A toy model in $M_n(\mathbb{C})$ for selective measurements in QM

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Abstract

The non-selective and selective measurements of a self-adjoint observable A in quantum mechanics are interpreted as 'jumps' of the state of the measured system into a decohered or pure state characterized by the spectral projections of A. However, one may try to describe the measurement results as asymptotic states of a dynamical process, where the non-unitarity of time evolution arises as an effective description of the interaction with the measuring device. The dynamics we present is a two-step dynamics: the first step is the non-selective measurement or decoherence, which is known to be described by the linear Lindblad equation, where the generator of the time evolution is the generator of a semigroup of unit preserving completely positive maps. The second step is a process from the resulted decohered state to a pure state, which is described by an effective non-linear toy model dynamics: the pure states arise as asymptotic fixed points, and their emergent probabilities are the relative volumes of their attractor regions.

Content

Measurements in quantum mechanics

- self-adjoint observable $M = \sum_{m\in\sigma(M)} mP_m \in \mathcal{B}(\mathcal{H})$
- prepared state $\omega\colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}$
  - non-selective measurement:
    $\omega \mapsto \omega \circ \Phi_M, \quad \Phi_M(A) := \sum_{m\in\sigma(M)} P_m A P_m \in \langle M\rangle'$
    (H): "jump" into the commutant $\langle M\rangle' \subset \mathcal{B}(\mathcal{H})$ containing the generated abelian subalgebra $\langle M\rangle$
    (S): an $\langle M\rangle$-decohered repreparation of a state
  - selective measurement:
    $\omega \mapsto \omega \circ \Phi_m, \quad \Phi_m(A) := P_m A P_m/\omega(P_m)$ with probability $\omega(P_m)$
    (H): "jump" into a spectral projecion P_m of M with probability $\omega(P_m)$
    (S): "jump" into a pure state of M with probability $\omega(P_m)$
    probability $\omega(P_m)$ = relative frequency in repeated experiments with prepared state ω

- both measurement "jumps" destroy unitary implemented dynamics
(H): $\alpha\colon (\mathbb{R},+) \to \mathrm{Aut}\,\mathcal{B}(\mathcal{H})$, such that $\alpha_t(A) := U_t^* A U_t, \ U_t \in \mathcal{U}(\mathcal{H})$
(S): $\omega_t := \omega \circ \alpha_t, \ t \in \mathbb{R}$
and are not unitary implementable, selective is not even deterministic
- however both Φ_M and Φ_m are completely positive (CP) maps
$\Phi \otimes \mathrm{Id}_n\colon \mathcal{B}(\mathcal{H}) \otimes M_n \to \mathcal{B}(\mathcal{H}) \otimes M_n$ is positive (linear) $\forall\, n \in \mathbb{N}$

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Connection between CP maps and subsystems in QM

S=subsystem and the E=environment in QM: $\mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_E) \simeq \mathcal{B}(\mathcal{H}_S) \otimes \mathcal{B}(\mathcal{H}_E)$

- full system $\to$ subsystem
  if $U_t \in \mathcal{U}(\mathcal{H}_S \otimes \mathcal{H}_E), t \in \mathbb{R}$ is a unitary dynamics on the full system then

  $$\mathcal{B}(\mathcal{H}_S) \ni A \mapsto \Phi_t(A) := \mathrm{Tr}_E \left[(\mathbf{1}_S \otimes \rho_E) U_t^* (A \otimes \mathbf{1}_E) U_t \right] \in \mathcal{B}(\mathcal{H}_S)$$

  unit preserving CP map on $\mathcal{B}(\mathcal{H}_S)$ $\forall\, t \in \mathbb{R}$
  $\Rightarrow$ one may look for a "CP-dynamics" on the subsystem instead of a unitary one
- subsystem $\to$ extended (= full) system
  If Φ unit preserving σ-weakly continuous CP map on $\mathcal{B}(\mathcal{H}_S) \Rightarrow$
  $\exists\, \mathcal{H}_E$ and V isometry on $\mathcal{H}_S \otimes \mathcal{H}_E$ such that $\forall\, \rho_E \in \mathcal{S}(\mathcal{H}_E)$

  $$\Phi(A) = \mathrm{Tr}_E \left[(\mathbf{1}_S \otimes \rho_E) V^* (A \otimes \mathbf{1}_E) V \right], \quad A \in \mathcal{B}(\mathcal{H}_S)$$

  (V can be made unitary by a ρ_E-dependent further extension of $\mathcal{H}_E$)
  $\Rightarrow$ every CP map on the subsystem comes from a restriction of a isometric/unitary sandwiching on a full system

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Generator of a special CP dynamics: Lindblad operator

- restriction on CP dynamics: special family of CP maps
  • form a semigroup: $\Phi_t \circ \Phi_s = \Phi_{t+s}$; $t, s \in \mathbb{R}_+$,
  • has a bounded generator L: $\Phi_t = \exp(tL)$
  latter is not a restriction if $\mathcal{B}(\mathcal{H}) = \mathcal{M}_n(\mathbb{C})$
- Theorem (Lindblad; 1976) on the generator of a CP_1 semigroup
  Let $L\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ bounded linear *-map.
  $\Phi_t := \exp(tL) \in \mathrm{CP}_1(\mathcal{B}(\mathcal{H}))_\sigma, t \geq 0 \Leftrightarrow L$ has the form

$$L(A) = i[H, A] + \sum_k V_k^* A V_k - \frac{1}{2}\{V_k^* V_k, A\}, \quad A \in \mathcal{B}(\mathcal{H}),$$

  where $H = H^*$; V_k, $\sum_k V_k^* V_k \in \mathcal{B}(\mathcal{H})$.
- Lindblad equation: generalization of the Schrödinger equation
  • $\omega\colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ normal state with density matrix ρ: $\omega(A) = \mathrm{Tr}\,(\rho A)$
  • $H \leftrightarrow S$ picture change: $\mathrm{Tr}\,(\hat{L}(\rho)A) := \mathrm{Tr}\,(\rho L(A))$

$$\frac{d\rho}{dt} = \hat{L}(\rho) := -i[H, \rho] + \sum_k V_k \rho V_k^* - \frac{1}{2}\{V_k^* V_k, \rho\}.$$

linear first order differential equation on density matrices

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GP effective one particle state in Bose–Einstein condensation

- Trapped interacting N-boson Hamiltonian in 3D: $\mathcal{H}^{\otimes N}, \mathcal{H} := L^2(\mathbb{R}^3)$

$$\tilde{H}_N = \sum_{j=1}^{N} (-\Delta_{\mathbf{r}_j} + V_{ext}(\mathbf{r}_j)) + \sum_{i<j}^{N} V_N(\mathbf{r}_i - \mathbf{r}_j)$$

  • $0 < V_{ext}(\mathbf{r}) \to \infty, |\mathbf{r}| \to \infty$
  • $0 < V_N(\mathbf{r}) = V_N(|\mathbf{r})| = N^2 V(N|\mathbf{r}|)$
  smooth with compact support and scattering length $a = a_0/N$

- Conjectured effective one-particle description: Gross–Pitaevskii equation and energy functional in $\mathcal{H}$

$$i\partial_t \varphi(t) = -\Delta\varphi(t) + \sigma|\varphi(t)|^2\varphi(t), \quad \varphi(t) \in \mathcal{H}, \|\varphi\| = 1$$

$$E_{GP}(\varphi) := \int d^3r (|\nabla\varphi(\mathbf{r})|^2 + V_{ext}(\mathbf{r})|\varphi(\mathbf{r})|^2 + 4\pi a_0|\varphi(\mathbf{r})|^4), \ \|\varphi\| = 1$$

- Theorem (Lieb, Seiringer; 2002) on BE-condensation
  Let ψ_N be the ground state of $\tilde{H}_N$ and let $\gamma_N^{(k)}, 1 \leq k \leq N$ be its k-particle marginal density operator. Let $\sigma := 8\pi Na$ in the GP equation and let φ_{GP} be the minimizer of E_{GP}. Then

$$\gamma_N^{(k)} \to |\varphi_{GP}\rangle\langle\varphi_{GP}|^{k\otimes}, \quad N \to \infty$$

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GP effective nonlinear dynamics after Bose–Einstein condensation

- N-particle Hamiltonian with trap removed

$$H_N = \sum_{j=1}^{N} -\Delta_{\mathbf{r}_j} + \sum_{i<j}^{N} V_N(\mathbf{r}_i - \mathbf{r}_j)$$

- Theorem (Erdős, Schlein, Yau; 2007) on GP-dynamics
  Let $\psi_N(t)$ be the solution of the Schrödinger equation $i\partial_t\psi_N(t) = H_N\psi_N(t)$ with initial condition $\psi_N(0) := \psi_N$ and let $\gamma_N^{(1)}(t)$ be its one-particle marginal density. Then for any $t \geq 0$

$$\gamma_N^{(1)}(t) \to |\varphi(t)\rangle\langle\varphi(t)|, \quad N \to \infty$$

pointwise for compact operators on $\mathcal{H}$, where φ_t solves the GP-equation

$$i\partial_t\varphi(t) = -\Delta\varphi(t) + 8\pi a_0|\varphi(t)|^2\varphi(t)$$

with initial condition $\varphi(0) := \varphi_{GP}$.

Two types of effective dynamics in selective measurements (SM)

- Instead of "jumps" try a "very fast" dynamical description of SM:
  SM result should be an asymptotic state of an effective dynamics caused by the interaction of the measured (sub)system with the measuring device
  - no modification of "fundamental" dynamics of quantum theories
  - technical restriction: measured (sub)systems live in finite dimensional Hilbert spaces $\Rightarrow M = M^* = \sum_{m\in\sigma(M)} mP_m \in \mathcal{B}(\mathcal{H}) \simeq M_n(\mathbb{C})$
- two types of effective dynamics for density matrices (S-picture) $\rho(t) \in S_n := M_n(\mathbb{C})_{+1}$ in two asymptotic steps
  1. linear deterministic CP_1-dynamics with M-decohered asymptotic state (non-selective measurement) :

$$\rho_0 \to \lim_{t\to\infty} \rho(t) =: \rho_\infty = \sum_{m\in\sigma(M)} P_m \rho_0 P_m$$

  2. "randomly chosen" nonlinear deterministic dynamics with M-pure asymptotic states P_m in $S_M := S_{n|\langle M\rangle}$ with probability $p_m := \mathrm{Tr}\,(\rho_0 P_m)$

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  1. linear deterministic CP_1-dynamics with M-decohered asymptotic state (non-selective measurement) :

$$\rho_0 \to \lim_{t\to\infty} \rho(t) =: \rho_\infty = \sum_{m\in\sigma(M)} P_m\rho_0 P_m$$

  2. "randomly chosen" nonlinear deterministic dynamics with M-pure asymptotic states P_m in $S_M := S_{n|\langle M\rangle}$ with probability $p_m := \mathrm{Tr}\,(\rho_0 P_m)$

$$S_M \ni \rho_{\infty|\langle M\rangle} =: \mu_0 \to \lim_{t\to\infty} \mu(t) =: \mu_\infty = P_m$$

1. CP_1 dynamics with specific Lindblad generator in SM

Describing M-decoherence, that is a non-selective measurement of $M = M^* = \sum_{m\in\sigma(M)} mP_m \in M_n(\mathbb{C})$, one can rely on previous works: Baumgartner, Narnhofer; 2008, Weinberg; 2016

- Proposition The set of asymptotic states of a Lindblad evolution

$$\frac{d\rho}{dt} = \hat{L}(\rho) := -i[H,\rho] + \sum_k V_k\rho V_k^* - \frac{1}{2}\{V_k^* V_k, \rho\}.$$

is equal to $\Phi_M(S_n)$ iff $\{H, V_k, V_k^*\}'' = \langle M\rangle$. Moreover,

$$\lim_{t\to\infty} \exp(t\hat{L})(\rho_0) =: \rho_\infty = \Phi_M(\rho_0) := \sum_{m\in\sigma(M)} P_m\rho_0 P_m$$

Proof hint:

• P projection is 'conserved', $P = \exp(tL)(P), t \geq 0$ iff $P \in \{H, V_k, V_k^*\}'$ $\Rightarrow \{H, V_k, V_k^*\}' = \Phi_M(M_n(\mathbb{C})) = \langle M\rangle'$, i.e. the choice $\{H, V_k, V_k^*\}'' = \langle M\rangle$ leads to the required set of possible asymptotic states (the invariant states)

• $\{H, V_k, V_k^*\}'' = \langle M\rangle$ is abelian, hence $\hat{L} = \tilde{L}(-H, V_k^*)$ is a generator of CP_1 maps $\Rightarrow$ $\hat{\Phi}_t, t \geq 0$ are norm one maps

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1. CP_1 dynamics with specific Lindblad generator in SM

Proof hint continued:

• $\hat{L}\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is not selfadjoint (or normal in general wrt the scalar product on $M_n(\mathbb{C})$ given by the trace), but the generalized eigenvalue problem $(\hat{L} - \lambda)^k = 0$ (in Jordan blocks), hence the time evolution can be solved: $\mathrm{Re}\,\lambda \leq 0$ for $k = 1$ and $\mathrm{Re}\,\lambda < 0$ for $k > 1$, because $\hat{\Phi}_t$ is a norm one map

• nontrivial H-eigenvalues, $\mathrm{Re}\,\lambda = 0, \mathrm{Im}\,\lambda \neq 0$ are excluded, because $\{H, V_k, V_k^*\}'' = \langle M \rangle$ is abelian

$\Rightarrow$ all initial states lead to asymptotic states, which should be invariant states

2. Nonlinear effective dynamics in selective measurements

Aim: "randomly chosen" nonlinear deterministic dynamics on $S_M := S_{n|\langle M\rangle}$ which results M-pure asymptotic states P_m with probability $p_m := \mathrm{Tr}\,(\rho_0 P_m)$

$$S_M \ni \mu_0 := \rho_{\infty|\langle M\rangle} \to \lim_{t\to\infty} \mu(t) =: \mu_\infty = P_m$$

- S_M, states on $\langle M\rangle$: convex combinations of spectral projections of M
  non-selective measurement preserves probability p_m
  $\mathrm{Tr}\,(\mu_0 P_m) = \mathrm{Tr}\,(\rho_\infty P_m) := \sum_{m'\in\sigma(M)} \mathrm{Tr}\,(P_{m'}\rho_0 P_{m'} P_m) = \mathrm{Tr}\,(\rho_0 P_m) =: p_m$
  $\Rightarrow$ two-step dynamics is consistent with experiment,
  second step initial state: $\mu_0 = \sum_{m\in\sigma(M)} p_m P_m$
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  $F(-, \mu_{ext}) \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$
  • unique effective GP-dynamics: uniquely given initial state $|\psi_N\rangle\langle\psi_N|$ of the full system in the inverse image of the initial state $\gamma_N^{(1)} \simeq |\varphi_{GP}\rangle\langle\varphi_{GP}|$ of the 'measured' subsystem
  • effective description of the interaction change between the measuring device and the measured system caused by the choice of the initial state of the full system in the inverse image of the initial state μ_0 of the measured system: uniform distribution of $\mu_{ext} \in S_M$ with respect to the Lebesgue measure in $\mathbb{R}^n$ (simplest choice)

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2. Nonlinear toy model dynamics for state purification

- nonlinar dynamics on $S_M = \{\sum_{i=1}^n p_i P_i \,|\, 0 \leq p_i \leq 1, \sum p_i = 1\}$

$$\frac{d\mu}{dt} = F(\mu, \mu_{ext}) := f(\mu, \mu_{ext}) - \mu \mathrm{Tr}\, f(\mu, \mu_{ext}), \quad \mu \in S_M$$
$$f(\mu, \mu_{ext}) := \alpha\mu(\lambda\mu - \mu_{ext}) \qquad (1)$$

- $\alpha > 0$ ”evolution strength”
- $\lambda = \lambda(\mu, \mu_{ext}) := \max\{\kappa \in [0,1] \,|\, \mu_{ext} - \kappa\mu \geq 0\}$,

that is $\mu_{ext} \equiv \sum_i s_i P_i$ is the convex combintion $\mu_{ext} = \lambda\mu + \sum_{i \neq j} \lambda_i P_i$

- Theorem on the fixpoint structure of the dynamics (1)

If the external density matrix $\mu_{ext} \in S_M$ is chosen uniformly with respect to the Lebesgue measure on S_M then the asymptotic state $\mu_\infty := \lim_{t\to\infty} \mu(t)$ of the dynamics (1) on S_M with initial condition $\mu_0 = \sum_{i=1}^n p_i P_i$ is equal to P_i with probability p_i.

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$$\begin{aligned} \frac{d\mu}{dt} = F(\mu, \mu_{ext}) &:= f(\mu, \mu_{ext}) - \mu \mathrm{Tr}\, f(\mu, \mu_{ext}), \quad \mu \in S_M \\ f(\mu, \mu_{ext}) &:= \alpha\mu(\lambda\mu - \mu_{ext}) \end{aligned} \tag{1}$$

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2. Fixpoint structure of the nonlinear toy model dynamics

Proof hint

• Picard–Lindelöf theorem on first order differential equations on a region with Lipschitz continuity

$$\|F(\mu,\mu_{ext})-F(\tilde{\mu},\mu_{ext})\|_\infty \leq (4+\frac{6}{s_j})\|\mu-\tilde{\mu}\|_\infty,\ \ \mu,\tilde{\mu}\in K_j(\mu_{ext})\cap S_M$$

⇒ unique integral curves within $K_j(\mu_{ext})\cap S_M$

• for $\mu_{ext}=\lambda\mu+\sum_{i\neq j}\lambda_i P_i$ (with $0\neq\lambda\lambda_1\dots\lambda_{j-1}\lambda_{j+1}\dots\lambda_n$) the tangent vector

$$F(\mu,\mu_{ext})=\sum_{i\neq j} p_i\lambda_i(\mu-P_i)\in K_j(\mu)$$

⇒ integral curves remain in S_M and tends to the fixpoint P_j as $t\to\infty$

• uniform choice of μ_{ext} within S_M with ‘repeated’ initial state $\mu_0=\sum_i p_i P_i$

⇒ probability (= relative frequency in ‘repeated experiments’) of the asymptotic state P_j is the relative volume of the simplices $S_j(\mu_0)$ and S_M:

$$\frac{V(S_j(\mu_0))}{V(S_M)}\equiv\frac{V(\langle\mu_0,P_1,\dots,P_{j-1},P_{j+1},\dots,P_n\rangle)}{V(\langle P_1,\dots,P_n\rangle)}=p_j$$

Closing remarks

- Unbounded or continuous spectra $M = \int_{\sigma(M)} m dE(m)$
  (e.g. position operator Q in $\mathbb{R}$)
  Write $\mathbb{R} \supseteq \sigma(M)$ as a partition of finitely many spectral intervals:
  $P_1 := E(m_1), \ldots, P_i := E(m_i) - E(m_{i-1}), \ldots, P_n := \mathbf{1} - E(m_{n-1})$
- Joint measurements of commuting operators $M^{(1)}$ and $M^{(2)}$
  (e.g. position operators Q_1, Q_2, Q_3 in $\mathbb{R}^3$)
  Products of commuting spectral projections: $P^{(1)}_{m_1} P^{(2)}_{m_2}, m_i \in \sigma(M^{(i)})$
- Experimental verification of the dynamical nature of measurements:
  slow measuring process and quick swith-in/swith-off possibility of the measuring device are needed
  instead of the distribution map between $t = 0$ to $t = \infty$:
  μ_0 and uniform μ_{ext} on $S_M \mapsto \mu_\infty = P_i$ with probability p_i
  make a swith-off and a swith-in at an intermediate time $0 < T < \infty$
  $\Rightarrow$ intermediate final distribution of μ_T as initial distribution μ_T with (new) uniform μ_{ext} may lead to a different (numerically calculable) asymptotic distibution of μ_∞
- one may try one-step dynamics: $d\rho/dt = \hat{L}(\rho) + F(\rho, \mu_{ext})$

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