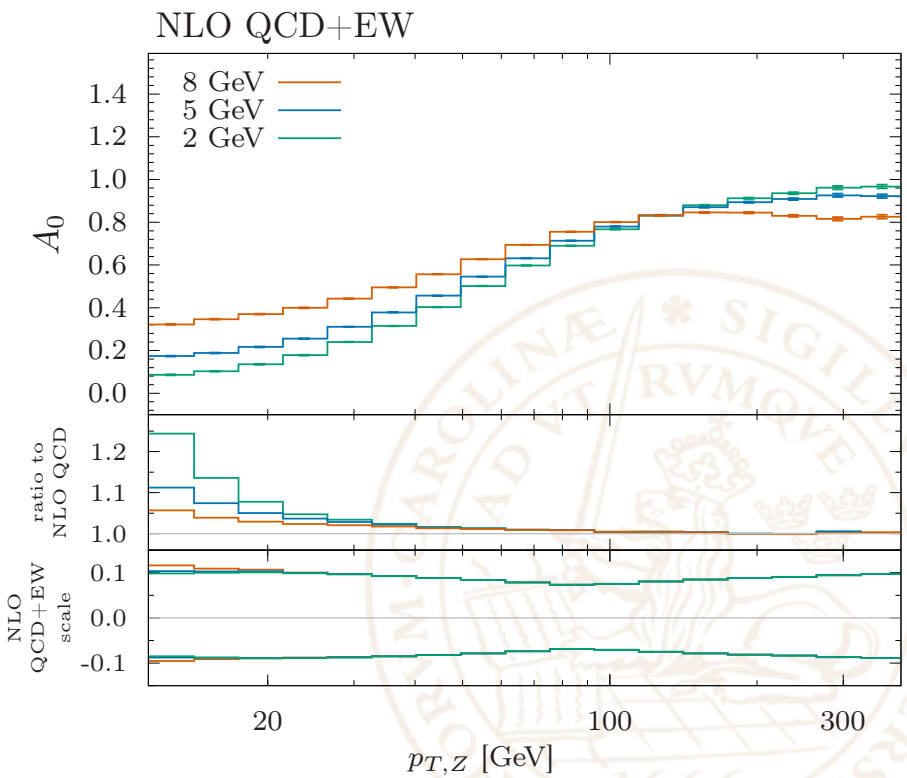
- Angles θ, Φ are the angles of the (negatively charged) lepton l^-

[7] J.-C. Peng *et al.*, arXiv:1511.08932

Decay coefficient for Z+jet: results

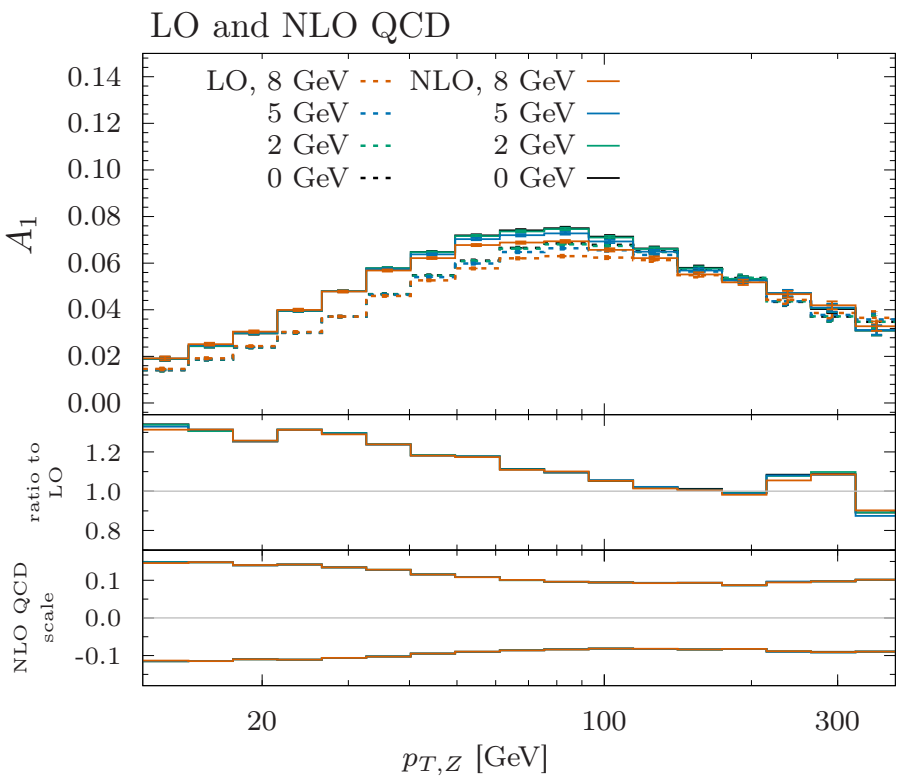
- Distributions for A_0
- Negligible electroweak corrections

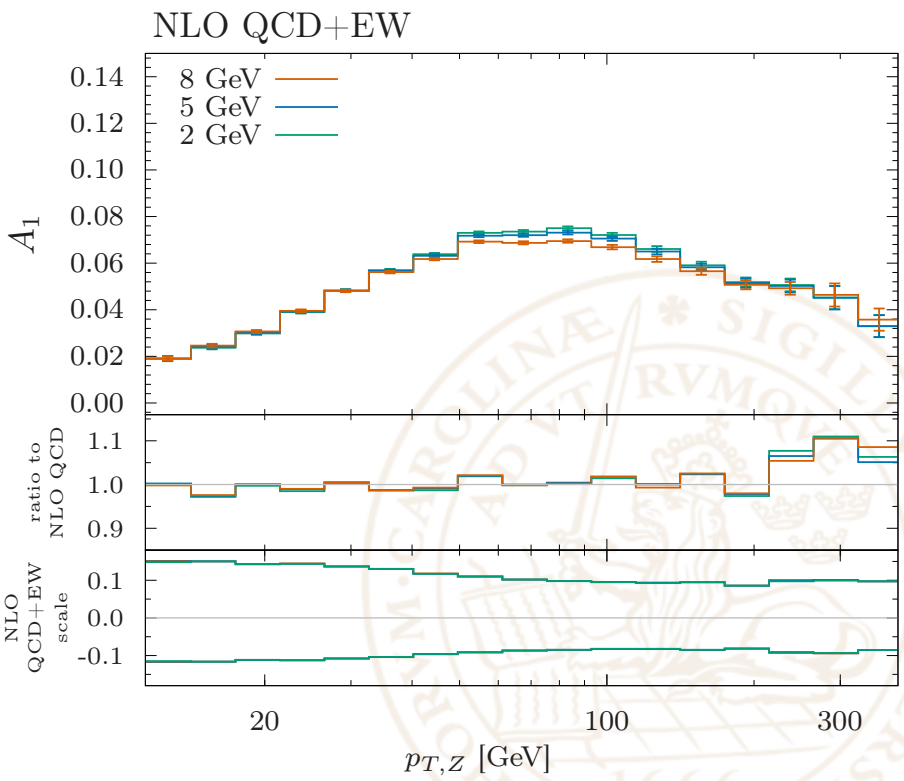




Decay coefficient for Z+jet: results

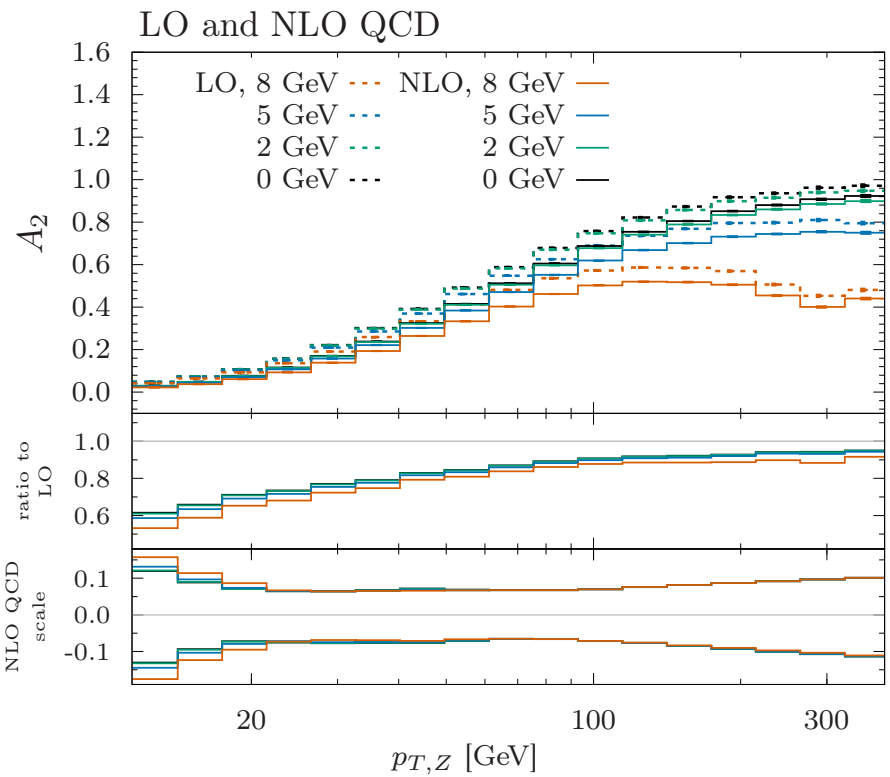
- Distributions for A_1
- Negligible electroweak corrections

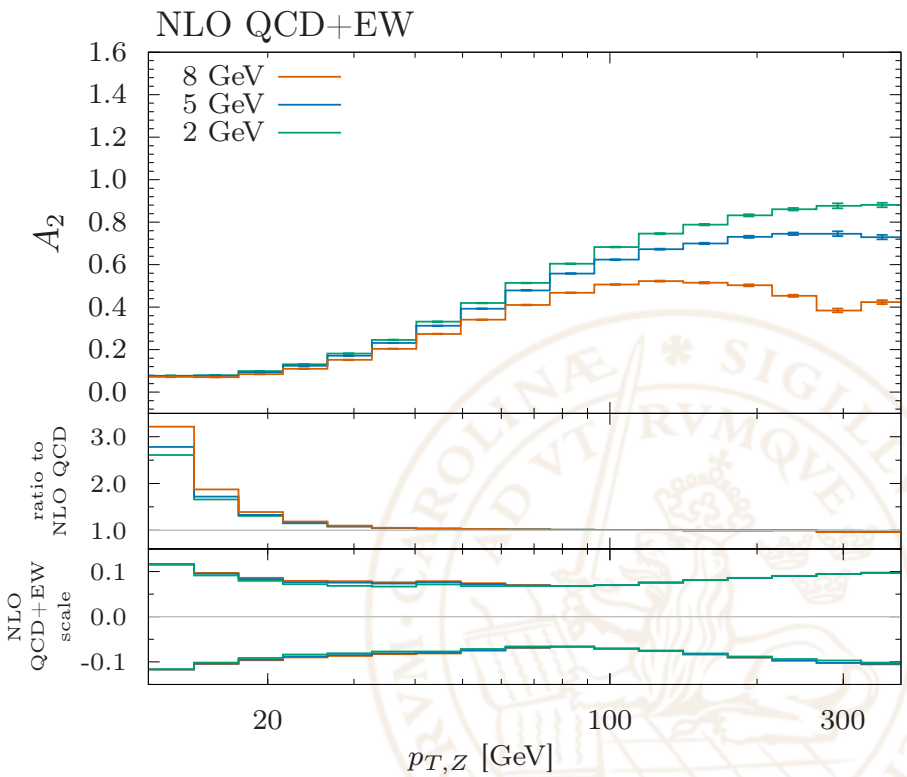




Decay coefficients for Z+jet: results

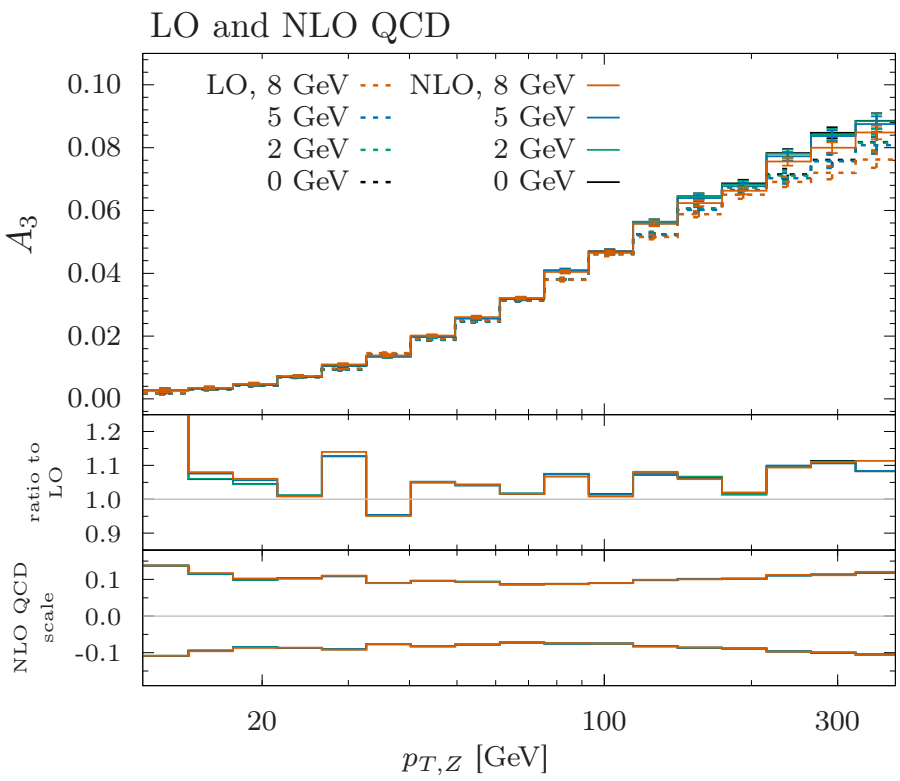
- Distributions for A_2

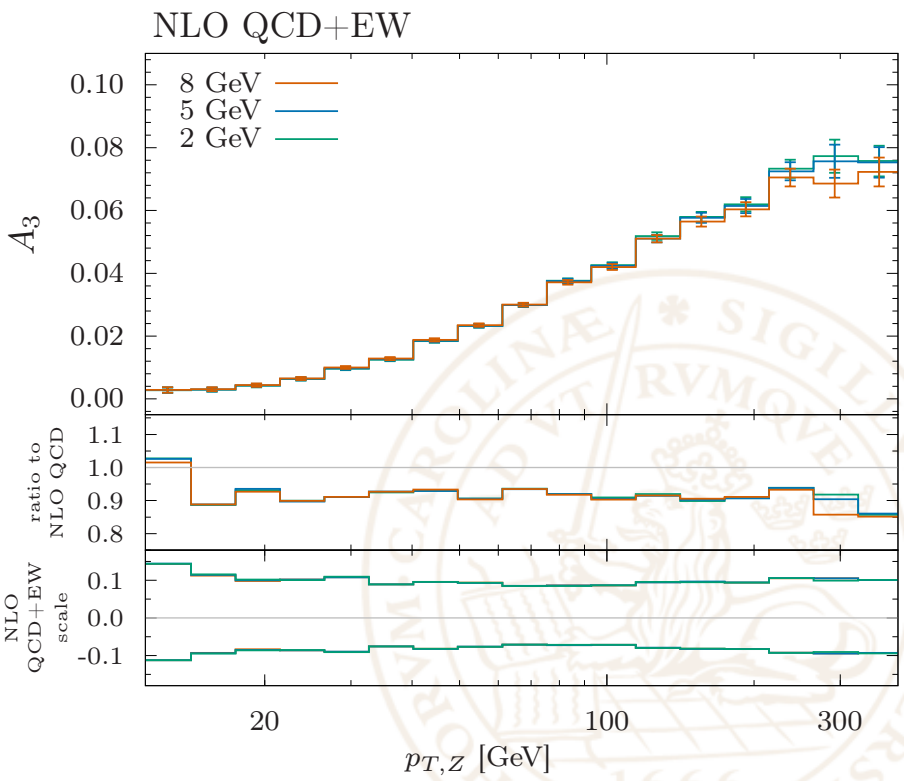




Decay coefficients for Z+jet: results

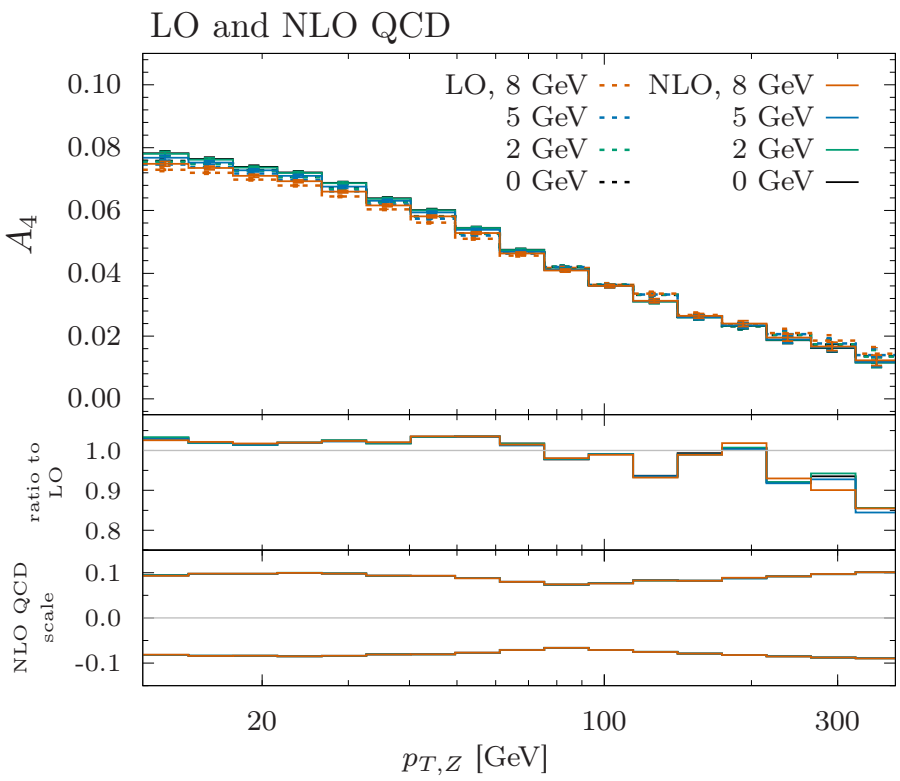
- Distributions for A_3
- Same -10% electroweak corrections

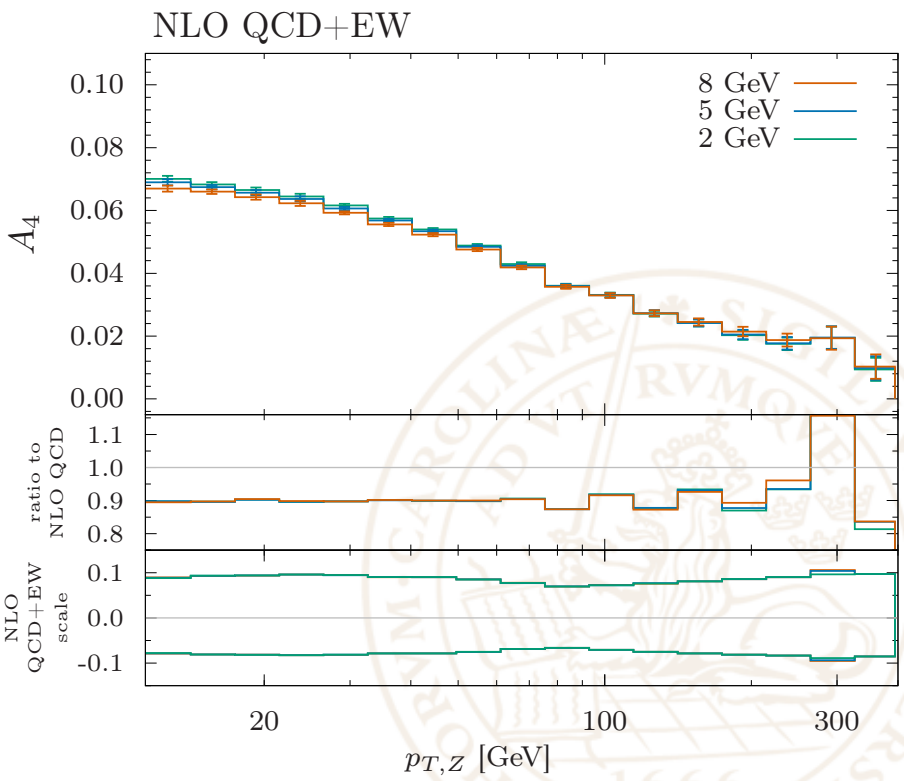




Decay coefficients for Z+jet: results

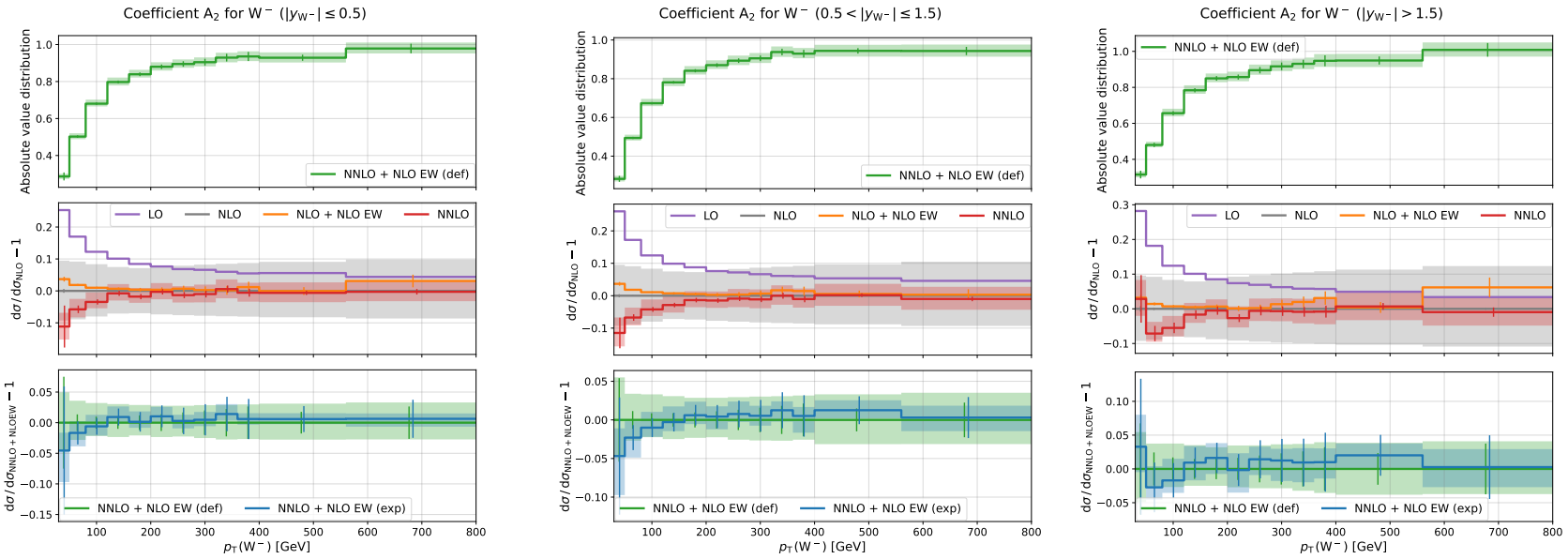
- Distributions for $\boldsymbol{A_4}$
- Same -10% electroweak corrections





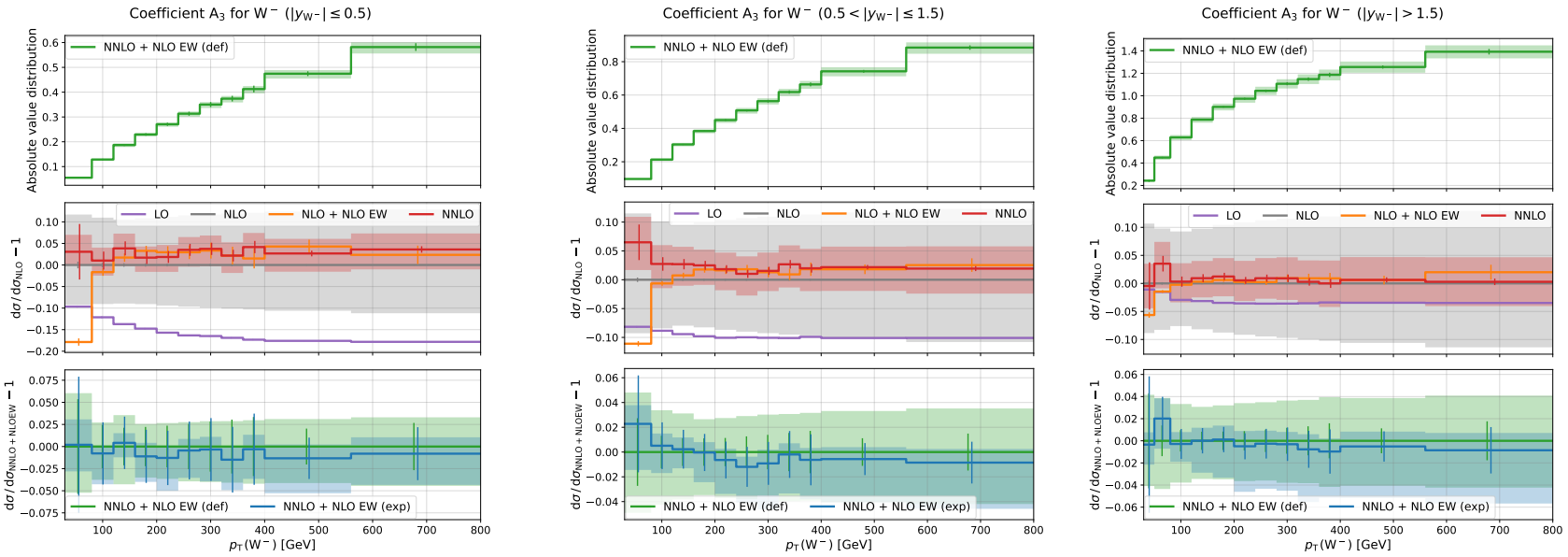
Decay coefficients for W+jet: results

- The coefficients A_2 in various rapidity bins



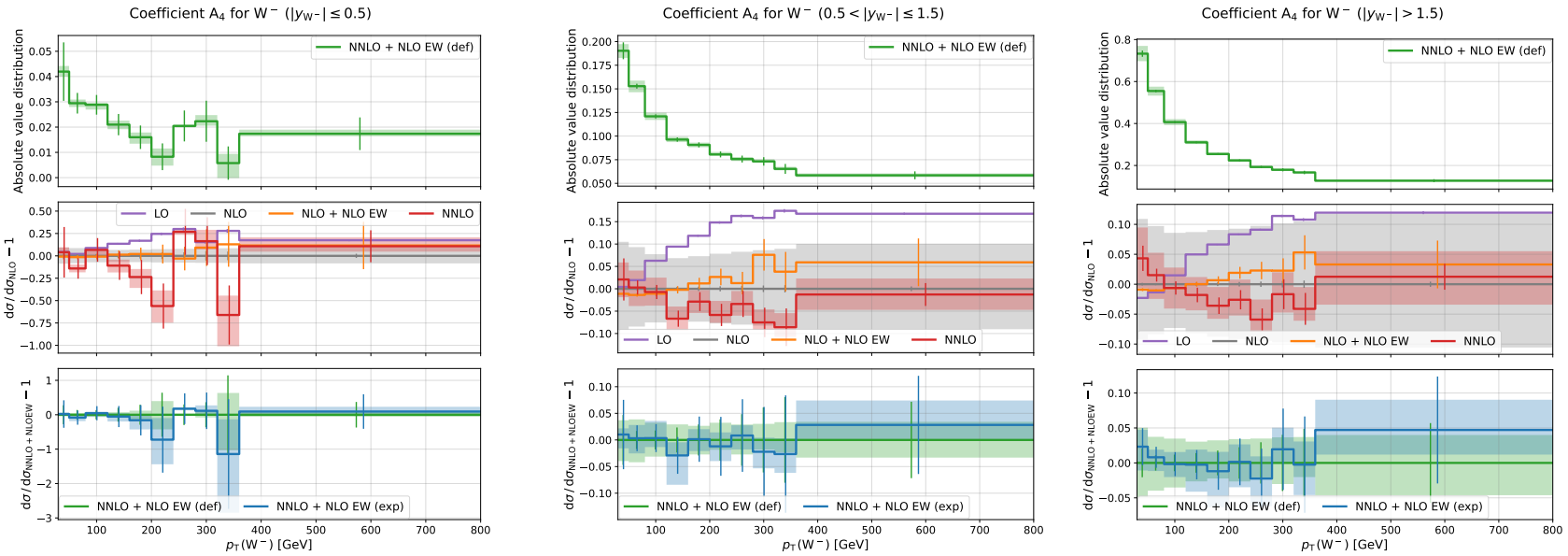
Decay coefficients for W+jet: results

- The coefficients A_3 in various rapidity bins



Decay coefficients for W+jet: results

- The coefficients A_4 in various rapidity bins



Colour matrix: useful identities

- Recall: Fierz identity

$$(T^a)_{ij}(T^a)_{kl} = T_R\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right) \tag{23}$$

set group index $T_R = 1$

Notation

Use $\mathcal{R}, \mathcal{Q}, \mathcal{S}, \mathcal{P}...$ to denote strings of fundamental generators

$$\mathcal{R} = T^{a_1}\,T^{a_2}...T^{a_r} \quad , \quad \tilde{\mathcal{R}} = T^{a_r}\,T^{a_{r-1}}...T^{a_1} \quad , \quad \mathrm{len}(\mathcal{R}) = r \tag{24}$$

Colour matrix: useful identities

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- Variations of the Fierz identity:

$$\text{Rule I:} \qquad \mathrm{Tr}[T^a\mathcal{R}]\mathrm{Tr}[T^a\mathcal{S}] = \mathrm{Tr}[\mathcal{R}\mathcal{S}] - \frac{1}{N_c}\mathrm{Tr}[\mathcal{R}]\mathrm{Tr}[\mathcal{S}], \tag{25}$$

$$\text{Rule II:} \qquad \mathrm{Tr}[\mathcal{R}\,T^a\mathcal{Q}\,T^a\mathcal{S}] = \mathrm{Tr}[\mathcal{Q}]\mathrm{Tr}[\mathcal{R}\mathcal{S}] - \frac{1}{N_c}\mathrm{Tr}[\mathcal{R}\mathcal{Q}\mathcal{S}], \tag{26}$$

$$\text{Rule IIb:} \qquad \mathrm{Tr}[\mathcal{R}\,T^a\,T^a\mathcal{S}] = N_c\mathrm{Tr}[\mathcal{R}\mathcal{S}] + \mathcal{O}(1/N_c) \tag{27}$$

Colour matrix: fundamental decomposition

For n-gluon amplitudes

- Matrix-element

$$\mathcal{M} = g^{n-2} \sum_{\sigma \in S_{n-1}} \mathrm{Tr}[T^{a_1} T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n-1)}}] \mathcal{A}(1, \sigma(1), \ldots, \sigma(n-1)) \tag{28}$$

Not a minimal set

Dual amplitudes are related by the Kleiss-Kuijf relation (dual Ward identity)

$$\mathcal{A}(1,2,3,4....,n) + \mathcal{A}(2,1,3,4,...,n) + ... + \mathcal{A}(2,3,4,...,1,n) = 0 \tag{29}$$

- Squared matrix-element

$$|\mathcal{M}|^2 = (g^2)^{n-2} \sum_{k,l=1}^{(n-1)!} C_{kl} \mathcal{A}(1, \sigma_k(1), \ldots, \sigma_k(n-1)) \left(\mathcal{A}(1, \sigma_l(1), \ldots, \sigma_l(n-1))\right)^*$$

- Colour matrix (size $(n-1)! \times (n-1)!$):

$$C_{kl} = \sum_{\text{col.}} \mathrm{Tr}[T^{a_1} T^{a_{\sigma_k(1)}} \ldots T^{a_{\sigma_k(n-1)}}] \left(\mathrm{Tr}[T^{a_1} T^{a_{\sigma_l(1)}} \ldots T^{a_{\sigma_l(n-1)}}]\right)^* \tag{30}$$

Colour matrix: fundamental decomposition

For n-gluon amplitudes

- Colour matrix

$$C_{kl} = \sum_{\text{col.}} \mathrm{Tr}[T^{a_1} T^{a_{\sigma_k(1)}} \dots T^{a_{\sigma_k(n-1)}}] \left(\mathrm{Tr}[T^{a_1} T^{a_{\sigma_l(1)}} \dots T^{a_{\sigma_l(n-1)}}]\right)^* \tag{31}$$

- LC: $\mathcal{O}(N_c^n)$, NLC: $\mathcal{O}(N_c^{n-2})$
- Leading-colour in all diagonal elements $\sigma_k = \sigma_l$

$$\left(N_c - \frac{1}{N_c}\right)^n + (N_c^2 - 1)\left(\frac{-1}{N_c}\right)^n = N_c^n + \mathcal{O}(N_c^{n-2}) \tag{32}$$

- Diagonal colour factors contain also NLC contribution

Colour matrix: fundamental decomposition

For n-gluon amplitudes

- NLC: for permutations which are related by a block interchange:[8]

$$\sigma_k \sim \mathcal{R}\mathcal{Q}_1\mathcal{S}\mathcal{Q}_2\mathcal{P} \quad , \quad \sigma_l \sim \mathcal{R}\mathcal{Q}_2\mathcal{S}\mathcal{Q}_1\mathcal{P} \tag{33}$$

with special cases if $\mathcal{S} = \mathbb{1}$

- If $\mathrm{len}(\mathcal{Q}_1) = \mathrm{len}(\mathcal{Q}_2) = 1$: $-N_C^{n-2} + \mathcal{O}(N_C^{n-4})$.
- If $\mathrm{len}(\mathcal{Q}_1) = 1$ *or* $\mathrm{len}(\mathcal{Q}_2) = 1$: $-N_C^{n-2} + \mathcal{O}(N_C^{n-4})$ if $\mathrm{len}(\mathcal{R}) = 1$ and $\mathrm{len}(\mathcal{P}) = 0$, otherwise not NLC
- If $\mathrm{len}(\mathcal{Q}_{1,2}) > 1$: $+N_C^{n-2} + \mathcal{O}(N_C^{n-4})$ if $\mathrm{len}(\mathcal{R}) \neq 1$ and $\mathrm{len}(\mathcal{P}) \neq 0$, otherwise not NLC

[8] A. Labane, aXiv:2008.13640

Colour matrix: results

For n-gluon amplitudes

Including the adjoint decomposition: matrix size $(n-2)! \times (n-2)!$

all-gluon n	Fundamental	Colour-flow	Adjoint
4	6 (6)	6 (6)	2 (2)
5	11 (24)	16 (24)	5 (6)
6	24 (120)	36 (120)	18 (24)
7	50 (720)	71 (720)	93 (120)
8	95 (5040)	127 (5040)	583 (720)
9	166 (40320)	211 (40320)	4162 (5040)
10	271 (362880)	331 (362880)	31649 (40320)
11	419 (3628800)	496 (3628800)	-
12	620 (39916800)	716 (39916800)	-
13	885 (479001600)	1002 (479001600)	-
14	1226 (6227020800)	1366 (6227020800)	-

Adjoint decomposition

For n-gluon amplitudes

- The amplitude is now

$$\mathcal{M} = \sum_{\sigma \in S_{n-2}} \left(F^{a_{\sigma(2)}} \ldots F^{a_{\sigma(n-1)}}\right)_{a_1 a_n} \mathcal{A}(1, \sigma(1), \ldots, \sigma(n), n), \tag{34}$$

  with $(F^a)_{bc} = if^{abc}$
- Minimal basis: $(n-2)!$ independent dual amplitudes
- Smaller colour matrix: but **LC not only on diagonal!**
- No found algorithm (yet) to get NLC elements

Colour-flow decompositions

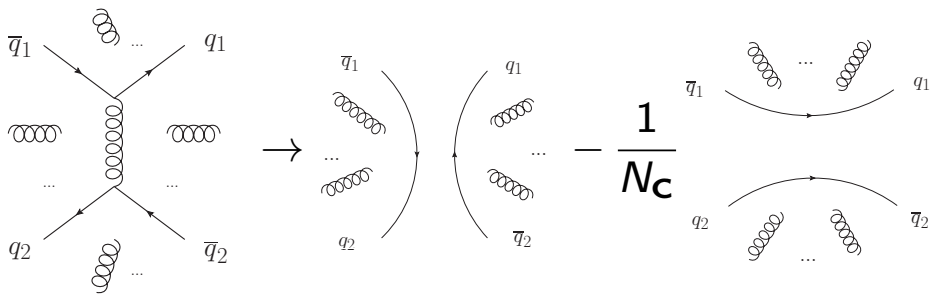
For one quark line plus n-gluon amplitudes: the full projection of U(1) gluons

$$
\begin{aligned}
\mathcal{M}_{1qq} = & \sum_{\sigma\in S_n} \delta^{i_q}_{j_{\sigma(\mathbf{1})}} \delta^{i_{\sigma(\mathbf{1})}}_{j_{\sigma(\mathbf{2})}} \dots \delta^{i_{\sigma(n-\mathbf{1})}}_{j_{\sigma(n)}} \delta^{i_{\sigma(n)}}_{j_q} \mathcal{A}_{1qq}(q,\sigma(1),\dots,\sigma(n),\overline{q}) \\
& + \left(\frac{-1}{N}\right) \sum_{\sigma\in S_n} \delta^{i_q}_{j_{\sigma(\mathbf{1})}} \delta^{i_{\sigma(\mathbf{1})}}_{j_{\sigma(\mathbf{2})}} \dots \delta^{i_{\sigma(n-\mathbf{1})}}_{j_q} \delta^{i_{\sigma(n)}}_{j_{\sigma(n)}} \mathcal{A}_{1qq}(q,\sigma(1),\dots,\sigma(n-1),\overline{q},\sigma(n)) \\
& + \left(\frac{-1}{N}\right)^2 \frac{1}{2!} \sum_{\sigma\in S_n} \delta^{i_q}_{j_{\sigma(\mathbf{1})}} \delta^{i_{\sigma(\mathbf{1})}}_{j_{\sigma(\mathbf{2})}} \dots \delta^{i_{\sigma(n-\mathbf{2})}}_{j_q} \delta^{i_{\sigma(n-\mathbf{1})}}_{j_{\sigma(n-\mathbf{1})}} \delta^{i_{\sigma(n)}}_{j_{\sigma(n)}} \\
& \qquad \mathcal{A}_{1qq}(q,\sigma(1),\dots,\sigma(n-2),\overline{q},\sigma(n-1),\sigma(n)) \\
& + \dots \\
& + \left(\frac{-1}{N}\right)^n \frac{1}{n!} \sum_{\sigma\in S_n} \delta^{i_q}_{j_q} \delta^{i_{\sigma(\mathbf{1})}}_{j_{\sigma(\mathbf{1})}} \dots \delta^{i_{\sigma(n-\mathbf{1})}}_{j_{\sigma(n-\mathbf{1})}} \delta^{i_{\sigma(n)}}_{j_{\sigma(n)}} \mathcal{A}_{1qq}(q,\overline{q},\sigma(1),\dots,\sigma(n)).
\end{aligned}
$$

Fundamental decompositions

For two **distinct flavour** quark pairs plus n-gluon amplitudes

- Now we have *two single colour lines* $\rightarrow$ internal U(1) gluon
- The internal gluon is decomposed into U($N_\mathbf{c}$) and U(1) part



(35)

- The two "quark-ordered" amplitudes

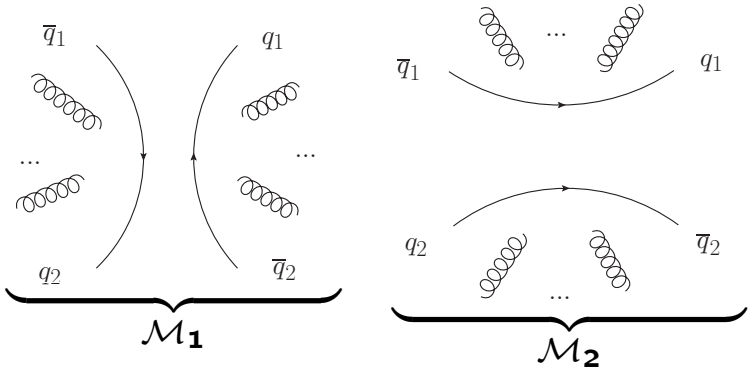
$$\mathcal{M}_{2qq} = \mathcal{M}_1 - \frac{1}{N_\mathbf{c}} \mathcal{M}_2 \tag{36}$$

- Decomposed as

$$\mathcal{M}_1 = \sum_{\sigma \in S_n} \sum_{n_\mathbf{1}=0}^{n} c_1(\sigma, n_1) \mathcal{A}_1(\sigma, n_1) \quad , \quad \mathcal{M}_2 = \sum_{\sigma \in S_n} \sum_{n_\mathbf{1}=0}^{n} c_2(\sigma, n_1) \mathcal{A}_2(\sigma, n_1)$$

Fundamental decompositions

For two **distinct flavour** quark pairs plus n-gluon amplitudes



(37)

- The colour factors

$$c_1(\sigma) = \left(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n_1)}}\right)_{i_1 j_2} \left(T^{a_{\sigma(n_1+1)}} \dots T^{a_{\sigma(n)}}\right)_{i_2 j_1} \tag{38}$$

$$c_2(\sigma) = \left(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n_1)}}\right)_{i_1 j_1} \left(T^{a_{\sigma(n_1+1)}} \dots T^{a_{\sigma(n)}}\right)_{i_2 j_2} \tag{39}$$

- The squared amplitude

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2} \sum_{\sigma_k, \sigma_l \in S_{n+1}} \begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) \end{pmatrix} \begin{pmatrix} c_1(\sigma_k) c_1(\sigma_l)^* & -c_1(\sigma_k) c_2(\sigma_l)^* / N_C \\ -c_2(\sigma_k) c_1(\sigma_l)^* / N_C & c_2(\sigma_k) c_2(\sigma_l)^* / N_C^2 \end{pmatrix} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \end{pmatrix} \tag{40}$$

Fundamental decompositions

For two **distinct flavour** quark pairs plus n-gluon amplitudes

- Note: **not all diagonal elements the same type now!**
- Leading-colour $\mathcal{O}(N_{\mathbf{C}}^{n+2})$:

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2} \sum_{\sigma_k,\sigma_l \in S_{n+\mathbf{1}}}$$

$$\begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) \end{pmatrix} \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^* & -c_1(\sigma_k)c_2(\sigma_l)^*/N_{\mathbf{C}} \\ -c_2(\sigma_k)c_1(\sigma_l)^*/N_{\mathbf{C}} & c_2(\sigma_k)c_2(\sigma_l)^*/N_{\mathbf{C}}^2 \end{pmatrix} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \end{pmatrix}$$

  if $\sigma_k = \sigma_l$
- NLC terms $\mathcal{O}(N_{\mathbf{C}}^{n})$, investigate block-by-block: appears in each block

Fundamental decompositions

For two same flavour quark pairs plus n-gluon amplitudes

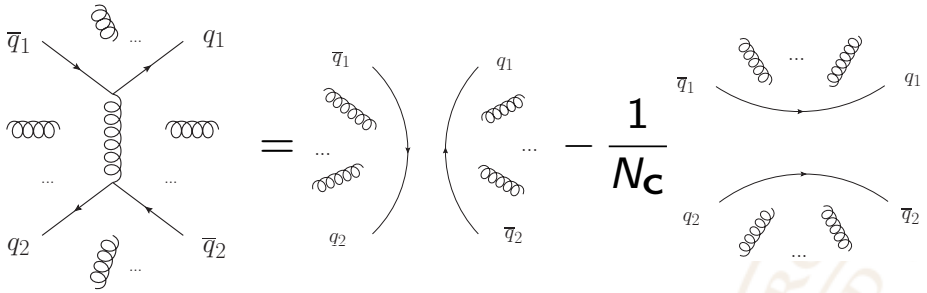
- Both a t- and s-channel contribution

$$\mathcal{M}_{2qq}(\overline{q}q\overline{q}q+ng) = \hat{\mathcal{M}}(\overline{q}_1 q_1 \overline{q}_2 q_2 + ng) - \hat{\mathcal{M}}(\overline{q}_1 q_2 \overline{q}_2 q_1 + ng) \tag{41}$$

(minus sign from Fermi statistics)

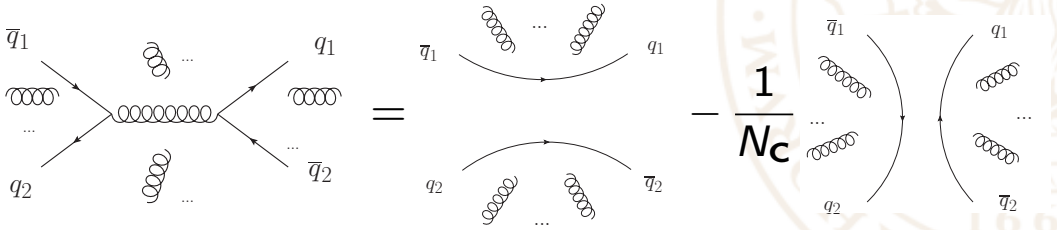
- Decomposed

$\hat{\mathcal{M}}(\overline{q}_1 q_1 \overline{q}_2 q_2 + ng) =$



(42)

$\hat{\mathcal{M}}(\overline{q}_1 q_2 \overline{q}_2 q_1 + ng) =$



Fundamental decompositions

For two **same flavour** quark pairs plus n-gluon amplitudes

- So then

$$\mathcal{M}_{2qq} = \left(1 + \frac{1}{N_\mathbf{C}}\right)(\mathcal{M}_1 - \mathcal{M}_2). \tag{43}$$

- Squared-matrix:

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2}\left(1 + \frac{1}{N_\mathbf{C}}\right)^2 \sum_{\sigma_k,\sigma_l \in S_{n+\mathbf{1}}} \begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) \end{pmatrix} \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^* & c_1(\sigma_k)c_2(\sigma_l)^* \\ c_2(\sigma_k)c_1(\sigma_l)^* & c_2(\sigma_k)c_2(\sigma_l)^* \end{pmatrix} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \end{pmatrix} \tag{44}$$

- Colour factors include an extra factor $\left(1 + \frac{1}{N_\mathbf{C}}\right)^2$ here
  $\rightarrow$ LC: $\mathcal{O}(N_\mathbf{C}^{n+2})$, non-zero $\mathcal{O}(N_\mathbf{C}^{n+1})$

Fundamental decompositions

For two **same flavour** quark pairs plus n-gluon amplitudes

- Note: **diagonal elements symmetrized now!**
- Leading-colour $\mathcal{O}(N_c^{n+2})$:

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2}\left(1+\frac{1}{N_c}\right)^2 \sum_{\sigma_k,\sigma_l\in S_{n+1}} \begin{pmatrix}\mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k)\end{pmatrix} \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^* & c_1(\sigma_k)c_2(\sigma_l)^* \\ c_2(\sigma_k)c_1(\sigma_l)^* & c_2(\sigma_k)c_2(\sigma_l)^* \end{pmatrix} \begin{pmatrix}\mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^*\end{pmatrix} \quad (45)$$

  if $\sigma_k = \sigma_l$
- NLC terms $\mathcal{O}(N_c^{n+1}) + \mathcal{O}(N_c^n)$, investigate block-by-block: appears in every block

Colour decompositions

For two **distinct flavour** quark pairs plus n-gluon amplitudes

- Same set of dual amplitudes as for fundamental decomposition

$$\mathcal{M}_{2qq} = \mathcal{M}_1 - \frac{1}{N_\mathbf{c}} \mathcal{M}_2 \tag{46}$$

- Once again, external gluons are projected out

$$\mathcal{M}_1 \to \mathcal{M}_1 - \frac{1}{N_\mathbf{c}} \sum_{\bar{\sigma} \in \bar{S}_{n+\mathbf{1}}} c_1^1(\bar{\sigma}) \mathcal{A}_1^1(\bar{\sigma}), \tag{47}$$

- For NLC, it turns out that a single U(1) projection is enough
- Colour factor for this dual amplitude

$$c_1^1(\bar{\sigma}) \quad = \delta^{i_{q_\mathbf{1}}}_{j_{\sigma(\mathbf{1})}} \ldots \delta^{i_{\sigma(n)}}_{j_{q_\mathbf{1}}} \delta^{i_{\sigma(n+\mathbf{1})}}_{j_{\sigma(n+\mathbf{1})}}, \tag{48}$$

with colourless external U(1) indices

Colour decompositions

For two **distinct flavour** quark line plus n-gluon amplitudes

- Matrix element

$$\mathcal{M}_{2qq} = \sum_{\sigma\in S_{n+1}} c_1(\sigma)\mathcal{A}_1(\sigma) - \frac{1}{N_c}\sum_{\sigma\in S_{n+1}} c_2(\sigma)\mathcal{A}_2(\sigma) - \frac{1}{N_c}\sum_{\bar\sigma\in \bar S_{n+1}} c_1^1(\bar\sigma)\mathcal{A}_1^1(\bar\sigma),$$

- Squared matrix-element

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n-2}\sum_{\sigma_k,\sigma_l}\begin{pmatrix}\mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) & \mathcal{A}_1^1(\bar\sigma_k)\end{pmatrix}$$

$$\begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^\dagger & -c_1(\sigma_k)c_2(\sigma_l)^\dagger/N_c & -c_1(\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_c \\ -c_2(\sigma_k)c_1(\sigma_l)^\dagger/N_c & c_2(\sigma_k)c_2(\sigma_l)^\dagger/N_c^2 & c_2(\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_c^2 \\ -c_1^1(\bar\sigma_k)c_1(\sigma_l)^\dagger/N_c & c_1^1(\bar\sigma_k)c_2(\sigma_l)^\dagger/N_c^2 & c_1^1(\bar\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_c^2 \end{pmatrix}\begin{pmatrix}\mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \\ \mathcal{A}_1^1(\bar\sigma_l)^*\end{pmatrix},$$

- Leading-colour (N_c^{n+2}) for $\sigma_k = \sigma_l$
- NLC (N_c^n) needs a careful analysis block-by-block

Colour decompositions

For two **same flavour** quark line plus n-gluon amplitudes

- Very similar to the distinct flavour case, but we also need to U(1) project the $\mathcal{M}_2$ amplitude

$$\mathcal{M}_2 \to \mathcal{M}_2 - \frac{1}{N_C} \sum_{\bar{\sigma} \in \bar{S}_{n+1}} c_2^1(\bar{\sigma}) \mathcal{A}_2^1(\bar{\sigma}), \tag{49}$$

- 2: Squared matrix-element

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n-2} \left(1 + \frac{1}{N_C}\right)^2$$

$$\sum_{\sigma_k, \sigma_l} \begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) & \mathcal{A}_1^1(\bar{\sigma}_k) & \mathcal{A}_2^1(\bar{\sigma}_k) \end{pmatrix} \mathbb{C} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \\ \mathcal{A}_1^1(\bar{\sigma}_l)^* \\ \mathcal{A}_2^1(\bar{\sigma}_l)^* \end{pmatrix} \tag{50}$$

(Again, colourr factors no longer monomials in N_C)

Colour decompositions

For two **same flavour** quark line plus n-gluon amplitudes

- Colour matrix

$$\mathbb{C} = \begin{pmatrix} c_{\mathbf{1}}(\sigma_k)c_{\mathbf{1}}(\sigma_l)^\dagger & -c_{\mathbf{1}}(\sigma_k)c_{\mathbf{2}}(\sigma_l)^\dagger & -c_{\mathbf{1}}(\sigma_k)c_{\mathbf{1}}^{\mathbf{1}}(\bar{\sigma}_l)^\dagger/N_{\mathbf{C}} & c_{\mathbf{1}}(\sigma_k)c_{\mathbf{2}}^{\mathbf{1}}(\bar{\sigma}_l)^\dagger/N_{\mathbf{C}} \\ -c_{\mathbf{2}}(\sigma_k)c_{\mathbf{1}}(\sigma_l)^\dagger & c_{\mathbf{2}}(\sigma_k)c_{\mathbf{2}}(\sigma_l)^\dagger & -c_{\mathbf{2}}(\sigma_k)c_{\mathbf{1}}^{\mathbf{1}}(\bar{\sigma}_l)^\dagger/N_{\mathbf{C}} & -c_{\mathbf{2}}(\sigma_k)c_{\mathbf{2}}^{\mathbf{1}}(\bar{\sigma}_l)^\dagger/N_{\mathbf{C}} \\ -c_{\mathbf{1}}^{\mathbf{1}}(\bar{\sigma}_k)c_{\mathbf{1}}(\sigma_l)^\dagger/N_{\mathbf{C}} & c_{\mathbf{1}}^{\mathbf{1}}(\bar{\sigma}_k)c_{\mathbf{2}}(\sigma_l)^\dagger/N_{\mathbf{C}} & c_{\mathbf{1}}^{\mathbf{1}}(\bar{\sigma}_k)c_{\mathbf{1}}^{\mathbf{1}}(\bar{\sigma}_l)^\dagger/N_{\mathbf{C}}^{\mathbf{2}} & -c_{\mathbf{1}}^{\mathbf{1}}(\bar{\sigma}_k)c_{\mathbf{2}}^{\mathbf{1}}(\bar{\sigma}_l)^\dagger/N_{\mathbf{C}}^{\mathbf{2}} \\ c_{\mathbf{2}}^{\mathbf{1}}(\bar{\sigma}_k)c_{\mathbf{1}}(\sigma_l)^\dagger/N_{\mathbf{C}} & -c_{\mathbf{2}}^{\mathbf{1}}(\bar{\sigma}_k)c_{\mathbf{2}}(\sigma_l)^\dagger/N_{\mathbf{C}} & -c_{\mathbf{2}}^{\mathbf{1}}(\bar{\sigma}_k)c_{\mathbf{1}}^{\mathbf{1}}(\bar{\sigma}_l)^\dagger/N_{\mathbf{C}}^{\mathbf{2}} & c_{\mathbf{2}}^{\mathbf{1}}(\bar{\sigma}_k)c_{\mathbf{2}}^{\mathbf{1}}(\bar{\sigma}_l)^\dagger/N_{\mathbf{C}}^{\mathbf{2}} \end{pmatrix}. \tag{51}$$

- Leading-colour: $\mathcal{O}(N_{\mathbf{C}}^{n+1})$ on first two block diagonal elements
- NLC $\mathcal{O}(N_{\mathbf{C}}^{n})$ is examined block-by-block

Non-zero elements without phase-space symmetrisation

$q\bar{q}\,Q\bar{Q} + ng$												Fundamental: $\mathcal{A}_1 \mid \mathcal{A}_2$ types
n	$\min(n_1, n-n_1)$ 0		1		2		3		4		5	
0	2	2										(2)
1	3	3										(4)
2	7	4	6	5								(12)
3	15	5	15	7								(48)
4	31	6	32	9	33	10						(240)
5	60	7	62	11	64	13						(1440)
6	108	8	111	13	114	16	115	17				(10080)
7	182	9	186	15	190	19	192	21				(80640)
8	290	10	295	17	300	22	303	25	304	26		(725760)
9	441	11	447	19	453	25	457	29	459	31		(7257600)
10	645	12	652	21	659	28	664	33	667	36	668	37 (79833600)

Table: Number of non-zero elements in a single row of the colour matrix for $q\bar{q}\,Q\bar{Q} + ng$ (distinct flavours) up to NLC accuracy, $\mathcal{O}(N_c^n)$ in the fundamental representation.

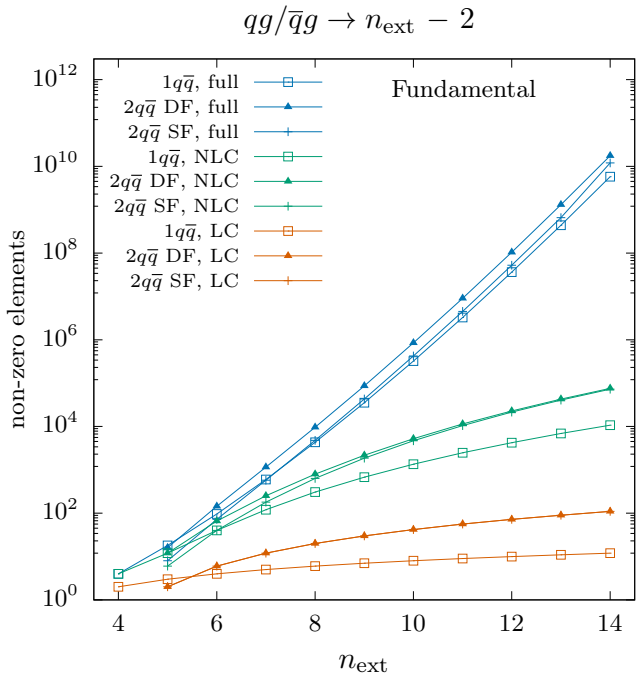
Non-zero elements without phase-space symmetrisation

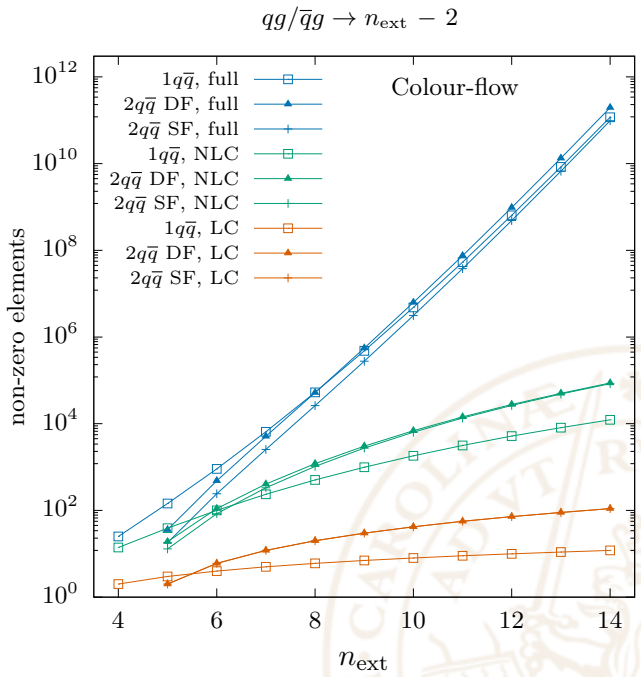
$q\bar{q}\,Q\overline{Q}+ng$												Colour-flow: $\mathcal{A}_1$, $\mathcal{A}_1^1$ | $\mathcal{A}_2$ types
n	$\min(n_1, n-n_1)$ 0		1		2		3		4		5	
0	2 , -	2										(2)
1	5 , 3	3										(6)
2	11 , 4	4	12 , -	5								(22)
3	23 , 5	5	25 , 5	7								(98)
4	45 , 6	6	48 , 6	9	49 , -	10						(522)
5	82 , 7	7	86 , 7	11	88 , 7	13						(3262)
6	140 , 8	8	145 , 8	13	148 , 8	16	149 , -	17				(23486)
7	226 , 9	9	232 , 9	15	236 , 9	19	238 , 9	21				(191802)
8	348 , 10	10	355 , 10	17	360 , 10	22	363 , 10	25	364 , -	25		(1753618)
9	515 , 11	11	523 , 11	19	529 , 11	25	533 , 11	29	535 , 11	31		(17755382)
10	737 , 12	12	746 , 12	21	753 , 12	28	758 , 12	33	761 , 12	36	762 , -	37 (197282022)

Table: Number of non-zero elements in a single row of the colour matrix for $q\bar{q}\,Q\overline{Q}+ng$ (distinct flavours) up to NLC accuracy in the colour-flow representation

Colour matrix: results for qg initiated

- Blue: full colour Green: NLC Red: LC

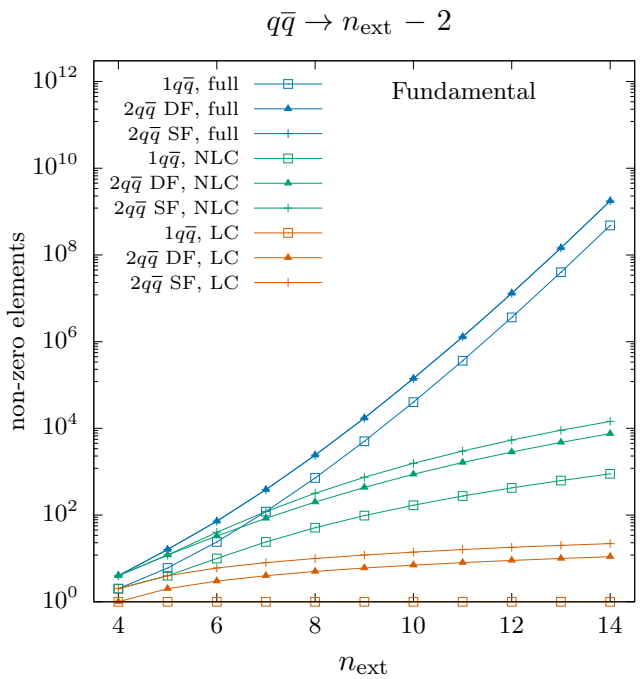


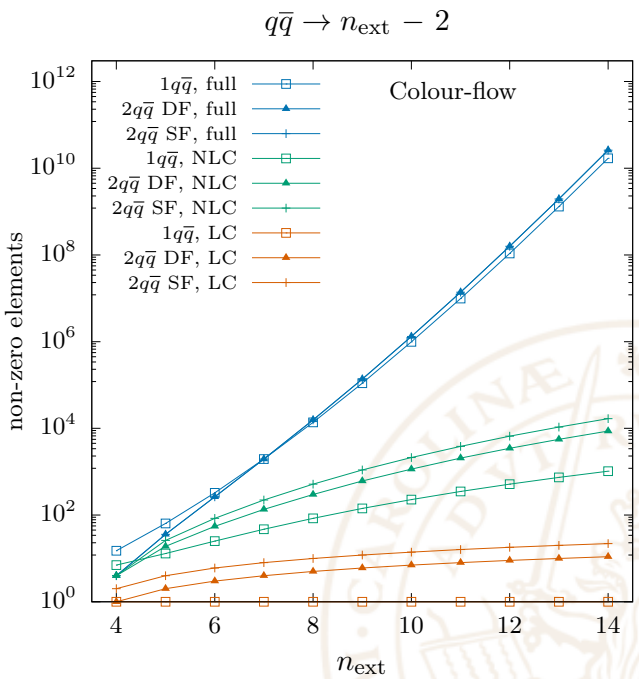


- Factorial growth for full-colour
- Polynomial scaling with n_{ext} for both LC and NLC ($\sim n_{\text{ext}}^4$)

Colour matrix: results for $q\bar{q}$ initiated

- Blue: full colour Green: NLC Red: LC

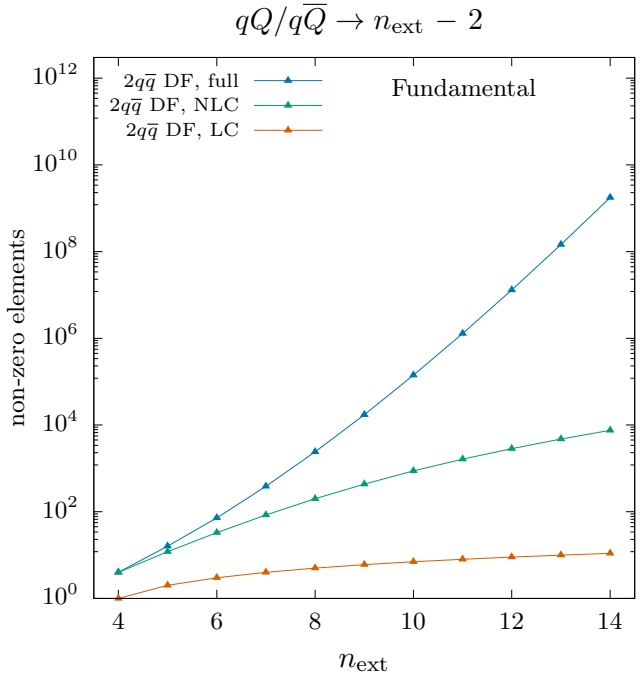


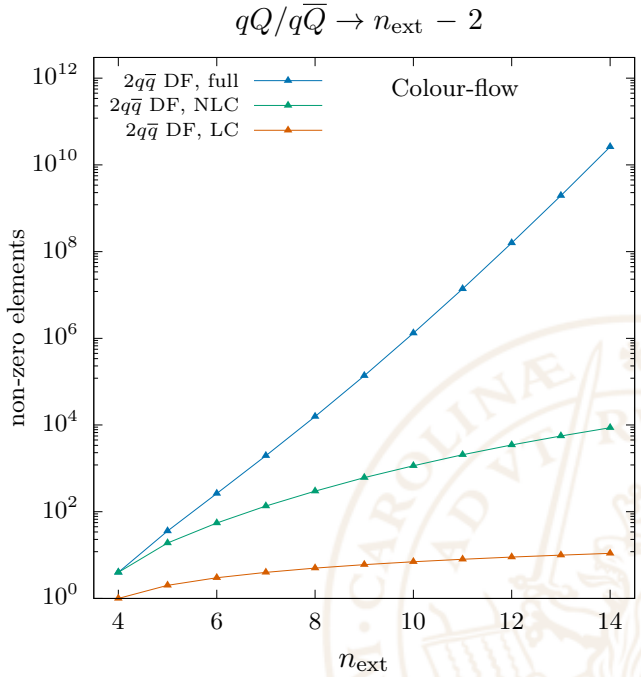


- Colour-flow very slightly less efficient than fundamental decomposition

Colour matrix: results for $qQ/q\overline{Q}$ initiated

- Blue: full colour Green: NLC Red: LC

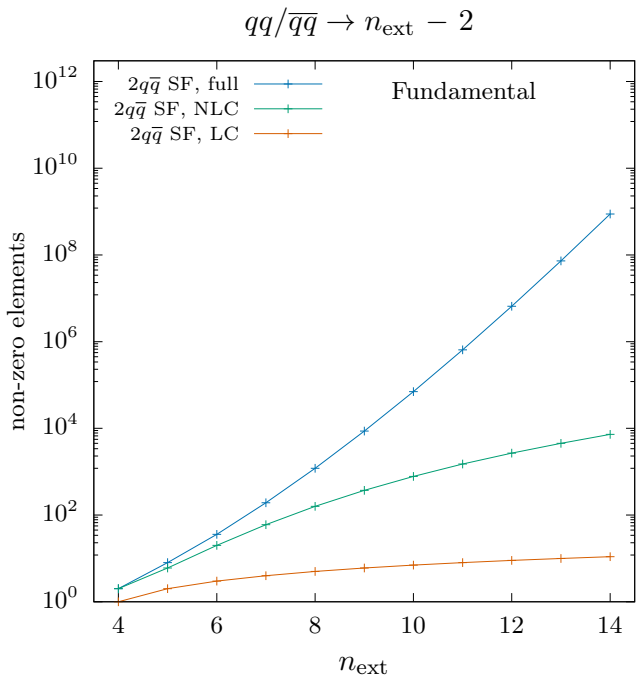


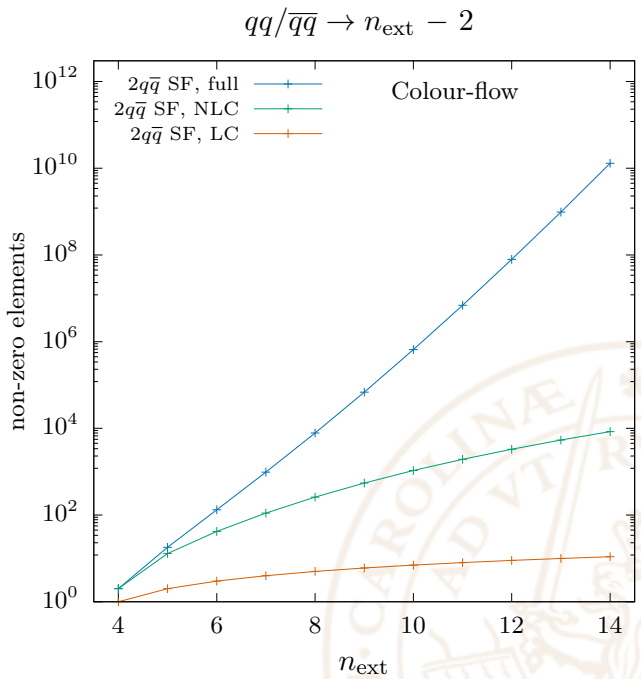


- Already a good efficiency improvement for NLC at $n_{\text{ext}} \sim 6$

Colour matrix: results for $qq/\overline{qq}$ initiated

- Blue: full colour Green: NLC Red: LC





- Same-flavour case has slightly more elements to consider because of symmetrisation of $\mathcal{M}_{1,2}$ amplitudes